On Approximations by Polynomial and Trigonometrical Splines of the Fifth Order

I.G. BUROVA
St. Petersburg State University
Mathematics and Mechanics Faculty
198504, Universitetsky prospekt, 28,
Peterhof, St. Petersburg
RUSSIA
i.g.burova@spbu.ru, burovaig@mail.ru

T.O. EVDOKIMOVA
St. Petersburg State University
Mathematics and Mechanics Faculty,
198504, Universitetsky prospekt, 28,
Peterhof, St. Petersburg
RUSSIA
t.evdokimova@spbu.ru

Abstract: Here we consider several approaches for constructing approximations of a function by the polynomial and the trigonometric splines of the fifth order. We compare the approximations to the left, the right and the middle minimal polynomial splines, the approximations to the left, the right and the middle minimal trigonometrical splines, the approximations to the left, the middle polynomial integro-differential splines, and the approximation to the left, the right and the middle trigonometrical integro-differential splines. The quadrature formulas are represented. The results of some calculations are done.

Key–Words: Polynomial splines, Trigonometrical splines, Integro-Differential Splines, Interpolation.

1 Introduction

Nowadays, there are many different splines for solving different problems [1–10]. B-splines, conic splines, cubic polynomial and nonpolynomial splines, and box spline functions can be used for interpolation or approximation of scattered data, simulation of the heart waveform, plotting surfaces and etc.

Minimal splines are intended for the approximation and interpolation functions. If we know the values of the functions in grid nodes we can construct the approximation on every net interval separately. In the next sections we compare the results of the approximations to the minimal splines (see [11]), to the polynomial integro-differential splines, and to the non-polynomial integro-differential splines (see [12]) of the fifth order. Polynomial integro-differential splines were first used by Kireev V.I. [13].

In general, construction of solutions of delay differential equations is much more complicated than the construction of solutions of ordinary differential equations [14–21]. We have the Cauchy problem for a numerical solution on each interval. The solution on the interval requires the solution from the previous interval. Some necessary values may be missing among the calculated values, but they may be obtained by interpolating [20]. Interpolation should use the positions of the discontinuities of the derivatives. The application splines of the fifth order for the delay problem is presented in the last section. We need to use the values of the function only in the given interval so we use the approximations with the left and the right basic splines.

2 Splines of the fifth order

We consider the grid of equidistant nodes with the step h

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Let the function $u(x)$ be such that $u \in C^5([a, b])$. We have to use the interpolation nodes only on the interval $[a, b]$. Therefore we can use polynomial boundary-minimal splines (see [11]). Right boundary-minimal splines are used on the left side on $[a, b]$ and left boundary-minimal splines are used on the right side on the interval.

We denote by $\widetilde{u}(x)$ an approximation by the polynomial minimal splines:

$$\widetilde{u}(x) = \sum_k u(x_k)\omega_k(x),\ x \in [x_k, x_{k+1}],$$

an approximation by the trigonometric minimal splines:

$$\widetilde{u}(x) = \sum_k u(x_k)w_k(x),\ x \in [x_k, x_{k+1}],$$

an approximation by the polynomial integro-differential splines:

$$\widetilde{u}(x)=\sum_k \int_{x_{k-i_2}}^{x_{k+i_1}} u(t)\,dt\,\omega_k^{<-i_2,i_1>}(x),\ x\in(x_k,x_{k+1}),$$

an approximation by the trigonometric integro-differential splines:

$$\widetilde{u}(x)=\sum_k \int_{x_{k-i_2}}^{x_{k+i_1}} u(t)\,dt\,w_k^{<-i_2,i_1>}(x),\ x\in(x_k,x_{k+1}),$$

where i_1, i_2 are integer numbers, $\omega_k(x)$, $w_k(x)$, $\omega_k^{<-i_2,\,+i_1>}(x)$, $w_k^{<-i_2,\,i_1>}(x)$ we determine from the system:

$$\widetilde{u}(x)=u(x),\ \ u(x)=\varphi_i(x),\ i=1,2,3,4,5. \quad (1)$$

Here $\varphi_i(x)$, $i=1,2,3,4,5$, is Chebyshev system on $[a,\,b]$, $\varphi_i\in C^5([x_0,\,x_n])$.

In polynomial case we take $\varphi_i(x)=x^{i-1}$, $i=1,2,3,4,5$; in trigonometric case we take $\varphi_1(x)=1$, $\varphi_2(x)=\sin(x)$, $\varphi_3(x)=\cos(x)$, $\varphi_4(x)=\sin(2x)$, $\varphi_5(x)=\cos(2x)$.

3 Approximation by the middle polynomial integro-differential splines

Let us take an approximation for $u(x)$, $x\in(x_k,x_{k+1})$, in the form:

$$\tilde{u}(x)=\int_{x_{k-2}}^{x_{k-1}} u(t)dt\,\omega_k^{<-2,-1>}(x)+$$

$$+\int_{x_{k-1}}^{x_k} u(t)dt\,\omega_k^{<-1,0>}(x)+\int_{x_k}^{x_{k+1}} u(t)dt\,\omega_k^{<0,1>}(x)+$$

$$+\int_{x_{k+1}}^{x_{k+2}} u(t)dt\,\omega_k^{<1,2>}(x)+\int_{x_{k+2}}^{x_{k+3}} u(t)dt\,\omega_k^{<2,3>}(x), \quad (2)$$

where $\omega_k^{<s,s+1>}(x)$, $s=-2,\,-1,\,0,\,1,\,2$, we find from the system (1).

Let us take $\varphi_i(x)=x^{i-1}$, $i=1,\,2,\,3,\,4,\,5$. If we put $x=x_k+th$, $t\in(0,\,1)$, then we obtain:

$$\omega_k^{<-2,-1>}(x_k+th)=\frac{5t^4-6+15t^2+10t-20t^3}{120h}, \quad (3)$$

$$\omega_k^{<-1,0>}(x_k+th)=\frac{10t^4-27-15t^2+75t-30t^3}{60h}, \quad (4)$$

$$\omega_k^{<0,1>}(x_k+th)=\frac{30t^3-47+60t^2-15t^4-75t}{60h}, \quad (5)$$

$$\omega_k^{<1,2>}(x_k+th)=\frac{13-45t^2+10t^4+5t-10t^3}{60h}, \quad (6)$$

$$\omega_k^{<2,3>}(x_k+th)=\frac{5t^4+4-15t^2}{120h}. \quad (7)$$

Let us take $\widetilde{U}(x)$, $x\in(a,b)$, such that $\widetilde{U}(x)=\widetilde{u}(x)$, $x\in(x_j,x_{j+1})$, $j=2,3,\ldots,n-3$.

We put $\|f\|_{(x_i,x_{i+1})}=\sup_{x\in(x_i,x_{i+1})}|f(x)|$,

$$\|f\|=\|f\|_{X(a,b)}=\max_i \sup_{x\in(x_i,x_{i+1})}|f(x)|.$$

Theorem 1. Let function $u(x)$ be such that $u\in C^5([a,b])$. For approximation $u(x)$, $x\in(x_k,x_{k+1})$ by (2), (3)–(7) we have:

$$|\widetilde{u}(x)-u(x)|\le K_1h^5\|u^{(5)}\|_{(x_{k-2},x_{k+3})}, \quad (8)$$

$$R_1=\|\widetilde{U}-u\|_{X(x_2,x_{n-2})}\le K_1h^5\|u^{(5)}\|_{X(x_0,x_n)}, \quad (9)$$

$K_1=0.028$.

Proof. Inequality (8) follows from the relations (3)–(7) and Taylor formula with the remainder term in Lagrange form. Here the next inequalities were used: $|\omega_k^{<-2,-1>}(x)|\le 1/(20h)$, $|\omega_k^{<0,1>}(x)|\le \frac{1067}{960h}$, $|\omega_k^{<-1,0>}(x)|\le 9/(20h)$, $|\omega_k^{<1,2>}(x)|\le 9/(20h)$, $|\omega_k^{<2,3>}(x)|\le 1/(20h)$.

Inequality (9) follows from (8).

4 Approximation by middle polynomial minimal splines of Lagrange type

Consider the case of the middle minimal splines. Let us take an approximation for $u\in C^5([a,b])$, $x\in[x_k,x_{k+1}]$, in the form:

$$\tilde{u}(x)=u(x_{k-2})\,\omega_{k-2}(x)+u(x_{k-1})\,\omega_{k-1}(x)+$$

$$+u(x_k)\,\omega_k(x)+u(x_{k+1})\,\omega_{k+1}(x)+ \quad (10)$$

$$+u(x_{k+2})\,\omega_{k+2}(x),$$

where $supp\,\omega_j=[x_{j-2},x_{j+3}]$, ω_{k+s}, $s=-2,\,-1,\,0,\,1,\,2$, we find from the system (1).

Let us take $\varphi_i(x)=x^{i-1}$, $i=1,\,2,\,3,\,4,\,5$.

