

Heat energy allocation models: Croatian case study

ALEKSANDAR HATZIVELKOS
 University of Applied Sciences Velika Gorica
 Department of Mathematics
 Zagrebačka 5, 10 410 Velika Gorica
 Croatia
 hatzi@vvg.hr

Abstract: This paper addresses the allocation methods used for allocation of the heat energy in systems with partial distribution of the heat allocators. Mathematical definition and analysis of three heat energy allocation methods is given. Two of them were officially legislated in Croatia, and in use since year 2008. Third one is proposed in works of Hatzivelkos. Properties of allocation methods are introduced: consistency, monotonicity and local consistency. While consistency, as a global property can be viewed as a necessary allocation property, special attention is given to later two properties, monotonicity and local consistency. Those properties describe allocation methods from the perspective of a consumer. Finally, mathematical analysis of allocation methods behaviour in worst case scenario is given, i.e. scenario that produces the greatest error for observed allocation model.

Key-Words: Heat energy allocation model

1 Introduction

More than 150 000 households in Croatia have centralized heating system with heated water. Most of the heat energy consumers are connected to one common heat meter located in the heating substation of the building. These are the buildings built before 2001 in which the piping did not provide for individual metering of heat energy for each apartment. To make cost allocation fairer for customers on a common heat meter, there is a possibility of installing heat cost allocators. This article addresses the allocation methods used for allocation of the heat energy in systems with partial distribution of the heat allocators.

Hopefully, results will stimulate use of methods which can guarantee desirable properties of the heat energy allocation. Such methods can reinforce public confidence in fairness of individual energy consumption metering, even in situations where not all apartments in building have heat allocators installed.

2 Heat allocation models

A models for heat energy allocation among apartments connected to common heat meter presented in this article are based upon two assumptions:

1. All of the electronic heat allocators placed on heat emitters are of the same type. Therefore, their numerical values are comparable.

2. Not all apartments need to have electronic heat allocators placed on heat emitters, but apartments that do, have to have them placed on each heat emitter.

In this article we will use following notification:

1. There are m apartments connected to common heat meter. First k of them (k being smaller than m) have electronic heat allocators placed on each heat emitter. For those k apartments, with N_1, N_2, \dots, N_k we will denote the sum of consumption readings on heat allocators on all heat emitters in those apartments in a given time period (for instance, in one month).
2. With A_1, A_2, \dots, A_m we will denote apartment living areas.
3. With E_{total} we will denote total amount of heat energy read on the common heat meter in a given time period (for instance, in one month).
4. With E_1, E_2, \dots, E_m we will denote an amount of energy allocated to each apartment in a given time period (for instance, in one month). It is clear that $E_{total} = \sum_{i=1}^m E_i$.
5. Finally, with α_A we will denote a factor which is determining a portion of energy total which is considered to be a energy loss or a common consumption in a building (energy used for heating of common areas). That portion of energy should

be allocated with respect of an area share of the apartments.

First we will define model for energy allocation which was officially used for a heat energy allocation in Croatia (with minor modifications) from year 2008. to year 2015. [7] [8] [9] [10] [11]. Due to its structure, we shall call this model "Static allocation model", and it is given in following definition:

Definition 1 (Static allocation model, SAM)

For all $m - k$ apartments without installed heat allocators, their allocated energy equals to

$$E_i = E_{total} \cdot \alpha_w \cdot \frac{A_i}{\sum_{j=k+1}^m A_j} \quad (1)$$

for all $i \in \{k+1, \dots, m\}$. For all k apartments with installed heat allocators, their allocated energy equals to

$$E_i = E_{total}(1 - \alpha_w) \left[\alpha_A \frac{A_i}{\sum_{j=1}^k A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^k N_j} \right] \quad (2)$$

for all $i \in \{1, \dots, k\}$, where α_w is a factor which defines the share of energy total that is allocated to all apartments without installed heat allocators:

$$\alpha_w = l \cdot \frac{\sum_{i=k+1}^m A_i}{\sum_{i=1}^m A_i}, \quad (3)$$

factor l being equal to 2.

Factor l which appears in the equation (3) in Definition 1, as part of the coefficient α_w (so called *correction factor*) changed several times in last eight years (from $l = 1$ in year 2008., $l = 1.25$ in year 2011., $l = 1.5$ in year 2014., to $l = 2$ in year 2015.) [7] [9] [11] [13]. With factor α_w model actually tries to determine the fixed share of the total energy used in apartments without the heat allocators. Because of this approach, we call model "static". Of course, share of apartments without the heat allocators in total energy could not be fixed, and it changes from one month to another. Furthermore, in some circumstances, the share of energy used in apartments without the heat allocators can be far greater than α_w . We will discuss this in detail later in article.

As we can see from equation (1), the share of the total energy which is allocated to all apartments without the heat allocators is further allocated to apartments according to their area share. On the other hand, from equation (2) we can see that remaining energy is allocated to apartments with heat allocators as a convex combination of their share in area, and their share in the total number of consumption

readings.

In year 2015. model for heat energy allocation was changed [12]. Although model kept its basic structure, new threshold was introduced. Therefore, model is named as "Static allocation model with threshold". That model, which was in use from April to November 2015. is given in following definition:

Definition 2 (SAM-wT) We define threshold value T :

$$N_t = \frac{\sum_{j=1}^k N_j}{k}, \quad T = \frac{\sum_{j=1}^k A_j}{\sum_{j=1}^k N_j} \quad (4)$$

For all $m - k$ apartments without installed heat allocators, their allocated energy equals to

$$E_i = E_{total} \cdot \alpha_w \cdot \frac{A_i}{\sum_{j=k+1}^m A_j} \quad (5)$$

for all $i \in \{k+1, \dots, m\}$. For all k apartments with installed heat allocators, their allocated energy equals to

$$E_i = E_{total}(1 - \alpha_w) \left[\beta_A \frac{A_i}{\sum_{j=1}^k A_j} + (1 - \beta_A) \frac{N_i}{\sum_{j=1}^k N_j} \right] \quad (6)$$

for all $i \in \{1, \dots, k\}$, where α_w is a factor which defines the share of energy total that is allocated to all apartments without installed heat allocators

$$\alpha_w = 2 \cdot \frac{\sum_{i=k+1}^m A_i}{\sum_{i=1}^m A_i}$$

and β_A equals to

$$\beta_A = \begin{cases} \alpha_A & \text{if } T \leq 0.3 \\ T & \text{if } T > 0.3 \end{cases} \quad (7)$$

As we can see from equation (4), threshold value T was defined as a share in area of the apartments with heat allocators, of the apartments which consumption readings are greater than average consumption reading, N_t .

The allocation for apartments without heat allocators, remained the same as in a SAM model, defined with factor α_w . Allocation for apartments with heat allocators changed, and now it depends on a value of T . If T is less than 0.3, energy for those apartments is allocated the same way it was allocated in SAM. But if T exceeds 0.3, then instead of α_A ,

we use T as the split ratio between the part of energy allocated according to area share, and the part of energy allocated according to consumption readings share. This means, that if T is big, say 0.8, then 80% of energy will be allocated according to apartments area share, while only smaller part (20%) will be allocated according to consumption readings share.

To conclude with legislated models, we should point out that in November 2015. model for the allocation of heat energy in Croatia yet again changed [13]. The latest change abandoned usage of SAM-wT model, and returned to SAM model, with one difference. Value $l = 2$ in Definition 1 is not legislated by government anymore, but now it can be freely formed.

Let us finally define third model for heat energy allocation, which was proposed in works of Hatzivelkos [3][4].

Definition 3 (Dynamic allocation model, DAM)

For all $m - k$ apartments without installed heat allocators, we define their number of impulses (consumption readings) as

$$N_i = \alpha_N \cdot A_i \cdot \max_{j=1, \dots, k} \left(\frac{N_j}{A_j} \right) \quad (8)$$

for all $i \in \{k + 1, \dots, m\}$, where is $\alpha_N \geq 1$ predefined ("motivation" or "punishment") factor, and it represents factor for which allocation of energy for an apartment without heat allocators should be greater than the allocation of energy for an apartment with heat allocators (with respect to apartments areas). Now we have

$$E_i = E_{total} \left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j} \right] \quad (9)$$

for all $i \in \{1, \dots, m\}$.

We will call this model "dynamic" because it does not prescribe fixed share of total energy consumed in apartments without the heat allocators. Rather than that, DAM model establishes relations between apartments itself, therefore acting on a micro level. Relation between an apartment with heat allocators and one without heat allocators is defined by equation (8). Motivation for that relationship, where an apartment without heat allocators is assigned (relative) consumption reading greater or equal to maximal consumption reading among apartments with heat allocators, comes from stand that since we

can not know the level of consumption in apartments without heat allocators, it is only fair to assume that it is maximal.

Let us now see how all three models allocate energy in following example:

Example 4 The areas and consumption readings of ten apartments are given in the following table. First eight apartments have the heat allocators, while last two do not. Let us say that total energy that should be allocated to those apartments equals to $E_{total} = 1000$ MWh, and that factor α_A (which determines common part of energy, that should be allocated according to areas of the apartments) equals to $\alpha_A = 0.1$.

Apartment	1.	2. – 4.	5. – 7.	8.	9. – 10.
Area	50	50	50	50	50
Impulses	80	20	10	0	–

In this example all areas are the same, so that difference in allocation methods would be more visible. If we use SAM method (see Definition 1), we get $\alpha_w = 0.4$, and so allocation to apartments without heat allocators (see equation (1)) equals to $E_9 = E_{10} = 200$. We calculate the allocations for apartments with the heat allocators from equation (2). See Table 1.

In case of SAM-wT model, average number of consumption reading N_t equals to $N_t = 21.25$. Therefore, all but the first apartment have consumption readings less than N_t , and threshold value T equals to $T = 0.875$. This means that value for β_A equals to $\beta_A = T = 0.875$. The allocations for apartments without the heat allocators remain same as in SAM model, but allocations for other apartments change, with lowest allocations significantly rising. See Table 1.

Finally, if we use DAM model (with $\alpha_N = 1.1$), consumption readings prescribed for apartments 9. and 10. equals to $N_9 = N_{10} = 88$. Usage of equation (9) gives following allocation:

As we can see in Table 1, in SAM model apartment 1. (with heat allocators) is assigned greater share of energy than apartments 9. and 10. (without heat allocators). SAM-wT model in this example produces allocation in which apartment 1. is allocated just 40% more energy than apartments 5., 6. and 7., although its energy consumption in eight times greater. Those observations will be in the focus of Section 3, in which we will analyze properties of the defined models for energy allocation.