If we put $x = x_k + th$, $t \in [0, 1]$, then we have

$$\omega_{k-2}(x_k + th) = \frac{t(t-1)(t-2)(t+1)}{24}, \quad (11)$$

$$\omega_{k-1}(x_k + th) = \frac{-t(t-1)(t-2)(t+2)}{6}, \quad (12)$$

$$\omega_k(x_k + th) = \frac{(t-1)(t-2)(t+2)(t+1)}{4}, \quad (13)$$

$$\omega_{k+1}(x_k + th) = \frac{-t(t-2)(t+2)(t+1)}{6}, \quad (14)$$

$$\omega_{k+2}(x_k + th) = \frac{t(t-1)(t+2)(t+1)}{24}. \quad (15)$$

Theorem 2. Let function u be such that $u \in C^5([a,b])$. For approximation $u(x)$, $x \in [x_k, x_{k+1}]$ by (10), (11)–(15) we have:

$$|\widetilde{u}(x) - u(x)| \leq K_2 h^5 \|u^{(5)}\|_{[x_{k-2},x_{k+2}]}, \quad (16)$$

$$R_2 = \|\widetilde{U} - u\|_{[x_2,x_{n-2}]} \leq K_2 h^5 \|u^{(5)}\|_{[x_0,x_n]}, \quad (17)$$

$K_2 = 0.012$.

Proof. Inequality (16) follows from the inequality:

$$|\widetilde{u}(x) - u(x)|_{[x_k,x_{k+1}]} \leq \frac{1}{5!} \max_{[x_{k-2},x_{k+2}]} |u^{(5)}(x)| \times$$

$$\times \max_{[x_k,x_{k+1}]} |(x - x_{k-2})(x - x_{k-1}) \times$$

$$\times (x - x_k)(x - x_{k+1})(x - x_{k+2})|.$$

Inequality (17) follows from (16).

Table 1 shows the actual errors of approximation of the functions. Here $R_M^{<P>}$ is the actual error of approximation by the splines (2), (3)–(7), R_M^P is the actual error of approximation by the splines (10), (11)–(15) on $(-1, 1)$ when $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 1. Actual errors of approximations by the splines (2), (3)–(7), and by the splines (10), (11)–(15).

$u(x)$	$R_M^{<P>}$	R_M^P
$1/(1 + 25x^2)$	0.0167	0.0124
$\sin(x)$	$0.166 \cdot 10^{-6}$	$0.118 \cdot 10^{-6}$
$\sin(3x)$	$0.393 \cdot 10^{-4}$	$0.284 \cdot 10^{-4}$
x^5	$0.20 \cdot 10^{-4}$	$0.142 \cdot 10^{-4}$

Figure 1 shows the error of approximation of the function $1/(1+25x^2)$ by the middle minimal polynomial middle splines (10), (11)–(15).

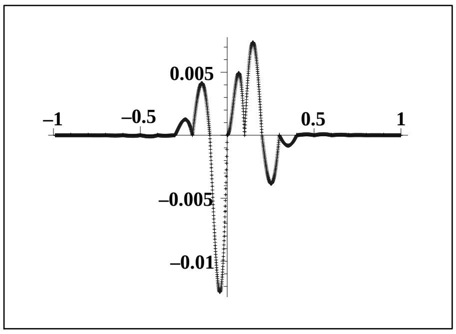

Figure 1: Plot of the error of approximation $u(x) = 1/(1+25x^2)$ by the middle polynomial spline (10), (11)–(15)

4.1 Quadrature formula

From the approximation by the minimal polynomial middle splines (10), (11)–(15) on $[x_k, x_{k+1}]$ we receive:

$$\int_{x_k}^{x_{k+1}} \widetilde{u}(t)dt = h\Big(\tfrac{11}{720} u(x_{k-2}) - \tfrac{37}{360} u(x_{k-1}) +$$

$$+ \tfrac{19}{30} u(x_k) + \tfrac{173}{360} u(x_{k+1}) - \tfrac{19}{720} u(x_{k+2})\Big). \quad (18)$$

Now in (2) we can use (18). We have:

$$\int_{x_k}^{x_{k+1}} u(t)dt = \int_{x_k}^{x_{k+1}} \widetilde{u}(t)dt + r,$$

where $r = 0$ if $u(x) = x^{i-1}$, $i = 1, 2, 3, 4, 5$.

4.2 Left polynomial splines

For $x \in [x_k, x_{k+1}]$ we take $\tilde{u}(x)$ in the form:

$$\tilde{u}(x) = u(x_{k-3})\, \omega_{k-3}(x) + u(x_{k-2})\, \omega_{k-2}(x) +$$
$$+ u(x_{k-1})\, \omega_{k-1}(x) + u(x_k)\, \omega_k(x) + \quad (19)$$
$$+ u(x_{k+1})\, \omega_{k+1}(x),$$

where $supp\, \omega_j = [x_{j-1}, x_{j+4}]$, $\omega_{k+s}(x)$, $s = -3, -2, -1, 0, 1$, we find from the system (1).

Let us take $\varphi_i(x) = x^{i-1}$, $i = 1, 2, 3, 4, 5$.

If we put $x = x_k + th$, $t \in [0, 1]$, then we have:

$$\omega_{k+1}(x_k + th) = \frac{t(t+1)(t+2)(t+3)}{24}, \quad (20)$$

$$\omega_k(x_k + th) = \frac{-(t^2-1)(t+2)(t+3)}{6}, \quad (21)$$

$$\omega_{k-1}(x_k + th) = \frac{t(t-1)(t+2)(t+3)}{4}, \quad (22)$$

$$\omega_{k-2}(x_k + th) = \frac{-t(t+3)(t^2-1)}{6}, \quad (23)$$

$$\omega_{k-3}(x_k+th)=\frac{t(t^2-1)(t+2)}{24}. \quad (24)$$

Let us take $\widetilde{U}(x)$, $x\in(a,b)$, such that $\widetilde{U}(x)=\widetilde{u}(x)$, $x\in(x_j,x_{j+1})$, $j=3,4,\dots,n-1$.

Theorem 3. Let function $u(x)$ be such that $u\in C^5([a,b])$. For approximation $u(x)$, $x\in[x_k,x_{k+1}]$ by (19), (20)–(24) we have the estimation:

$$|\widetilde{u}(x)-u(x)|\le K_2h^5\|u^{(5)}\|_{[x_{k-3},x_{k+1}]}, \quad (25)$$

$$R_2=\|\widetilde{U}-u\|_{[x_3,x_n]}\le K_2h^5\|u^{(5)}\|_{[x_0,x_n]}, \quad (26)$$

where $K_2=0.0303$.

Proof. The inequality (25) follows from the next relation:

$$|\widetilde{u}(x)-u(x)|_{[x_k,x_{k+1}]}\le\frac{1}{5!}\max_{[x_{k-3},x_{k+1}]}|u^{(5)}(x)|\times$$

$$\times\max_{[x_k,x_{k+1}]}|(x-x_{k-3})(x-x_{k-2})\times$$

$$\times(x-x_{k-1})(x-x_k)(x-x_{k+1})|.$$

Inequality (26) follows from (25).

Figure 2 shows the error of approximation of the function $1/(1+25x^2)$ by the left polynomial splines (19), (20)–(24).

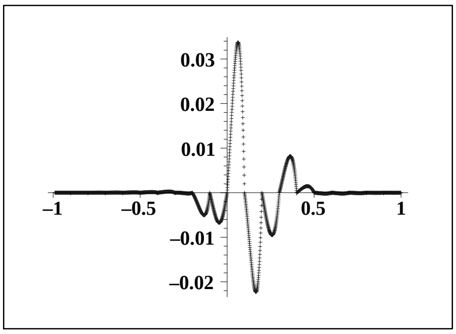

Figure 2: Plot of the error of approximation $u(x)=1/(1+25x^2)$ by the left polynomial spline (19), (20)–(24)

4.3 Quadrature formula

From the approximation by the minimal polynomial left splines (19), (20)–(24) on $[x_k,x_{k+1}]$ we receive the next formula:

$$\int_{x_k}^{x_{k+1}}\widetilde{u}(t)dt=h\Big(-\tfrac{19}{720}u(x_{k-3})+\tfrac{53}{360}u(x_{k-2})-$$

$$-\tfrac{11}{30}u(x_{k-1})+\tfrac{323}{360}u(x_k)+\tfrac{251}{720}u(x_{k+1})\Big). \quad (27)$$

Now in (2) we can use (27). We have:

$$\int_{x_k}^{x_{k+1}}u(t)dt=\int_{x_k}^{x_{k+1}}\widetilde{u}(t)dt+r,$$

where $r=0$ if $u(x)=x^{i-1}$, $i=1,2,3,4,5$.

4.4 Right polynomial splines

For $x\in[x_k,x_{k+1}]$ we take $\tilde{u}(x)$ in the form:

$$\tilde{u}(x)=u(x_k)\,\omega_k(x)+u(x_{k+1})\,\omega_{k+1}(x)+$$

$$+u(x_{k+2})\,\omega_{k+2}(x)+u(x_{k+3})\,\omega_{k+3}(x)+ \quad (28)$$

$$+u(x_{k+4})\,\omega_{k+4}(x),$$

where $supp\,\omega_j=[x_{j-4},x_{j+1}]$, $\omega_{k+s}(x)$, $s=1,2,3,4,5$, we find from the system (1) for $\varphi_i(x)=x^{i-1}$, $i=1,2,3,4,5$.

If we put $x=x_k+th$, $t\in[0,1]$, then we have:

$$\omega_k(x_k+th)=\frac{(t-4)(t-3)(t-2)(t-1)}{24}, \quad (29)$$

$$\omega_{k+1}(x_k+th)=\frac{-t(t-2)(t-3)(t-4)}{6}, \quad (30)$$

$$\omega_{k+2}(x_k+th)=\frac{t(t-1)(t-3)(t-4)}{4}, \quad (31)$$

$$\omega_{k+3}(x_k+th)=\frac{-t(t-1)(t-2)(t-4)}{6}, \quad (32)$$

$$\omega_{k+4}(x_k+th)=\frac{t(t-1)(t-2)(t-3)}{24}. \quad (33)$$

Let us take $\widetilde{U}(x)$, $x\in(a,b)$, such that $\widetilde{U}(x)=\widetilde{u}(x)$, $x\in(x_j,x_{j+1})$, $j=0,1,\dots,n-5$.