No.	A_i	N_i	SAM E_i	SAM-wT E_i	DAM E_i
1	50	80	261.62	100.92	218.09
2	50	20	71.03	74.45	62.02
3	50	20	71.03	74.45	62.02
4	50	20	71.03	74.45	62.02
5	50	10	39.26	70.04	36.01
6	50	10	39.26	70.04	36.01
7	50	10	39.26	70.04	36.01
8	50	0	7.50	65.63	10.00
9	50	–	200.00	200.00	238.90
10	50	–	200.00	200.00	238.90

Table 1: Example of SAM, SAM-wT and DAM allocations

3 Properties of the energy allocation models

First thing that we expect from every model of energy allocation is consistency. This means that algorithm which assigns values of energy allocation to apartments, must do it in a way that the sum of all allocated energy is equal to total energy consumption reading. We will formalize this in following definition.

Definition 5 We say an energy allocation model is consistent if

$$\sum_{i=1}^m E_i = E_{total}.$$

Theorem 6 (Consistency) Models for energy allocation SAM, SAM-wT and DAM are consistent.

Proof: Let us first take a look at DAM model. Allocation to all apartments is given by equation (9). Let us sum up those values over all apartments:

$$\begin{aligned} \sum_{i=1}^m E_i &= \sum_{i=1}^m E_{total} \left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j} \right] \\ &= E_{total} \sum_{i=1}^m \left[\alpha_A \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j} \right] \\ &= E_{total} \left[\sum_{i=1}^m \alpha_A \frac{A_i}{\sum_{j=1}^m A_j} + \sum_{i=1}^m (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j} \right] \\ &= E_{total} \left[\frac{\alpha_A}{\sum_{j=1}^m A_j} \sum_{i=1}^m A_i + \frac{1 - \alpha_A}{\sum_{j=1}^m N_j} \sum_{i=1}^m N_i \right] \\ &= E_{total} [\alpha_A + 1 - \alpha_A] = E_{total}. \end{aligned}$$

For SAM model, if we sum all allocations for apartments without heat allocators, given in equation (1) in Definition 1 we will get

$$\sum_{j=k+1}^m E_j = E_{total} \cdot \alpha_w.$$

On the other hand, if we sum all allocations for apartments with heat allocators, given in equation (2) we will get

$$\sum_{j=1}^k E_j = E_{total} \cdot (1 - \alpha_w).$$

Therefore, if we sum all allocations, we have

$$\sum_{j=1}^m E_j = E_{total} \cdot (1 - \alpha_w) + E_{total} \cdot \alpha_w = E_{total}.$$

Proof for SAM-wT model is carried out in the same way. If we sum allocations for apartments without heat allocators, given in equation (5) in Definition 2, and then allocations for all apartments with heat allocators given in equation (6) we would get the same conclusion as in the case of SAM model. Introduction of the threshold value T, and the factor β_A doesn't change argument. \square

Now it is time to analyze other properties of defined allocation models. One of the first, is concept of monotonicity, which is given in following definition.

Definition 7 We say an energy allocation model is monotone if when all consumers maintain same level of energy consumption, except for one consumer who increases its energy consumption, then energy allocation to that consumer will increase.

Monotonicity is important property for the heat energy allocation models. It ensures that energy allocation will follow most basic expectation - if you spend more, you will pay more. Unfortunately, not all of defined heat allocation models satisfy that property.

Theorem 8 (Monotonicity) Models for energy allocation SAM and DAM are monotone, while SAM-wT is not.

Proof: To prove that DAM model is monotone, suppose that consumption readings in one of the apartments with heat allocators increased from N_i to $N_i + \Delta N$, while consumption in all other apartments remained the same. This means that total energy consumption also increased from E_{total} to $E_{total} + k \cdot \Delta N$

for some factor $k > 0$. Let us compare energy allocations E_i before and after that increase. We want to prove that

$$(E_{total} + k \cdot \Delta N) \cdot \left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum_{j=1}^m N_j} \right] >$$

$$> E_{total} \cdot \left[\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j} \right]$$

This leads to

$$E_{total}(1 - \alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum_{j=1}^m N_j} + k \cdot \Delta N \cdot$$

$$\left(\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j} + (1 - \alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum_{j=1}^m N_j} \right) > E_{total}(1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j}$$

We will prove that first summand on the left side of inequation is greater than right side of inequation:

$$E_{total}(1 - \alpha_A) \frac{N_i + \Delta N}{\Delta N + \sum_{j=1}^m N_j} > E_{total}(1 - \alpha_A) \frac{N_i}{\sum_{j=1}^m N_j}$$

$$\Rightarrow \frac{N_i + \Delta N}{\Delta N + \sum_{j=1}^m N_j} > \frac{N_i}{\sum_{j=1}^m N_j}$$

$$\Rightarrow (N_i + \Delta N) \cdot \sum_{j=1}^m N_j > N_i \cdot \left(\sum_{j=1}^m N_j + \Delta N \right)$$

Last inequality holds when $\sum_{j=1}^m N_j > N_i$. The only time this inequality does not hold is when consumption in all other apartments (other than i-th one) equals to zero. But in that case, the original inequality holds because of factor $\alpha_A \cdot \frac{A_i}{\sum_{j=1}^m A_j}$ which does not depend on other apartments consumption readings.

On the other hand, if consumption is increased in an apartment without heat allocators, then all consumption readings in DAM model will remain the same. But increase of E_{total} then leads to increase of all allocations, including one in an apartment which increased consumption.