Theorem 4. Let function $u(x)$ be such that $u\in C^5([a,b])$. For approximation u, $x\in[x_k,x_{k+1}]$ by (28), (29)–(33) we have

$$|\widetilde{u}(x)-u(x)|\le K_2h^5\|u^{(5)}\|_{[x_k,x_{k+4}]}, \quad (34)$$

$$R_2=\|\widetilde{U}-u\|_{[x_0,x_{n-4}]}\le K_2h^5\|u^{(5)}\|_{[x_0,x_n]}, \quad (35)$$

$K_2=0.0303$.

Proof. Inequality (34) follows from the relation:

$$|\widetilde{u}(x)-u(x)|_{[x_k,x_{k+1}]}\le\frac{1}{5!}\max_{[x_k,x_{k+4}]}|u^{(5)}(x)|\times$$

$$\times\max_{[x_k,x_{k+1}]}|(x-x_k)(x-x_{k+1})\times$$

$$\times(x-x_{k+2})(x-x_{k+3})(x-x_{k+4})|.$$

Inequality (35) follows from (34).

Table 2 shows the actual errors of approximation by the left and right splines. Here R_L^P is the actual error of approximation by the splines (19), (20)–(24), and R_R^P is the actual error of approximation by the splines (28), (29)–(33) on $(-1, 1)$ when $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 2. The actual errors of approximations by the left splines (19), (20)–(24) and the actual errors of approximations by the right splines (28), (29)–(33).

$u(x)$	R_L^P	R_R^P
$1/(1+25x^2)$	0.0337	0.0337
$\sin(x)$	$0.302\cdot 10^{-6}$	$0.302\cdot 10^{-6}$
$\sin(3x)$	$0.724\cdot 10^{-4}$	$0.724\cdot 10^{-4}$
x^5	$0.363\cdot 10^{-4}$	$0.363\cdot 10^{-4}$

Figure 3 shows the error of approximation of the function $1/(1+25x^2)$ by the right polynomial splines (28), (29)–(33).

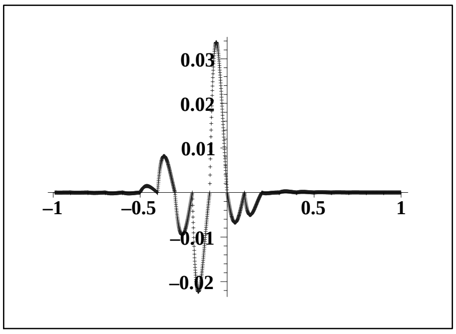

Figure 3: Plot of the error of approximation $u(x) = 1/(1+25x^2)$ by the right polynomial splines (28), (29)–(33)

4.5 Quadrature formula

From the approximation by the minimal polynomial right splines (28), (29)–(33) on $[x_k, x_{k+1}]$ we receive:

$$\int_{x_k}^{x_{k+1}} \widetilde{u}(t)dt = h\Big(\tfrac{251}{720}u(x_k) + \tfrac{323}{360}u(x_{k+1}) -$$

$$-\tfrac{11}{30}u(x_{k+2}) + \tfrac{53}{360}u(x_{k+3}) - \tfrac{19}{720}u(x_{k+4})\Big). \quad (36)$$

Now in (2) we can use (36). We have:

$$\int_{x_k}^{x_{k+1}} u(t)dt = \int_{x_k}^{x_{k+1}} \widetilde{u}(t)dt + r,$$

where $r = 0$ if $u(x) = x^{i-1}$, $i = 1, 2, 3, 4, 5$.

5 Middle trigonometrical splines

Let us take an approximation for $u \in C^5([a, b])$, $x \in [x_k, x_{k+1}]$, in the form:

$$\tilde{u}(x) = u(x_{k-2})\, w_{k-2}(x) + u(x_{k-1})\, w_{k-1}(x) +$$
$$+u(x_k)\, w_k(x) + u(x_{k+1})\, w_{k+1}(x) + \quad (37)$$
$$+u(x_{k+2})\, w_{k+2}(x),$$

where $supp\, w_j = [x_{j-2}, x_{j+3}]$, w_{k+s}, $s = -2, -1, 0, 1, 2$, we find from the system (1), where $\varphi_1(x) = 1$, $\varphi_2(x) = \sin(x)$, $\varphi_3(x) = \cos(x)$, $\varphi_4(x) = \sin(2x)$, $\varphi_5(x) = \cos(2x)$.

If we put $x = x_k + th$, $x \in [x_k, x_{k+1}]$, $t \in [0, 1]$, then:

$$w_{k-2}(x_k+th) = \frac{S_1}{\sin(h/2)\sin(h)\sin(3h/2)\sin(2h)}, \quad (38)$$
$$S_1 = \sin\big(\tfrac{th+h}{2}\big)\sin\big(\tfrac{th}{2}\big)\sin\big(\tfrac{h-th}{2}\big)\sin\big(h - \tfrac{th}{2}\big),$$

$$w_{k-1}(x_k+th) = -\frac{S_2}{\sin^2(h/2)\sin(h)\sin(3h/2)}, \quad (39)$$
$$S_2 = \sin\big(\tfrac{th}{2} + h\big)\sin\big(\tfrac{th}{2}\big)\sin\big(\tfrac{h-th}{2}\big)\sin\big(h - \tfrac{th}{2}\big),$$

$$w_k(x_k+th) = \frac{S_3}{\sin^2(h)\sin^2(h/2)}, \quad (40)$$

$$S_3 = \sin\big(\tfrac{th}{2}+h\big)\sin\big(\tfrac{th+h}{2}\big)\sin\big(\tfrac{h-th}{2}\big)\sin\big(h - \tfrac{th}{2}\big),$$

$$w_{k+1}(x_k+th) = \frac{S_4}{\sin(3h/2)\sin(h)\sin^2(h/2)}, \quad (41)$$

$$S_4 = \sin\big(\tfrac{th}{2} + h\big)\sin\big(\tfrac{th+h}{2}\big)\sin\big(\tfrac{th}{2}\big)\sin\big(h - \tfrac{th}{2}\big),$$

$$w_{k+2}(x_k+th) = \frac{S_5}{\sin(2h)\sin(3h/2)\sin(h)\sin(h/2)}, \quad (42)$$
$$S_5 = \sin\big(\tfrac{th}{2} + h\big)\sin\big(\tfrac{th+h}{2}\big)\sin\big(\tfrac{th}{2}\big)\sin\big(\tfrac{th-h}{2}\big).$$

It can be shown that for the polynomial basic splines $\omega_s(x)$ (10) and trigonometrical basic splines $w_j(x)$ (37) the next relation is fulfilled $w_j(x_k+th) = \omega_j(x_k+th) + O(h^2)$, $j = k-2, k-1, \ldots, k+2$.

Figure 4 shows the error of approximation of the function $1/(1+25x^2)$ by the trigonometrical splines (37), (38)–(42).

Theorem 5. The error of the approximation by the splines (37), (38)–(42) is the next:

$$|\widetilde{u}(x) - u(x)| \le Kh^5 \|4u' + 5u''' + u^V\|_{[x_{k-2}, x_{k+2}]}, \quad (43)$$

where $x \in [x_k, x_{k+1}]$, $K = 0.1$.

Proof. The function $u(x)$ on $[x_k, x_{k+1}]$ can be written in the form (see [12]) $u(x) = \frac{2}{3}\int_{x_k}^{x}(4u'(\tau)+$

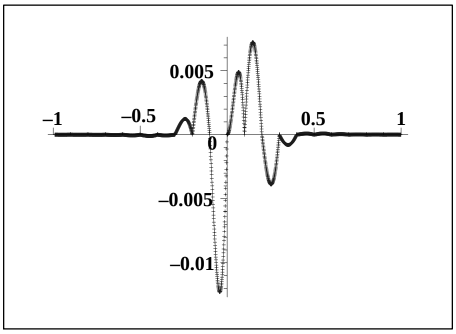

Figure 4: Plot of the error of approximation $1/(1+25x^2)$ by the trigonometrical splines (37), (38)–(42).

$5u'''(\tau)+u^V(\tau))\sin^4(x/2-\tau/2)d\tau+c_1+c_2\sin(x)+c_3\cos(x)+c_4\sin(2x)+c_5\cos(2x)$, where c_i, $i=1,2,$ 3, 4, 5 are arbitrary constants. We have: $|w_{k-2}(x)|\le 0.08$, $|w_{k-1}(x)|\le 0.21$, $|w_k(x)|\le 1$, $|w_{k+1}(x)|\le 1$, $|w_{k+2}(x)|\le 0.08$. Using the method from [12] we obtain (43).

5.1 Left trigonometrical splines

For $x\in[x_k,x_{k+1}]$ we take $\tilde{u}(x)$ in the form:

$$\tilde{u}(x)=u(x_{k-3})\,w_{k-3}(x)+u(x_{k-2})\,w_{k-2}(x)+$$
$$+u(x_{k-1})\,w_{k-1}(x)+u(x_k)\,w_k(x)+ \quad (44)$$
$$+u(x_{k+1})\,w_{k+1}(x),$$

where $supp\,\omega_j=[x_{j-1},x_{j+4}]$, $w_{k+s}(x)$, $s=-3,-2,-1,0,1$, we find from the system (1).