Proof of the monotonicity for the SAM model follows the similar argument as one for DAM model. But when we analyze SAM-wT model, we can see that same argument does not hold. Increase of energy consumption in one apartment with heat allocators, leads to increase of an average consumption reading N_t , and consequently possible increase of value T (see equation (4) in Definition 2). Increase of T can lead to increase of β_A (see equation (7) in Definition

2), which can significantly change outcome of energy allocation, putting much more weight on areas of the apartments instead of consumption readings. This can be seen in Example 9. \square

Example 9 Allocation is being made for a house with ten apartments. First eight apartments have the heat allocators, while last two do not. Let us say that total energy that should be allocated to those apartments equals to $E_{total} = 1000$ MWh, and that factor α_A (which determines common part of energy, that should be allocated according to areas of the apartments) equals to $\alpha_A = 0.1$. Let us suppose that one point in consumption readings represent 7.5 MWh of consumed energy (which is consistent with the consumption in last two apartments being similar to consumption in first apartment). SAM-wT model is allocating heat energy as shown in following table:

No.	A_i	N_i	SAM-wT E_i	N'_i	SAM-wT E'_i
1	50	29	205.73	31	95.74
2	50	10	75.85	10	76.01
3	50	10	75.85	10	76.01
4	50	10	75.85	10	76.01
5	50	10	75.85	10	76.01
6	50	10	75.85	10	76.01
7	50	0	7.50	0	66.61
8	50	0	7.50	0	66.61
9	50	—	200.00	—	203.00
10	50	—	200.00	—	203.00

As we can see, increase of consumption reading for two points in first apartment resulted with decrease of its energy allocation. Before that consumption increase, average consumption reading was equal to $N_t = 9.88$, and so we had $T = 0.25$, and $\beta_A = 0.1$. After increase of consumption reading in first apartment, average consumption reading rose to $N'_t = 10.13$, and consequently we have $\beta_A = T = 0.875$.

Although not being monotone is severe disadvantage for a energy allocation model, it is not easily open to manipulation. Consumers does not have information about current consumption and distribution of consumption readings, so it is very hard to manipulate own consumption in order to take advantage of lack of monotonicity. There is, however, another property which is very visible form the perspective of the consumer - a comparison with other consumers within the same building.

Definition 10 We say an energy allocation model is locally consistent if its allocations to apartments, E_i , satisfy following:

1. For allocations E_i and E_j allocated to two apartments with same area and installed heat allocators, there is $E_i < E_j$ iff $N_i < N_j$.
2. For allocations E_i and E_j allocated to two apartments with installed heat allocators and same consumption readings, there is $E_i < E_j$ iff $A_i < A_j$.
3. For allocations E_i and E_j allocated to two apartments without installed heat allocators, there is $E_i < E_j$ iff $A_i < A_j$.
4. For allocations E_i and E_j allocated to two apartments with same area, where first one is assigned to an apartment with heat allocators, and second one is assigned to an apartment without heat allocators, there is $E_i \leq E_j$.

Properties in Definition 10 establish consistency of an allocation model from the perspective of a consumer. For a consumer, it is natural to expect lower allocation than the one allocated to the same size apartment with higher consumption readings (1). On the other hand, if two apartments have the same consumption readings, it is also natural to expect lower allocation to one with smaller area, due to lower participation in common energy expenses (2). For consumers without heat allocators, apartment area should be deciding factor for allocated amount of energy (3). Finally, if we compare allocations for the same size apartments, where one of them has, and other has not heat allocators, it is only fair to assume (without any additional knowledge, such as an energy efficiency of apartments) that the apartment without allocators consumes at least the same amount of the energy as an apartment with heat allocators (4). Therefore, properties from Definition 10 play important role in establishing public confidence in *fairness* of an allocation model.

Theorem 11 [Local consistency] Model for energy allocation DAM is locally consistent, while models SAM and SAM-wT are not.

Proof: We will first prove that DAM is locally consistent. For some two apartments with heat allocators, properties (1) and (2) in Definition 10

follow from equation (9) in Definition 3. Property (3) is satisfied because of equation (8) in Definition 3; assigned consumption readings are formed as an area of an apartment multiplied with a fixed value. Finally, property (4) also follows from equation (8) in Definition 3, because consumption reading assigned to an apartment without heat allocator is always great or equal to consumption reading of an apartment with maximal (relative to its size) consumption reading among all apartments with the heat allocators.

For SAM model we can easily show that it satisfies properties (1), (2) and (3) from Definition 10, which follows from equations (1) and (2) in Definition 1. But SAM does not satisfy property (4) of local consistency. As we can see from Table 1 (Example 4), SAM allocated greater value to the 1st than to the 9th and 10th apartment. Because of static structure of that model, one can easily find a distribution of consumption readings which produces allocation for an apartment with heat allocators that is *several times* greater than allocation to an apartment without heat allocators. To do that, the actual share of consumption by apartments without heat allocators should be much greater than α_w , which is something that can happen during warm months, when energy savings among apartments with heat allocators are high, and so consumption share for apartments without heat allocators rise. We analyze that scenario in Chapter 4.