Let us take $\varphi_1(x)=1$, $\varphi_2(x)=\sin(x)$, $\varphi_3(x)=\cos(x)$, $\varphi_4(x)=\sin(2x)$, $\varphi_5(x)=\cos(2x)$.

If we put $x=x_k+th$, $t\in[0,1]$, then we have:

$$w_{k+1}(x_k+th)=T_{k+1}/T_1, \quad (45)$$

$T_{k+1}=\sin\left(\frac{th+3h}{2}\right)\sin\left(\frac{th}{2}+h\right)\sin\left(\frac{th+h}{2}\right)\sin\left(\frac{th}{2}\right)$,
$T_1=\sin(2h)\sin\left(\frac{3h}{2}\right)\sin(h)\sin\left(\frac{h}{2}\right)$,

$$w_k(x_k+th)=T_k/T_2, \quad (46)$$

$T_k=\sin\left(\frac{th+3h}{2}\right)\sin\left(\frac{th}{2}+h\right)\sin\left(\frac{th+h}{2}\right)\sin\left(\frac{h-th}{2}\right)$,
$T_2=\sin\left(\frac{3h}{2}\right)\sin(h)\sin^2\left(\frac{h}{2}\right)$,

$$w_{k-1}(x_k+th)=T_{k-1}/T_3, \quad (47)$$

$T_{k-1}=\sin\left(\frac{th+3h}{2}\right)\sin\left(\frac{th}{2}+h\right)\sin\left(\frac{th}{2}\right)\sin\left(\frac{th-h}{2}\right)$,
$T_3=\sin^2(h)\sin^2\left(\frac{h}{2}\right)$,

$$w_{k-2}(x_k+th)=T_{k-2}/T_4, \quad (48)$$

$T_{k-2}=\sin\left(\frac{th+3h}{2}\right)\sin\left(\frac{th+h}{2}\right)\sin\left(\frac{th}{2}\right)\sin\left(\frac{h-th}{2}\right)$,
$T_4=\sin^2\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$,

$$w_{k-3}(x_k+th)=T_{k-3}/T_5, \quad (49)$$

$T_{k-3}=\sin\left(\frac{th}{2}+h\right)\sin\left(\frac{th+h}{2}\right)\sin\left(\frac{th}{2}\right)\sin\left(\frac{th-h}{2}\right)$,
$T_5=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)$.

Theorem 6. The error of the approximation by the splines (44), (45)–(49) is the next:

$$|\widetilde{u}(x)-u(x)|\le Kh^5\|4u'+5u'''+u^V\|_{[x_{k-3},x_{k+1}]}, \quad (50)$$

where $x\in[x_k,x_{k+1}]$, $K=0.5$.

Proof is similar to that done in the proof of Theorem 5. Here the next inequalities were used: $|w_{k-3}(x)|\le 0.12$, $|w_{k-2}(x)|\le 0.22$, $|w_{k-1}(x)|\le 0.36$, $|w_k(x)|\le 1$, $|w_{k+1}(x)|\le 1.06$.

5.2 Right trigonometrical splines

For $x\in[x_k,x_{k+1}]$ we take $\tilde{u}(x)$ in the form:

$$\tilde{u}(x)=u(x_k)\,w_k(x)+u(x_{k+1})\,w_{k+1}(x)+$$
$$+u(x_{k+2})\,w_{k+2}(x)+u(x_{k+3})\,w_{k+3}(x)+ \quad (51)$$
$$+u(x_{k+4})\,w_{k+4}(x),$$

where $supp\,w_j=[x_{j-4},x_{j+1}]$, w_{k+s}, $s=0,1,2,3,$ 4, we find from the system (1).

Let us take $\varphi_1(x)=1$, $\varphi_2(x)=\sin(x)$, $\varphi_3(x)=\cos(x)$, $\varphi_4(x)=\sin(2x)$, $\varphi_5(x)=\cos(2x)$.

If we put $x=x_k+th$, $t\in[0,1]$, then we have:

$$w_k(x_k+th)=T_k^R/T_{0R}, \quad (52)$$

$T_k^R=\sin\left(\frac{th}{2}-2h\right)\sin\left(\frac{th-3h}{2}\right)\sin\left(\frac{th}{2}-h\right)\sin\left(\frac{th-h}{2}\right)$,
$T_{0R}=\sin(2h)\sin(3h/2)\sin(h)\sin(h/2)$,

$$w_{k+1}(x_k+th)=T_{k+1}^R/T_{1R}, \quad (53)$$

$T_{k+1}^R=\sin\left(\frac{th}{2}-2h\right)\sin\left(\frac{th-3h}{2}\right)\sin\left(h-\frac{th}{2}\right)\sin\left(\frac{th}{2}\right)$,
$T_{1R}=\sin(3h/2)\sin(h)\sin^2(h/2)$,

$$w_{k+2}(x_k+th)=T_{k+2}^R/T_{2R}, \quad (54)$$

$T_{k+3}^R=\sin\left(\frac{th}{2}-2h\right)\sin\left(\frac{th-3h}{2}\right)\sin\left(\frac{th-h}{2}\right)\sin\left(\frac{th}{2}\right)$,
$T_{2R}=\sin^2(h)\sin^2(h/2)$,

$$w_{k+3}(x_k+th)=T_{k+3}^R/T_{3R}, \quad (55)$$

$T_{k+3}^R=\sin\left(\frac{th}{2}-2h\right)\sin\left(h-\frac{th}{2}\right)\sin\left(\frac{th-h}{2}\right)\sin\left(\frac{th}{2}\right)$,
$T_{3R}=\sin^2(h/2)\sin(h)\sin(h/2)$,

$$w_{k+4}(x_k+th)=T_{k+4}^R/T_{4R}, \quad (56)$$

$T_{k+4}^R=\sin\left(\frac{th-3h}{2}\right)\sin\left(\frac{th}{2}-h\right)\sin\left(\frac{th-h}{2}\right)\sin\left(\frac{th}{2}\right)$,
$T_{4R}=\sin(h/2)\sin(h)\sin(3h/2)\sin(2h)$.

Theorem 7. The error of the approximation by the splines (51), (52)–(56) is the next:

$$|\widetilde{u}(x) - u(x)| \leq Kh^5 \|4u' + 5u''' + u^V\|_{[x_k, x_{k+4}]},$$

where $x \in (x_k, x_{k+1}),\ K = 2$.

Proof is similar to that done in the proof of Theorem 5. Here the next inequalities were used:

$|w_k(x)| \leq 1$, $|w_{k+1}(x)| \leq 1.1$, $|w_{k+2}(x)| \leq 0.36$, $|w_{k+3}(x)| \leq 0.22$, $|w_{k+4}(x)| \leq 0.12$.

Table 3 shows the actual errors of approximation by the left and right trigonometrical splines. R_L^T is the actual error of approximation by the splines (44), (45)–(49). R_R^T is the actual error of approximation by the splines (51), (52)–(56) on $(-1, 1)$ when $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 3. Actual errors of approximations by the left splines (44), (45)–(49), and by the right splines (51), (52)–(56).

$u(x)$	R_L^T	R_R^T
$1/(1+25x^2)$	0.0333	0.0333
$\sin(x)$	0.0	0.0
$\sin(3x)$	$0.358 \cdot 10^{-4}$	$0.358 \cdot 10^{-4}$
x^5	$0.15 \cdot 10^{-3}$	$0.15 \cdot 10^{-3}$

Table 4 shows the actual and theoretical errors of approximation by the middle trigonometrical splines. R_M^T is the actual error of approximation by the splines (37), (38)–(42). $\mathcal{R}_M^T$ is the theoretical error of approximation by the splines (37), (38)–(42) on $(-1, 1)$ when $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 4. Actual errors of approximations by the middle splines (37), (38)–(42), and theoretical errors of approximation by the middle splines (37), (38)–(42).

$u(x)$	R_M^T	$\mathcal{R}_M^T$
$1/(1+25x^2)$	0.0123	0.918
$\sin(x)$	0.0	0.0
$\sin(3x)$	$0.141 \cdot 10^{-4}$	$0.36 \cdot 10^{-3}$
x^5	$0.517 \cdot 10^{-4}$	$0.13 \cdot 10^{-2}$

6 Approximation by polynomial integro-differential splines in special form

Let us take for $x \in (x_j, x_{j+1})$:

$$\widetilde{u}(x) = J_1 \omega_j^{<0,2>}(x) +$$

$$+ J_2 \omega_j^{<-1,1>}(x) + J_3 \omega_j^{<-2,1>}(x) +$$

$$+ J_4 \omega_j^{<-3,1>}(x) + J_5 \omega_j^{<-4,1>}(x), \quad (57)$$

where

$$J_1 = \int_{x_j}^{x_{j+2}} u(t)dt,\ J_2 = \int_{x_{j-1}}^{x_{j+1}} u(t)dt, \quad (58)$$

$$J_3 = \int_{x_{j-2}}^{x_{j+1}} u(t)dt,\ J_4 = \int_{x_{j-3}}^{x_{j+1}} u(t)dt, \quad (59)$$

$$J_5 = \int_{x_{j-4}}^{x_{j+1}} u(t)dt. \quad (60)$$

From $\widetilde{u}(x) = u(x)$, $u = x^{i-1}$, $i = 1, 2, 3, 4, 5$, we find $\omega_j^{<0,2>}(x)$, $\omega_j^{<-1,1>}(x)$, $\omega_j^{<-2,1>}(x)$, $\omega_j^{<-3,1>}(x)$, $\omega_j^{<-4,1>}(x)$.

So we have for $x = x_j + th$, $t \in (0, 1)$,

$$\omega_j^{<0,2>}(x_j + th) = \frac{30t + 75t^2 + 36t^3 + 5t^4 - 26}{384h}, \quad (61)$$

$$\omega_j^{<-1,1>}(x_j + th) = \frac{182t - 33t^2 - 76t^3 - 15t^4 + 62}{96h}, \quad (62)$$

$$\omega_j^{<-2,1>}(x_j + th) = \frac{6 - 898t - 69t^2 + 356t^3 + 85t^4}{384h}, \quad (63)$$

$$\omega_j^{<-3,1>}(x_j + th) = \frac{57t^2 - 25t^4 + 186t - 84t^3 - 14}{192h}, \quad (64)$$

$$\omega_j^{<-4,1>}(x_j + th) = \frac{55t^4 - 310t - 135t^2 + 140t^3 + 34}{1920h}. \quad (65)$$

Theorem 8. Suppose the function $u(x)$ be such that $u \in C^5([x_0, x_n])$, $\widetilde{u}(x)$ is given by (57)–(65). Then for $x \in (x_j, x_{j+1})$ we have:

$$|\widetilde{u}(x) - u(x)| \leq K_5 h^5 \|u^{(5)}\|_{(x_{j-4}, x_{j+2})}, \quad (66)$$

$$K_5 = 0.1625.$$

Proof. We have from (61)–(65):

$$|\omega_j^{<0,2>}(x)| \leq 120/(384h) = 0.3125/h,$$

$$|\omega_j^{<-1,1>}(x)| \leq 143.5734/(96h) \approx 1.4956/h,$$

$$|\omega_j^{<-2,1>}(x)| \leq 544.40331/(384h) \approx 1.4178/h,$$

$$|\omega_j^{<-3,1>}(x)| \leq 122.04525/(192h) \approx 0.6357/h,$$

$$|\omega_j^{<-4,1>}(x)| \leq 217.5077/(1920h) \approx 0.1133/h.$$

Representing $u(x)$ by the Taylor formula one obtains (66).

Let us take $\widetilde{U}(x)$, $x \in (a, b)$, such that $\widetilde{U}(x) = \widetilde{u}(x)$, $x \in (x_j, x_{j+1})$, $j = 4, 5, \ldots, n-3$.

Theorem 9. Suppose the hypothesis of the Theorem 3 is fulfilled. Then:

$$R = \|\widetilde{U} - u\|_{(x_4, x_{n-2})} \le Kh^5 \|u^{(5)}\|_{(a,b)}, \quad (67)$$

$$K = 0.1625.$$

Proof. Inequality (67) follows from the relation (66).

Table 5 shows the errors of approximation of functions by the splines (57)–(65) on $(-1, 1)$ when $h = 0.1$. The calculations of the actual error $R^{<P>}$ were done in Maple, Digits=15.

Table 5. The errors of approximation of functions by the splines (57)–(65)

N	$u(x)$	$R^{<P>}$
1	$1/(1+25x^2)$	0.08097
2	$\sin(x)$	$0.145 \cdot 10^{-5}$
3	$\sin(3x)$	$0.344 \cdot 10^{-3}$
4	x^5	$0.175 \cdot 10^{-3}$

Figure 5 shows the errors of approximation of the function $1/(1 + 25x^2)$ by the polynomial integro-differential splines (57)–(65) on $(-1, 1)$, $h = 0.1$.

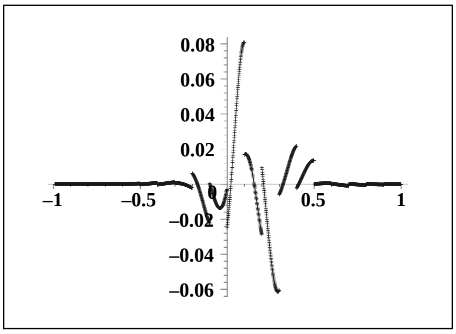

Figure 5: Plot of the error of approximation of the function $1/(1 + 25x^2)$ by the polynomial integro-differential splines (57)–(65).

7 Trigonometric integro-differential splines

Let us take for $x \in (x_j, x_{j+1})$:

$$\widetilde{u}(x) = \int_{x_{j-2}}^{x_{j-1}} u(t)dt\, w_j^{<-2,-1>}(x) +$$

$$+ \int_{x_{j-1}}^{x_j} u(t)dt\, w_j^{<-1,0>}(x) + \int_{x_j}^{x_{j+1}} u(t)dt\, w_j^{<0,1>}(x) +$$

$$+ \int_{x_{j+1}}^{x_{j+2}} u(t)dt\, w_j^{<1,2>}(x) + \int_{x_{j+2}}^{x_{j+3}} u(t)dt\, w_j^{<2,3>}(x), \quad (68)$$

where $w_j^{<-2,-1>}(x)$, $w_j^{<-1,0>}(x)$, $w_j^{<0,1>}(x)$, $w_j^{<1,2>}(x)$, $w_j^{<2,3>}(x)$ we find from $\widetilde{u}(x) = \varphi_i(x)$, $\varphi_1(x) = 1$, $\varphi_2(x) = \sin(x)$, $\varphi_3(x) = \cos(x)$, $\varphi_4(x) = \sin(2x)$, $\varphi_5(x) = \cos(2x)$.

So we have for $x = x_j + th$, $t \in (0, 1)$:

$$w_j^{<-2,-1>}(x_j + th) = \Big(2 \sin \tfrac{3h(t-1)}{2} \sin \tfrac{h(t+1)}{2} +$$
$$2(2\cos(h) + 1) \sin \tfrac{h(t+1)}{2} \sin \tfrac{h(1-t)}{2}\Big) \times$$
$$\times \left(2\sin(2h) - \sin(h) - h\cos(h)(2\cos(h) - 1)\right)^{-1}, \quad (69)$$

$$w_j^{<-1,0>}(x_j + th) = \Big(\sin(h) + 2\sin \tfrac{h}{2} \cos \tfrac{4th-h}{2} -$$
$$-4\cos(2h) \sin \tfrac{h(t-1)}{2} \cos \tfrac{h(t+1)}{2} +$$
$$+2\sin(th - 2h)\cos(h)\Big) \times$$
$$\times \Big(8\cos^3(h) + 2h\sin(2h)\cos(h) - 7\cos^2(h) -$$
$$-4\cos(h) - h\sin(h) + 3\Big)^{-1}, \quad (70)$$

$$w_j^{<0,1>}(x_j + th) = \Big(\cos(3h) + \cos(h) + 1 + \cos(2th) -$$
$$-2(\cos(h) + 1)(2\cos(h) - 1)\cos(th)\Big) \times$$
$$\times \Big(2h\cos(h)\cos(2h) - 2\sin(2h)\cos(h) -$$
$$-\tfrac{1}{2}\sin(2h) + h + 2\sin(h)\Big)^{-1}, \quad (71)$$

$$w_j^{<1,2>}(x_j + th) = \Big(4\cos(2h) \sin \tfrac{h(t+1)}{2} \cos \tfrac{h(t-1)}{2} -$$
$$-2\sin \tfrac{h(2t-1)}{2} \cos \tfrac{h(2t+1)}{2} + \sin(h + 2th) -$$
$$-2\sin(2h + th)\cos(h)\Big) \times$$
$$\times \Big(\cos(4h) - 2\cos(3h) + 4\cos(2h) - 2\cos^3(h) -$$
$$-\cos^2(h) + 2h\sin(2h)\cos(h) - h\sin(h)\Big)^{-1}, \quad (72)$$

$$w_j^{<2,3>}(x_j + th) = \Big(4\cos(h)\cos(th + \tfrac{h}{2})\cos \tfrac{h}{2} -$$
$$-\cos(h + 2th) - \cos(h)(2\cos(h) + 1)\Big) \times$$
$$\times \left(\sin(2h)\cos(h) + \sin(h) - 2h\cos^2(h) - h\cos(h)\right)^{-1}. \quad (73)$$

If we don't know the value of the integrals we can use quadrature formula. For example, from trigonometrical splines (37), (38)–(42) we obtain $\int\limits_{x_j}^{x_{j+1}} u(t)dt = u(x_{j-2})J_{-2}+u(x_{j-1})J_{-1}+u(x_j)J_0+u(x_{j+1})J_1 + u(x_{j+2})J_2$,

where

$$J_{-2} = \left(8\sin^2(h)(\cos(h)-\cos(2h))\cos(h)\right)^{-1}\times$$
$$\times\left(-2\sin(2h)+\sin(h)+h\cos(h)(2\cos(h)+1)\right),$$
$$J_{-1} = \left(4\sin^2(h)(\cos(h)-\cos(2h))\right)^{-1}\times$$
$$\times\Big(\sin(2h)\left(\tfrac{1}{2}+4\cos(h)\right)-$$
$$-3\sin(h)-h(\cos(h)+1)(4\cos^2(h)-1)\Big),$$
$$J_0 = \left(4\sin^2(h)(1-\cos(h))\right)^{-1}\times$$
$$\times(2\sin(h)+2h\cos(h)\cos(2h)+$$
$$+h-\sin(2h)\,(1/2+2\cos(h))),$$
$$J_1 = 1/(2\cos^4(h)-\cos^3(h)-3\cos^2(h)+\cos(h)+1)\times$$
$$(8\sin(h)\cos^3(h)-3\sin(h)\cos(h)-2\sin(h)\cos^2(h)+$$
$$3\sin(h)-4h\cos^3(h)+h-4h\cos^2(h)+h\cos(h))/4),$$
$$J_2 = \left(8\sin^2(h)(\cos(h)-\cos(2h))\cos(h)\right)^{-1}\times$$
$$\left(2h\cos^2(h)+h\cos(h)-\sin(2h)\cos(h)-\sin(h)\right).$$

It can be easily shown that between polynomial and trigonometric integro-differential splines the next relation is fulfilled $w_j^{<s,s+1>}(x_j+th) = \omega_j^{<s,s+1>}(x_j+th)+O(h^2)$.