We can only assume that this result was motivation for introduction of SAM-wT model. Threshold defined in Definition 2 is activated in case when greater number of apartments saves energy. SAM-wT responds to described situation that violates property (4) of local consistency in a way that it rises share of an area part of allocation in process of determining allocations. This way, the model for energy allocation cancels the basic purpose of the heat allocators - to allocate energy (predominately) upon consumption. Even so, SAM-wT did not succeed to satisfy property (4) of local consistency, as can be seen in Example 9, where allocation to first apartment (with heat allocators) equals to $E_1 = 205.73$, while allocation to tenth apartment (without heat allocators) equals to $E_{10} = 200$. \square

To show the extreme to which can go lack of local consistency for SAM model, we present following

example:

Example 12 Suppose there are 20 apartments in a building, each with area of 100m^2 . Two of them don't have heat allocators (we will denote their allocations with E_1 and E_2), while others have. Apartments without allocators consume 199 MWh each, one of the apartments with heat allocators consumes 2 MWh (we will denote its allocation with E_3), and all other apartments don't consume any energy. So, $E_{total} = 400$. Let us have $\alpha_A = 0.1$.

If we look at SAM allocation, we will see that apartments without heat allocators have been allocated $E_1 = E_2 = 40$, apartment with minimal energy consumption and heat allocators has been allocated $E_3 = 289.78$, and all other apartments with heat allocators have been allocated $E_i = 1.78$. So, third apartment, with energy consumption 100 times smaller than first (or second apartment) has allocation seven times greater then theirs.

If we use SAM-wT model for allocation on same example, we will get $E_1 = E_2 = 40$, $E_3 = 34.568$ and $E_i = 16.79$. Now we don't have such huge anomaly, as with SAM, but side effect is that all allocations tend to be close to allocation solely upon area share of an apartment (which would be $E_i = 20$ in this example). This way, main purpose of heat allocators - to allocate energy on basis of consumption - is being negated. Finally, if we use DAM, allocations would be $E_1 = E_2 = 125.75$, $E_3 = 114.50$ and $E_i = 2.00$ for $i \in \{4, \dots, 20\}$.

4 The analysis of allocation models accuracy

4.1 SAM model

As we mentioned before, SAM model through factor α_w actually tries to predetermine share of energy consumption of the apartments without heat allocators. So what is the level of consumption (compared with the state prior to installation of heat allocators) for which SAM model accurately allocates heat energy?

To answer that question, we will assume that apartments without heat allocators maintain the same level of heat energy consumption as before heat allocators were installed in the building. Another

assumption we are making, when talking about "accurate" allocation, is that we will assume that all apartments without heat allocators consume same amount of energy (relative to their size). This doesn't have to be so, because heat transfer between apartments changes after some of them consumes less energy. Also, different apartments have different energy consumption, due to different energy efficiency [6]. But if we want to stay within the same analysis framework, and if we don't want to introduce new class of information into the allocation models (such as database of energy efficiency factors for all apartments in building), we should accept those assumptions.

We will denote the consumption of the apartments with heat allocators before their installation with E_{with} , and consumption after allocators installation with E'_{with} . Similarly, energy consumption in the apartments without heat allocators we will denote with $E_{without}$ and $E'_{without}$. Now, from Definition 1 we have:

$$\begin{aligned} E_{without} &= E'_{without} & \Rightarrow \\ \frac{A_{without}}{A_{total}} \cdot E_{total} &= l \cdot \frac{A_{without}}{A_{total}} \cdot E'_{total} & \Rightarrow \\ E_{total} &= l \cdot E'_{total} & \Rightarrow \\ E_{without} + E_{with} &= l \cdot (E'_{without} + E'_{with}) \end{aligned}$$

If we denote with α' share of energy apartments with heat allocators are consuming (relative to consumption prior to installation of allocators), we have:

$$\begin{aligned} E_{without} + E_{with} &= l \cdot (E_{without} + \alpha' E_{with}) \\ E_{without} \cdot (1 - l) &= E_{with} \cdot (\alpha' l - 1) \\ \frac{A_{without}}{A_{total}} \cdot E_{total} \cdot (1 - l) &= \frac{A_{with}}{A_{total}} \cdot E_{total} \cdot (\alpha' l - 1) \\ A_{without} \cdot (1 - l) &= A_{with} \cdot (\alpha' l - 1) \end{aligned}$$

Now, let us with α_{with} denote total area share of the apartments with installed heat allocators.

$$\begin{aligned} A_{total} \cdot (1 - \alpha_{with}) \cdot (1 - l) &= A_{total} \cdot \alpha_{with} \cdot (\alpha' l - 1) \\ (1 - \alpha_{with}) \cdot (1 - l) &= \alpha_{with} \cdot (\alpha' l - 1) \end{aligned}$$

I we solve this equation for α' , we get

$$\alpha' = \frac{1}{l} + \left(1 - \frac{1}{l}\right) \left(1 - \frac{1}{\alpha_{with}}\right) \quad (10)$$

Equation (10) gives us factor of change (reduction) of the consumption of apartments with heat allocators, at which SAM model accurately allocates heat energy, as a function of two parameters: factor l from Definition 1 and an area share of apartments with heat allocators.

For instance, if we use $l = 2$ (as in Definition 1) and $\alpha_{with} = 0.9$ (which means that total area of the apartments with heat allocators accounts for 90% of total buildings area), we get $\alpha' = 0.4444$. This means that SAM allocation will be accurate if apartments with heat allocators consume 44.44% of energy they consumed prior to installation of heat allocators. Dependence between α' and α_{with} , for a value $l = 2$, is shown in Figure 1.