Table 6 shows the errors of approximation of functions by splines (68)–(73) on $(-1,1)$ when $h = 0.1$. The calculations of the actual error $R_M^{<T>}$ were done in Maple, Digits=15.

Table 6. The errors of approximation of functions by splines (68)–(73)

N	$u(x)$	$R_M^{<T>}$
1	$1/(1+25x^2)$	0.0165
2	$\sin(x)$	0.
3	$\sin(3x)$	$0.197\cdot 10^{-4}$
4	x^5	$0.6927\cdot 10^{-4}$

Theorem 10. The error of the approximation by the splines (51), (52)–(56) is the next:

$$|\widetilde{u}(x)-u(x)| \le Kh^5\|4u'+5u'''+u^V\|_{(x_{k-2},x_{k+3})}, \tag{74}$$

where $x\in(x_k,x_{k+1}),\ K=0.2$.

Proof is similar to that done in the proof of Theorem 5. Here the next inequalities were used: $|w_j^{<-2,-1>}(x)| \le 0.05/h$, $|w_j^{<-1,0>}(x)| \le 0.45/h$, $|w_j^{<0,1>}(x)| \le 1.12/h$, $|w_j^{<1,2>}(x)| \le 0.45/h$, $|w_j^{<2,3>}(x)| \le 0.05/h$.

8 Right integro-differential trigonometric splines

On the left side of $[a,\, b]$ the best approximation gives us the right basic splines. In each (x_k, x_{k+1}), $k = 0, 1, 2, 3, \ldots, n-5$ the approximation for $u(x)$ are presented in the form:

$$\tilde{u}(x) = \int\limits_{x_k}^{x_{k+1}} u(t)dt\, w_k^{<0,1>}(x)+$$
$$+\int\limits_{x_{k+1}}^{x_{k+2}} u(t)dt\, w_k^{<1,2>}(x)+\int\limits_{x_{k+2}}^{x_{k+3}} u(t)dt\, w_k^{<2,3>}(x)+$$
$$+\int\limits_{x_{k+3}}^{x_{k+4}} u(t)dt\, w_k^{<3,4>}(x)+\int\limits_{x_{k+4}}^{x_{k+5}} u(t)dt\, w_k^{<4,5>}(x), \tag{75}$$

where $w_k^{<s,s+1>}(x)$, $s = 0, 1, 2, 3, 4$, are determined from the conditions $\varphi_1(x) = 1$, $\varphi_2(x) = \sin(x)$, $\varphi_3(x) = \cos(x)$, $\varphi_4(x) = \sin(2x)$, $\varphi_5(x) = \cos(2x)$.

If we put $x = x_k + th$, $t \in (0,1)$, then we have

$$w_k^{<0,1>}(x_k+th) = -G_1^{<0,1>}G_2^{<0,1>}, \tag{76}$$

where

$$G_1^{<0,1>} = \big(8h\sin^3(h)\cos(h)(2\cos(h)+1)\times$$
$$\times(\cos(h)-1)\big)^{-1},$$
$$G_2^{<0,1>} = \Big(\tfrac{1}{2}\sin(2h)(2\cos(h)+1)-$$
$$-2h(\cos(h)+1)\cos(h)(4\cos^2(h)-1)\cos(th-h)+$$
$$+2h\sin(2h)\sin(2th)+4h(\cos(h)+1)\times$$
$$\times\cos^2(h)\cos(th)+h(2\cos(4h)-1)\cos(2th-2h)\Big),$$

$$w_k^{<1,2>}(x_k+th) = -G_1^{<1,2>}G_2^{<1,2>}, \tag{77}$$

where

$$G_1^{<1,2>}= \left(4h\sin^2(h)(2\cos(h)+1)(\cos(h)-1)^2\right)^{-1},$$
$$G_2^{<1,2>} = \Big(\sin^2(h)(4\cos^2(h)-1)-$$
$$-h((2\cos(h)+1)(2\cos(4h)+1)+2\cos(2h))\times$$
$$\times(\sin(2th-h)-\sin(2th))-h\sin(h)\times$$
$$\times(8\cos^2(h)\cos(2h)+2\cos(h)+1)\cos(th-h)+$$
$$+h(\sin(4h)+\sin(2h)+\sin(h))\cos(th)-$$
$$-2h\sin(h)(2\cos(4h)+2\cos(2h)+1)\cos(2th)\Big),$$

$$w_k^{<2,3>}(x_k+th) = -\frac{1+\cos(h)}{4h\sin^5(h)}G_1^{<2,3>}, \quad (78)$$

where

$$G_1^{<2,3>} = \Big(2h(\cos(2h)+\cos(h))\Big(\sin(2h)\sin(th)+$$
$$+\cos(th-\tfrac{h}{2})\left(\cos\left(\tfrac{h}{2}\right)-2\sin\left(\tfrac{3h}{2}\right)\sin(h)\right)\Big)-$$
$$-\tfrac{1}{2}\sin(4h)-\sin(h)-h\Big(4\cos(2h)\sin(h)\sin(2th)+$$
$$+\big(1-4\sin(3h)\sin(h)\big)\cos(2th-h)\Big)\Big),$$

$$w_k^{<3,4>}(x_k+th) = -G_1^{<3,4>}G_2^{<3,4>}, \quad (79)$$

where

$$G_1^{<3,4>}= \left(4h\sin^2(h)(2\cos(h)+1)(\cos(h)-1)^2\right)^{-1},$$
$$G_2^{<3,4>} = \Big(\sin^2(h)(4\cos^2(h)-1)-$$
$$-h\sin(th)\big(\cos(h)(2\cos(2h)+1)+\cos(2h)\big)-$$
$$-h\sin(th-h)\left(\cos(4h)+\cos(3h)+2\cos^2(h)\right)-$$
$$-2h\sin(4h)\cos(2th)+2h\big(4\cos^2(h)\cos(2h)+$$
$$+\cos(4h)+4\cos(h)\cos(2h)\big)\sin\left(\tfrac{h}{2}\right)\cos\left(2th-\tfrac{h}{2}\right)\Big),$$

$$w_k^{<4,5>}(x_k+th) = -G_2^{<4,5>}/G_1^{<4,5>}. \quad (80)$$

where
$G_1^{<4,5>}=8h\sin^3(h)\cos(h)(2\cos(h)+1)(\cos(h)-1)$,
$G_2^{<4,5>}=\frac{1}{2}\sin(2h)(2\cos(h)+1) - 2h(\cos(h)+1)\times \cos(h)\cos(th-2h)+h\cos(2th-4h)$.

Theorem 11. The error of the approximation by the splines (75), (76)–(80) is the next

$$|\widetilde{u}(x)-u(x)| \le Kh^5\|4u'+5u'''+u^V\|_{(x_k,x_{k+5})}, \quad (81)$$

where $x\in(x_k,x_{k+1})$, $K=2$.

Proof. The function $u(x)$ on (x_k,x_{k+1}) can be written in the form (see [12]):

$$u(x)=\frac{2}{3}\int_{x_k}^{x}\left(4u'(\tau)+5u'''(\tau)+u^V(\tau)\right)\sin^4\left(\frac{x-\tau}{2}\right)d\tau+$$
$c_1+c_2\sin(x)+c_3\cos(x)+c_4\sin(2x)+c_5\cos(2x)$, where c_i, $i=1,2,3,4,5$, are arbitrary constants.

Here the next inequalities were used: $|w_j^{<0,1>}(x)| \le 2.29/h$, $|w_j^{<1,2>}(x)| \le 2.72/h$, $|w_j^{<2,3>}(x)| \le 2.29/h$, $|w_j^{<3,4>}(x)| \le 1.05/h$, $|w_j^{<4,5>}(x)| \le 0.2/h$.

Using the method from [12]) we obtain from (75), (76)–(80), (81).

Let $\widetilde{U}(x)$, $x\in(a,b)$, be such that $\widetilde{u}_k(x)$ $\widetilde{U}(x)=\widetilde{u}_k(x)$, $x\in(x_k,x_{k+1})$, $k=0,1,\ldots,n-5$.

Theorem 12. For the error of approximation by trigonometric splines (75), (76)–(80) we have the next relation:

$$\|\widetilde{U}-u\|_{(x_0,x_{n-4})} \le Kh^5\|4u'+5u'''+u^V\|_{(x_0,x_n)},$$

$K=2$.

Proof. The proof follows from (81).

Table 7 shows the theoretical ($\mathcal{R}_R^{<T>}$) and the actual ($R_R^{<T>}$) errors of approximation by the splines (75), (76)–(80), they were found in Maple $Digits=25$ with $h=0.1$.

Table 7.