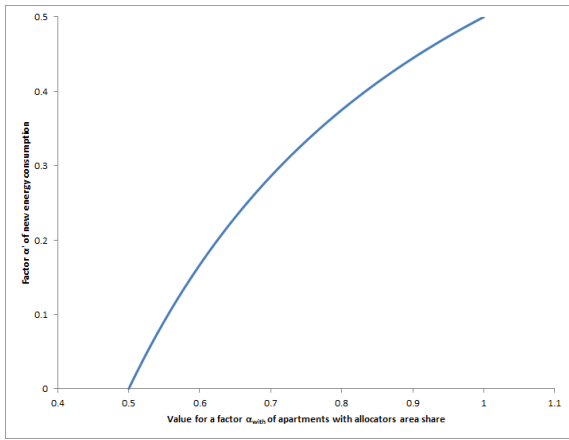


Figure 1: Dependence between α' and α_{with} for a value $l = 2$

Here we can see that, if we want SAM to provide accurate allocation, energy consumption (compared to the previous state without heat allocators) in the apartments with heat allocators must range from 0 (when there is only 50% of total building area with heat allocators) up to 0.5. Either way, if we want SAM to allocate energy accurately, savings in the apartments with heat allocators (for $l = 2$, as in Definition 1) must be significant.

Property described with equation (10) is something that requires information about share (in area) of apartments with installed heat allocators, and furthermore, the information that it provides is difficult to measure or estimate. Therefore, let us take look at total savings in building which provides accurate allocation for a SAM. Share of total energy consumption in building (compared to consumption prior

to installation of heat allocators) we will denote β' : $E'_{total} = \beta' \cdot E_{total}$. Now we have:

$$\begin{aligned} E'_{total} &= E'_{with} + E'_{without} \\ &= \alpha' \cdot \alpha_{with} \cdot E_{total} + (1 - \alpha_{with}) \cdot E_{total} \\ &= E_{total} \cdot (\alpha' \cdot \alpha_{with} + 1 - \alpha_{with}) \end{aligned}$$

For β' we have:

$$\begin{aligned} \beta' &= \frac{E'_{total}}{E_{total}} \\ &= \frac{E_{total} \cdot (\alpha' \cdot \alpha_{with} + 1 - \alpha_{with})}{E_{total}} \\ &= \left[\frac{1}{l} + \left(1 - \frac{1}{l}\right) \left(1 - \frac{1}{\alpha_{with}}\right) \right] \cdot \alpha_{with} + 1 - \alpha_{with} \\ &= \frac{1}{l} \end{aligned} \tag{11}$$

This result is global property of the model, which can easily be measured. If we want SAM model to accurately allocate energy within a building, total energy consumption in building must be at a level of $\beta' = \frac{1}{l}$ of energy consumed prior to allocators installation. Since in Definition 1 we have $l = 2$, building has to reduce its total heat energy consumption to one half of its prior consumption, for SAM model to be accurate.

This brings out the question: why is l set to 2? What was wrong with previous values for l ? Especially when recent studies on behaviour change and energy use show that up to 20% of the currently consumed energy can be saved through changing behaviour [1]. Such savings would mean that statistically most appropriate value for l would be 1.25. Furthermore, HEP-Toplinarstvo d.o.o., the biggest heat energy provider in Croatia states that saving in heat energy for buildings, after installation of heat allocators vary from 15% up to 30% [2], which would yield a value for l between 1.17 and 1.42 (if we want a SAM model to be accurate in most of the situations).

To answer that questions, we have to look at a behavior of SAM model in the case when change of the total consumption factor, β' , does not equal to $\frac{1}{l}$.

Let us first take a look at a SAM allocation for apartments without heat allocators at that level of consumption. SAM model allocates to all those apartments total amount of energy equal to $l \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}$, while allocation should be $\frac{1}{\beta'} \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}$, $\beta' \in [1 - \alpha_{with}, 1]$, for a model to be accurate.

The ratio of those two values equals to

$$\frac{l \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}}{\frac{1}{\beta'} \cdot \frac{A_{without}}{A_{total}} \cdot E_{total}} = l \cdot \beta', \quad (12)$$

where $\beta' \in [1 - \alpha_{with}, 1]$. Therefore, for $l = 2$, ratio of an energy allocated to apartments without heat allocators, and energy actually consumed in those apartments, equals to $2\beta'$. This means, that for $\beta' = 1$ (when no energy was saved compared to the consumption prior to installation of heat allocators, which can occur during very cold months, when all available heat energy is consumed), apartments without allocators are being allocated twice the consumed energy. On the level of consumption $\beta' = 0.75$ they are allocated 50% more energy than their consumption. On the other hand, if total savings are high (for instance during warmer weather period), and $\beta' = 0.25$, those apartments are being allocated only half the energy they spend. The extreme is when savings are maximal, and $\beta' = 1 - \alpha_{with}$. This means that all apartments with heat allocators are consuming no energy at all. But in this case, apartments without heat allocators are the one who benefit most from the situation. Although they consume all energy in the building, they will be allocated only $2 \cdot (1 - \alpha_{with})$ share of it.

Analysis for the apartments with heat allocators is more complex, since it depends on the distribution of the consumption readings. Because of that, we will analyze only "worst case scenario" for those apartments. This scenario occurs in situation when we maximize number (area) of the apartments with heat allocators and no energy consumption, because in that scenario, allocation for the rest of apartments with heat allocators will be greatest.