N	$u(x)$	$R_R^{<T>}$	$\mathcal{R}_R^{<T>}$
1	$\frac{1}{(1+25x^2)}$	0.116	6.226
2	$\sin(x)$	$0.865\cdot10^{-19}$	0
3	$\sin(3x)$	$0.19641\cdot10^{-3}$	$0.24\cdot10^{-2}$
4	x^5	$0.8764\cdot10^{-3}$	$0.88\cdot10^{-2}$

8.1 Trigonometric quadrature

Sometimes, if the values of the integrals are unknown, the next trigonometric quadrature may be useful.

From (44), (45)–(49) we obtain:

$$\int_{x_k}^{x_{k+1}} u(t)dt = u(x_k)J_0+u(x_{k+1})J_1+$$
$$+u(x_{k+2})J_2+u(x_{k+3})J_3+u(x_{k+4})J_4+r,$$

where:

$$J_0=Q_{01}/Q_{02},$$

$Q_{01} = h\cos(h) - 4\sin(2h)\cos(h)(\sin(h)^2 + \cos(h))+2\sin(2h)+\sin(h)+2h\cos^2(h)$,
$Q_{02}=8(\cos(h)-\cos(2h))\cos(h)\sin^2(h)$,

$$J_1=Q_{11}/Q_{12},$$

$Q_{11} = h+2\sin(2h)\sin^2(h)+\sin(4h)\cos(h)+\sin(h)(\cos(h)+1)+h\cos(h)(1-4\cos(h)-4\cos^2(h))$,

$Q_{12} = 4(\cos(h) - \cos(2h))\sin^2(h)$,

$$J_2 = -Q_{21}/Q_{22},$$

$Q_{21} = -h - 3\sin(h)\cos(h) + 4\sin(h)\cos^3(h) + 2\sin(h) - 4h\cos^3(h) + 2h\cos(h)$,
$Q_{22} = 4(\cos^3(h) - \cos^2(h) - \cos(h) + 1)$,

$$J_3 = Q_{31}/Q_{32}, \ \ J_4 = -Q_{41}/Q_{42},$$

$Q_{31} = (\sin(h) - h)(\cos(3h) + 4\cos(h) + \cos(2h)) - h(2\cos^2(h) - 2\cos(h))$,
$Q_{32} = 4(\cos(h) - \cos(2h))\sin^2(h)$, $Q_{41} = 2\sin(h)\cos^2(h) - h\cos(h) - 2h\cos^2(h) + \sin(h)$,
$Q_{42} = 8(\cos(h) - \cos(2h))\cos(h)\sin^2(h)$.

This formula such that $r = 0$, if $u = 1$, $\sin(x)$, $\cos(x)$, $\sin(2x)$, $\cos(2x)$.

9 Left integro-differential trigonometric splines

On the right side of $[a, b]$ the best approximation gives us the left basic splines. In each interval (x_k, x_{k+1}), $k = 0, 1, \dots, 4$ the approximation for $u(x)$ is presented in the form:

$$\tilde{u}(x) = \int\limits_{x_{k-4}}^{x_{k-3}} u(t)dt\, w_k^{<-4,-3>}(x) + \int\limits_{x_{k-3}}^{x_{k-2}} u(t)dt w_k^{<-3,-2>}(x) + \int\limits_{x_{k-2}}^{x_{k-1}} u(t)dt w_k^{<-2,-1>}(x) + \int\limits_{x_{k-1}}^{x_k} u(t)dt w_k^{<-1,0>}(x) + \int\limits_{x_k}^{x_{k+1}} u(t)dt w_k^{<0,1>}(x), \quad (82)$$

where $w_k^{<s,s+1>}(x)$, $s = 0, 1, \dots, 4$, are determined from the conditions from $\widetilde{u}(x) = \varphi_i(x)$, $\varphi_1 = 1$, $\varphi_2 = \sin(x)$, $\varphi_3 = \cos(x)$, $\varphi_4 = \sin(2x)$, $\varphi_5 = \cos(2x)$. If we put $x = x_k + th$, $t \in (0, 1)$, then we have:

$$w_k^{<-4,-3>}(x_k + th) = -P_2^{<-4,-3>}/P_1^{<-4,-3>}, \quad (83)$$

where:
$P_1^{<-4,-3>} = 4h\sin(2h)\sin^2(h)(2\cos(h) + 1)\times$
$\times(\cos(h) - 1)$,
$P_2^{<-4,-3>} = -2h(\cos(h) + 1)\cos(h)\cos(th + h) + \frac{1}{2}\sin(2h)(2\cos(h) + 1) + h\cos(2h + 2th)$,

$$w_k^{<-3,-2>}(x_k + th) = -P_2^{<-3,-2>}/P_1^{<-3,-2>}, \quad (84)$$

where:
$P_1^{<-3,-2>} = 4h\sin^2(h)(2\cos(h) + 1)(\cos(h) - 1)^2$,
$P_2^{<-3,-2>} = h\sin(h)\cos(th) - 2h\sin(2h)\cos(2th) + \sin^2(h)(4\cos(h)^2 - 1) + h(2\cos(2h) + 2\cos(h) + 1)\left(2\sin\left(\frac{h}{2}\right)\cos\left(2th + \frac{h}{2}\right) - \sin(h)\cos(th + h)\right)$,

$$w_k^{<-2,-1>}(x_k + th) = -P_2^{<-2,-1>}/P_1^{<-2,-1>}, \quad (85)$$

where:
$P_1^{<-2,-1>} = 4h\sin^3(h)(\cos(h) - 1)$,
$P_2^{<-2,-1>} = -4h\cos\left(\frac{3h}{2}\right)\cos^2\left(\frac{h}{2}\right)\cos\left(th + \frac{3h}{2}\right) + 2h\cos(2h)\cos(2th + h) + \frac{1}{2}\sin(4h) + \sin(h) - h\cos(2th - h)$,

$$w_k^{<-1,0>}(x_k + th) = -P_2^{<-1,0>}/P_1^{<-1,0>}, \quad (86)$$

where:
$P_1^{<-1,0>} = 4h\sin^3(h)(1 - \cos(h))(2\cos(h) + 1)$,
$P_2^{<-1,0>} = 2h\Big(-\sin(h)(2\cos(2h) + 1)\sin(2th) + (2\cos(2h)(2\cos(h) - 1) - 1)\cos\left(2th + \frac{h}{2}\right)\cos\left(\frac{h}{2}\right) + (\cos(h) + 1)\Big(\sin(2h)\sin(th) - \cos(th + 2h)\cos(2h) - \cos\left(\frac{3h}{2}\right)\cos\left(th - \frac{h}{2}\right)\Big)\Big) + \sin(h)(4\cos^2(h) - 1)\times$
$\times(\cos(h) + 1)$,

$$w_k^{<0,1>}(x_k + th) = -P_2^{<0,1>}/P_1^{<0,1>}, \quad (87)$$

where:
$P_1^{<0,1>} = 8h\sin^3(h)\cos(h)(2\cos(h) + 1)(\cos(h) - 1)$,
$P_2^{<0,1>} = -2h(\cos(h) + 1)\cos(h)\cos(th + 2h) + h\cos(4h + 2th) + \frac{1}{2}\sin(2h)(2\cos(h) + 1)$.

Theorem 13. The error of the approximation by the splines (82), (83)–(87) is the next:

$$|\widetilde{u}(x) - u(x)| \leq Kh^5\|4u' + 5u''' + u^V\|_{(x_{k-4}, x_{k+1})}, \quad (88)$$

where $x \in (x_k, x_{k+1})$, $K = 2.12$.

Proof is similar the proof of Theorem 11. Here the next inequalities were used:

$|w_k^{<-4,-3>}(x)| \leq 0.2/h$, $|w_k^{<-3,-2>}(x)| \leq 1.05/h$, $|w_k^{<-2,-1>}(x)| \leq 2.29/h$, $|w_k^{<-1,0>}(x)| \leq 2.72/h$, $|w_k^{<0,1>}(x)| \leq 2.29/h$.

Let $\widetilde{U}(x)$, $x \in (a, b)$, be such that $\widetilde{u}_k(x)$ $\widetilde{U}(x) = \widetilde{u}_k(x)$, $x \in (x_k, x_{k+1})$, $k = 4, 5, \dots, n - 1$.

Theorem 14. For the error of approximation by trigonometric splines (82), (83)–(87) we have the next relation:

$$\|\widetilde{U} - u\|_{(x_4, x_n)} \leq Kh^5\|4u' + 5u''' + u^V\|_{(x_0, x_n)},$$

$K = 2.12$.

Proof follows from (88).

Table 8 shows the theoretical ($\mathcal{R}_L^{<T>}$) and actual ($R_L^{<T>}$) errors of approximation of functions by trigonometric splines (82), (83)–(87), calculated in Maple for $Digits = 25$ and $h = 0.01$.

Table 8.