To make such analysis possible, we will work under assumption that maximal consumption in the apartments with heat allocators equals one in the apartment without them. Now, we have to denote (area) share of the apartments with heat allocators that consume maximal amount of heat energy (while others are consuming none). Let us denote that share with α_{mc} (as for "maximal consumption").

$$\alpha_{mc} = \frac{\text{Area of the apartments with heat allocators and maximal consumption}}{\text{Area of all apartments}}$$

Now we have

$$\alpha_{mc} = \beta' - \alpha_{without}$$

Why is this so? Because, in situation in which part of the apartments maintain full consumption, while

other part consumes none of the available energy, β' equals to area share of the apartments with full consumption. When we subtract share of the area without heat allocators (for which we assume full consumption), we get α_{mc} .

To that part of the building (apartments with heat allocators and maximal consumption), SAM model allocates all the energy

$$(1 - \alpha_w) \cdot \beta' \cdot E_{total},$$

which equals to (see equation (3)):

$$(1 - l \cdot \alpha_{without}) \cdot \beta' \cdot E_{total}$$

Now, we calculate allocation relative to the area of the apartments with heat allocators and maximal consumption:

$$\begin{aligned} \frac{(1 - l \cdot \alpha_{without}) \cdot \beta' \cdot E_{total}}{\alpha_{mc} \cdot A_{total}} &= \\ \frac{(1 - l \cdot \alpha_{without}) \cdot \beta' \cdot E_{total}}{\beta' - \alpha_{without}} \cdot \frac{E_{total}}{A_{total}} &= \\ \frac{1 - l \cdot \alpha_{without}}{1 - \frac{\alpha_{without}}{\beta'}} \cdot \frac{E_{total}}{A_{total}} &. \end{aligned} \quad (13)$$

From (13) we can see that if $\beta' = \frac{1}{l}$, factor multiplying $\frac{E_{total}}{A_{total}}$ equals to 1. This means that allocation is accurate, because all (relative) consumptions (where there is some) are equal. If energy savings are low (if β' is equal or close to 1), we can easily see that factor in question is lower than 1, which means that apartments with heat allocators and maximal consumption are being allocated smaller amount of energy than they consume.

The most interesting scenario, however, is the one in which β' tends to $\alpha_{without}$, which means that energy consumed in all apartments with heat allocators tends to zero.

In that case value in denominator tends to 0, so factor in equation (13) doesn't have upper bond. This analysis shows that there are consumption reading distributions among apartments with heat allocators, that can easily lead to extremely high energy allocation to some of those apartments, using the SAM model.

Anomaly derived from equation (13) could have been the reason why SAM-wT model was introduced. Situation when β' tends to $\alpha_{without}$ means that α_{mc} tends to 0 and share of the apartments with no (or very low) consumption tends to α_{with} . In such scenario value T from equation (4) in Definition 2 tends to 1, and consequently value β_A in equation (7) tends to 1. This eliminates extremely high allocations, because allocation tends to be done according to area (rather than consumption). But, in the same time, SAM-wT model negates very purpose of heat allocators.

4.2 DAM model

Let us now analyze DAM method for energy allocation. This method produces accurate allocation when N_i ,

$$N_i = \alpha_N \cdot A_i \cdot \max_{j=1, \dots, k} \frac{N_j}{A_j}, \quad i \in \{k+1, \dots, m\}$$

$$\frac{N_i}{A_i} = \alpha_N \cdot \max_{j=1, \dots, k} \frac{N_j}{A_j}, \quad i \in \{k+1, \dots, m\}$$

represents accurate consumption reading for energy consumed in the apartments without heat allocators. Let us denote consumption reading and area of an apartment which maximizes given fraction with N_{max} and A_{max} . Since we are assuming accurate allocation, for all consumption reading we have $N_i = \gamma E_i$ for some constant γ . Now we have

$$\frac{\gamma E_i}{A_i} = \alpha_N \cdot \frac{\gamma E_{max}}{A_{max}}, \quad i \in \{k+1, \dots, m\}.$$

So, if we want for an allocation to be accurate, there should be

$$E_{max} = \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i, \quad (14)$$

for all apartments without allocators, $i \in \{k+1, \dots, m\}$.

What happens if we have $E_{max} > \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i$? In that case, the apartments without allocators are assigned consumption readings N'_i :

$$\begin{aligned} N'_i &= \alpha_N \cdot A_i \cdot \frac{N_{max}}{A_{max}} \\ &= \alpha_N \cdot A_i \cdot \frac{1}{A_{max}} \cdot \gamma \cdot E_{max} \\ &> \alpha_N \cdot A_i \cdot \frac{1}{A_{max}} \cdot \gamma \cdot \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i \end{aligned}$$

So, we have $N'_i > \gamma \cdot E_i = N_i$. Therefore, in the case of an apartment with maximal relative consumption among those with heat allocators, consumes more energy than stated in equation (14) (where i -th apartment is one of the apartments without heat allocators), apartments without heat allocators are allocated more energy than their actual consumption is. Consequently, apartments with heat allocators are allocated smaller amount of energy than their actual consumption.

Analogously, in case when we have $E_{max} < \frac{A_{max}}{A_i} \cdot \frac{1}{\alpha_N} \cdot E_i$, we conclude that apartments with allocators are allocated greater amount of energy compared to their actual consumption, while apartments without heat allocators are allocated smaller amount of energy than their actual consumption.

Unfortunately, it is impossible to determine relation between E_{max} and E_i solely upon consumption readings on heat allocators, nor from the total energy consumption readings. No matter what are total saving in energy consumption, distribution of energy consumption within the building can produce both results, depending of actual consumption of just one apartment (the one that maximizes (relative) consumption).

Nevertheless, we can still give bound for maximal error of allocation model, through analyze of the worst case scenario. Based on the given analysis, worst case scenario for the DAM model of allocation occurs in case when $E_{max} \rightarrow 0$. Consequently, consumption in all apartments with heat allocators tends to zero. Therefore, we will analyze situation in which there is

$$\frac{E_{max}}{A_{max}} = \epsilon \ll \frac{E_i}{A_i}, \quad i = k+1, \dots, m \quad (15)$$

while energy consumption in all other apartments with heat allocators equals to zero. Furthermore, let us suppose (without loss of generality) that all apartments have equal area, so that we could have clearer readings of the results. So, we are working under assumption that $E_{max} = \epsilon \ll E_i$, where E_i is same amount of energy consumed in all apartments without heat allocators. In that case total energy consumption equals to:

$$E_{total} = \sum_{i=k+1}^m + \epsilon = (m-k) \cdot E_i + \epsilon$$

Let us with N_0 denote energy consumption reading in the apartment with energy consumption $E_{max} = \epsilon$. All other apartments with heat allocators have consumption reading equal to 0, while according to DAM apartments without heat allocators are assigned consumption readings $N_i = \alpha_N \cdot N_0$. Furthermore, let us have $\alpha_C = 0$, so that complete allocation is done according to consumption readings (and none according to apartments areas) which emphasize energy allocation error. Now we have:

$$N_{total} = (m - k) \cdot \alpha_N \cdot N_0 + N_0$$

Now, energy allocated to the apartment with maximal energy consumption among apartments with heat allocators equals to (see Definition 3):

$$\begin{aligned} E_{i'} &= E_{total} \cdot \frac{N_0}{N_{total}} = \\ ((m - k) \cdot E_i + \epsilon) \cdot \frac{N_0}{(m - k) \cdot \alpha_N \cdot N_0 + N_0} &= \\ \frac{(m - k) \cdot E_i + \epsilon}{(m - k) \cdot \alpha_N + 1} &\approx \frac{(m - k) \cdot E_i}{(m - k) \cdot \alpha_N + 1} \\ E_{i'} &= E_i \cdot \frac{1}{\alpha_N + \frac{1}{m-k}}. \end{aligned} \quad (16)$$

Since $\alpha_N \geq 1$, from equation (16) we see that allocation to an apartment with heat allocators is smaller than allocation to apartments without heat allocators, which is conformation of Property 11. But, in this worst case scenario allocation to an apartment with heat allocators can be close to an allocation to the apartment without heat allocators, even if its actual consumption equals to ϵ which is very low compared to the consumption on the apartment without heat allocators.

For instance, if we use $\alpha_N = 1.1$ in a building in which 10 apartments does not have heat allocators, we have $E_{i'} = 0.833E_i$, even if i'-th apartment consumes few percent of energy consumed in i-th apartment.

We should point out that this worst case scenario for a DAM model, is in the same time a worst case scenario for a SAM model (see analysis following equation (13) on page 9). But, while worst case scenario produces unbounded share of energy allocation for an apartment with heat allocators in SAM model,

the same scenario produces allocation for an apartment with heat allocators which is bounded by energy allocation share to the apartment without heat allocators.

5 Conclusion

In this article we presented and defined three methods for heat energy allocation with partial distribution of heat allocators. Two of them were legislated in Croatia during last seven years (SAM model defined in Definition 1 and SAM-wT model defined in Definition 2), while third method was proposed in works of Hatzivelkos (DAM model defined in Definition 3).

Moreover, we defined several properties of heat energy allocation models: consistency, monotonicity and local consistency. Consistency of the allocation model is necessary property; the sum of all allocated energy must be equal to consumed energy total. As we showed, all three allocation models are consistent. Monotonicity is property that ensures that greater energy consumption will always lead to greater energy allocation to that consumer (see Definition 7). But, while SAM and DAM allocation models are monotone, SAM-wT is not.

Finally, local consistency is property that is most visible from the perspective of the consumer (see Definition 10). This property ensures that consumer sees allocation as "fair", when consumers compare their energy allocations. As we showed, only DAM model satisfy local consistency, while SAM and SAM-wT do not.

In the last part of article, we analyzed the behavior of SAM and DAM allocation models in "worst case scenario", that is, for distributions of consumption readings for which allocation distribution differs the most from actual energy consumption distribution. We find out that in those situations SAM model allocate share of energy to the consumer with heat allocators, that is far greater than its actual energy consumption. Even more, that ratio (of allocated and consumed heat energy) is unbounded from above. On the other hand, DAM model even in the worst case scenario, produced allocation for a consumer with heat allocators, that is bounded from above with allocation to the consumers without heat allocators.

Even if those worst case scenarios are rare, such allocation anomaly can take high media coverage. Such situations then lead to public distrust in very notion of individual metering through heat allocators. Problem then outgrows the individual injustice done to one consumer. In the long run, repetition of such anomalies, backed with allocation model properties which are seen as unfair (such as lack of local consistency) can lead to public disapprove of the very concept of individual heat energy metering through heat allocators.

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