N	$f(x)$	$R_L^{<T>}$	$\mathcal{R}_L^{<T>}$
1	$\frac{1}{1+25x^2}$	0.11096	6.599
2	$\sin(x)$	$0.351 \cdot 10^{-19}$	0
3	$\sin(3x)$	$0.18757 \cdot 10^{-3}$	$0.2544 \cdot 10^{-2}$
4	x^5	$0.8373 \cdot 10^{-3}$	$0.9328 \cdot 10^{-2}$

We can use the next formula if the values of the integrals are unknown. From (51), (52)–(56) we obtain:

$$\int_{x_k}^{x_{k+1}} u(t)dt \approx u(x_{k-3})J_{-3} + u(x_{k-2})J_{-2}+$$

$$+u(x_{k-1})J_{-1} + u(x_k)J_0 + u(x_{k+1})J_1,$$

where:

$$J_{-3} = -Q_{-31}/Q_{-32}, \;\; J_{-2} = Q_{-21}/Q_{-22},$$

$Q_{-31} = 2\sin(h)\cos^2(h) + \sin(h) - 2h\cos(h)^2 - h\cos(h)$,
$Q_{-42} = 8\sin^2(h)^2(\cos(h) - \cos(2h))\cos(h)$,
$Q_{-21} = 2\sin(2h)\cos^2(h) + (\sin(h) - 2h)(\cos(h) + \cos(2h)) - h\cos(3h) - h$,
$Q_{-22} = 4\sin^2(h)(\cos(h) - \cos(2h))$,

$$J_{-1} = -Q_{-11}/Q_{-12}, \;\; J_0 = Q_{01}/Q_{02},$$

$Q_{-11} = (\sin(h)-h)(\cos(3h)+1)+\sin(h)-h\cos(h)$,
$Q_{-12} = 4\sin^2(h)(1-\cos(h))$, $Q_{01} = \sin(4h)\cos(h) - (\sin(h) + h)\cos(3h) - h + 2\left(\sin\left(\frac{3h}{2}\right) - 2h\cos\left(\frac{3h}{2}\right)\right)\cos\left(\frac{h}{2}\right)$
$Q_{02} = 4\sin^2(h)(\cos(h) - \cos(2h))$,

$$J_1 = Q_{11}/Q_{12},$$

$Q_{-11} = \sin(h) - 2\sin(h)\sin^2(2h) - \sin(4h) + 2h\cos^2(h) + h\cos(h)$,
$Q_{12} = 8\sin^2(h)(\cos(h) - \cos(2h))\cos(h)$.

10 Application for solving delay equation

We consider the delay differential equations:

$$y' = -y(t-1) \quad \text{for } t \geq 1, \qquad (89)$$

with constant history $y(t) = 1$, $0 \leq t \leq 1$.

The solution of equation (89) is such that y' becomes discontinuous at $x = 1$, y'' becomes discontinuous at $x = 2$, and so on.

Here, for solving this problem we apply approximation methods with the minimal trigonometric splines and the polynomial integro-differential splines.

Figures 6 and 7 show the errors of solution of the delay problem (89) when $h = 0.1$.

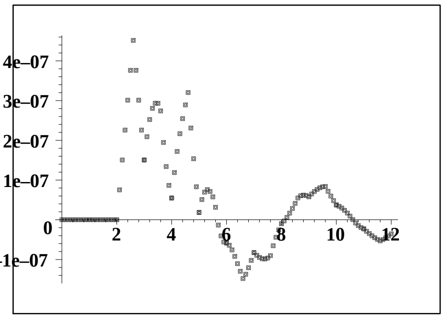

Figure 6: Plot of the error of solution of the delay problem (89) by trigonometric left and right splines.

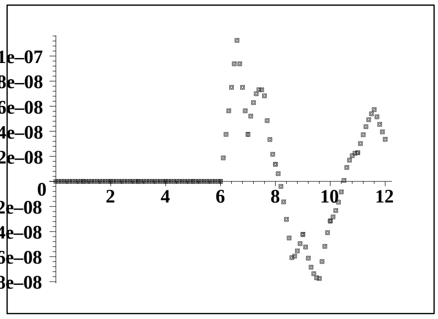

Figure 7: Plot of the error of solution of the delay problem (89) by polynomial splines (57) –(65)

11 Conclusion

Polynomial and nonpolynomial minimal and integro-differential splines are useful for solving different approximation problems. The results represented in the tables shows that the constants in the estimations of the errors of approximation represented in the theorems can be diminished. Here, these constants were calculated using the Taylor theorem with the remainder term in Lagrange form.

Further more, we are going to minimize the constants taking the remainder term of the Taylor theorem in another form, and to find a way for minimizing the constants in the estimations of the errors of approximation for nonpolynomial splines.

References:

[1] Sarfraz, M. Generating outlines of generic shapes by mining feature points. *WSEAS Transactions on Systems*. Vol. 13, 2014, pp. 584–595.

[2] Skala, V. Fast interpolation and approximation of scattered multidimensional and dynamic data

using radial basis functions. *WSEAS Transactions on Mathematics* . Vol. 12, Iss. 5, May 2013, pp. 501–511.

[3] Popoviciu, N., Boncut, M. On the theorem of curry & schoenberg and the relation between B-Spline and Box-Spline Functions. *Recent Researches in Computational Techniques, Non-Linear Systems and Control. Proc. of the 13th WSEAS Int. Conf. on MAMECTIS'11, NOLASC'11, CONTROL'11, WAMUS'11.* 2011, pp. 285–288.

[4] Kalovrektis, K., Ganetsos, T., Shammas, N.Y.A., Taylor, I., Andonopoulos, J. Development of a computerized ECG analysis model using the cubic spline interpolation method. *Recent Researches in Circuits, Systems and Signal Processing - Proc. of the 15th WSEAS Int. Conf. on Circuits, Part of the 15th WSEAS CSCC Multiconference, Proc. of the 5th Int. Conf. on CSS'11.* 2011, pp. 186–189.

[5] Popoviciu, N. A comparison between two box spline algorithms based on inductive method and geometric method. *Recent Researches in Computational Techniques, Non-Linear Systems and Control. Proc. of the 13th WSEAS Int. Conf. on MAMECTIS'11, NOLASC'11, CONTROL'11, WAMUS'11.* 2011, pp. 277–284.

[6] Furferi, R., Governi, L., Palai, M. , Volpe, Y. From unordered point cloud to weighted B-spline - A novel PCA-based method. *Applications of Mathematics and Computer Engineering - American Conf. on Applied Mathematics, AMERICAN-MATH'11, 5th WSEAS International Conference on Computer Engineering and Applications, CEA'11.* 2011, pp. 146–151.

[7] Caglar, H., Yilmaz, S., Caglar, N., Iseri, M. A non-polynomial spline solution of the one-dimensional wave equation subject to an integral conservation condition. *Proc. of the 9th WSEAS International Conf. on Applied Computer and Applied Computational Science, ACACOS'10.* 2010, pp. 27–30.

[8] Kuragano, T. Quintic B-spline curve generation using given points and gradients and modification based on specified radius of curvature (Article). *WSEAS Transactions on Mathematics.* Vol. 9, Iss. 2, 2010, pp. 79–89.

[9] Harus Kalis, Ojars Lietuvietis. On Finite Difference Approximations for Solving some Problems of Mathematical Physics with Periodical Boundary Conditions. *Advances in Applied and Pure Mathematics. Proc. of the 7-th International Conf. on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements (F-and-B'14).* Gdansk. Poland. May 15–17. 2014, pp. 109–117.

[10] Fengmin Chen, Patricia J.Y.Wong. On periodic discrete spline interpolation: Quintic and biquintic case. *Journal of Computational and Applied Mathematics*. 255, 2014, pp. 282-296.

[11] Burova I.G., Demyanovich Yu.K. Minimal Splines and theirs Applications. Spb.2010. (in Russian)

[12] Burova Irina. On Integro- Differential Splines Construction. *Advances in Applied and Pure Mathematics. Proc. of the 7-th International Conf. on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements (F-and-B'14).* Gdansk. Poland. May 15–17. 2014. pp.57–61.

[13] Kireev V.I. Panteleev A.V. Numerical methods in examples and tasks. Moscow. 2008. 480 p. (Russian)

[14] N.E.Mastorakis. An extended Crank-Nicholson method and its Applications in the Solution of Partial Differential Equations: 1-D and 3-D Conduction Equations. *Proceedings of the 10th WSEAS International Conference on APPLIED MATHEMATICS, Dallas, Texas, USA*, November 1-3, 2006, pp.134–143.

[15] Nicola Guglielmi, Ernst Hairer. Numerical approaches for state-dependent neutral delay equations with discontinuities. *Mathematics And Computers In Simulation.* Vol. 95, 2014, pp. 2 – 12.

[16] Kennedy, Benjamin B. A State-Dependent Delay Equation with Negative Feedback and Mildly Unstable Rapidly Oscillating Periodic Solutions. *Discrete and Continuous Dynamical System*. Series B 18.6 (2013). pp. 1633–1650.

[17] Borges, Ricardo; Calsina, Angel; Cuadrado, Silvi, Odo Diekmann. Delay equation formulation of a cyclin-structured cell population model, *Journal of Evolution Equations*. Vol. 14, Issue 4–5, 2014, pp. 841–862.

[18] Abdellatif Ben Makhlouf, Mohamed Ali Hammami. A comment on Exponential stability of nonlinear delay equation with constant decay rate via perturbed system method. *International Journal of Control, Automation and Systems.* Vol. 12, Issue 6, 2014, pp. 1352–1357.

[19] Bellman R.E., Cooke K.L. On the Computational solution of a class of functional differential equations. *J.Math. Anal. and its Appl.* Volume 12. 1965. pp.495–500.

[20] Hall G., Watt J.M. Modern Numerical Methods for Ordinary Differential Equations. 1976.

[21] Zhang Chenghui, Agarwal Ravi P., Bohner Martin And Li Tongxing. Oscillation of fourth-order delay dynamic equations. *Science China*. January 2015, Vol.58, No.1, pp. 143-160.