Parametric Sensitivity in Geoengineering and Controlling the Weather and Climate

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Abstract: - From the standpoint of control theory, the Earth's climate system (ECS) can be considered as a self-regulating feedback cybernetic system, in which the climate system itself represents control object and the role of controller is given to human operators. Mathematical modelling is one of the most suitable and reasonable instruments to study the ECS and to explore climate manipulation (geoengineering) and weather modification technologies. This is because of the extreme complexity of the ECS as physical object. Mathematical modelling allows describing large-scale interventions in the ECS parametrically. Consequently, the efficiency of geoengineering methods can be estimated by studying the sensitivity of ECS models with respect to variations in their input parameters that reflect the influence of external man-made forcing. In this paper, we will discuss the use of sensitivity analysis in climate engineering and controlling the weather and climate problems. Certain critical issues that can arise will be also considered.

Key-Words: - Geoengineering, climate manipulation, weather modification, climate change, optimal control, sensitivity analysis.

1 Introduction

Geoengineering also referred to as climate engineering, is the deliberate and purposeful large-scale modification of the Earth's climate system made by humans [1, 2]. Climate engineering has been suggested as a response measure on the global warming, which is currently acknowledged by both the international scientific community and majority of policymakers. Observations show that since the beginning of the 20-th century, the Earth's global average surface temperature has increased by almost 0.8 °C, with about two-thirds of the increase occurring since 1980. The IPCC (Intergovernmental Panel on Climate Change) Fifth Assessment Report (AR5) [3] provides a clear view of the up-to-date state of scientific knowledge regarding climate change. It is recognized, that mankind is causing global warming by anthropogenic CO_2 emissions generated by human activities through combustion of fossil fuel, mainly coal, oil, natural gas and wood. Due to the burning of fossil fuels and destruction of native forests, the concentration of carbon dioxide is increased from 280 to more than 400 parts per million (ppm) since the beginning of the so-called Industrial Revolution (~1750).

The most suitable solution to reduce global warming is sequestration the anthropogenic emissions of greenhouse gases (GHG) [1, 3]. However, this is unlikely achievable because the world economic growth and increasing population require more and more energy resources generating more and more GHG emissions. Carbon dioxide-free renewable energy resources and energy efficiency measures are at the moment not the alternative because they are very expensive and require long time to achieve tangible results. Considering these arguments, scientists and engineers proposed several solutions, known as geoengineering, to stabilize the global climate (e.g. [4-9]). Geoengineering technologies, in general, are divided on two main categories: carbon dioxide removal technologies (CDR), and solar radiation management (SRM). A number of geoengineering solutions are offered to date, however, all of them introduce uncertainties and unexpected consequences that must be explored.

Implementation and execution of climate engineering technologies is a purposeful process, i.e. the process which has a specific purpose and, therefore, outcome that can be formulated in various ways and must be achieved to successfully complete a geoengineering operation. In this respect, climate engineering is essentially the process of controlling the climate system that can be examined from the standpoint of control theory [10-13]. Therefore, the ECS can be considered as a self-regulating feedback cybernetic system, in which the climate system itself represents control object, and the role of controller is given to human operators.

The ECS is a complex, interactive, nonlinear dynamical system consisting of the atmosphere, hydrosphere, cryosphere, lithosphere and biosphere. The state of the ECS at a given time and place with respect to variables such as temperature, barometric pressure, wind velocity, moisture, precipitations is known as the weather. Climate is usually defined as "average weather" or, in other words, as an ensemble of states traversed by the climate system over a sufficiently long period of time. Commonly, this period corresponds to ~30 years, as defined by the World Meteorological Organization. Since the ECS is a unique physical object with a large number of specific features [14-17], control of physical and dynamical processes occurring in the ECS is an extremely complex and difficult problem. Synthesis of control systems of such natural objects represents a multidisciplinary research area, which is developed on the ideas and methods of optimal control theory, dynamical systems theory, technical cybernetics, climate physics and dynamics, and other academic disciplines. The success of climate engineering strongly depends on the understanding of physics, chemistry and dynamics of climate processes as well as the availability of enabling technologies. We also need to know the response of the ECS to geoengineering interventions.

To estimate the efficiency of climate modification methods and to assess their direct and indirect impacts on the ECS, we can use the method of mathematical/numerical modelling since the ability of laboratory simulations of the ECS are, with rare exceptions, very limited. In this framework, in order to explore the applicability of geoengineering methods and technologies to climate manipulation and weather modification, we can use sensitivity analysis since control actions may be considered as variations in the model parameters. Mathematical models of the ECS used in variety of applications are derived from a set of multidimensional nonlinear differential equations in partial derivatives, which are the equations of fluid dynamics and thermodynamics that describe dynamical, physical and chemical processes and cycles in the ECS. Mathematical models of the ECS are mostly deterministic with a large phase space dimension [18-24]. Equations that describe the evolution of the ECS cannot be solved analytically with an arbitrary set of initial conditions, but only numerically using various types of finite-dimensional approximations such, for example, as Galerkin projection or finite-difference methods. In mathematical models, large-scale intervention in the ECS can be described parametrically, or, in other words, effects of geoengineering actions can be parameterized using conventional parameterization schemes of sub-grid physical processes, first of all schemes that describe radiative processes (short-wave solar radiation, long-wave emissions of the Earth). Consequently, the efficiency of geoengineering methods can be estimated by studying the sensitivity of ECS models with respect to variations in their input parameters that reflect the influence of external man-made forcing [25-27].

Dynamical systems used to model ECS, are essentially nonlinear and under certain conditions they exhibit aperiodic oscillations, which are known as the phenomenon of deterministic chaos [28]. For such systems, the conventional methods of sensitivity analysis, such as forward and adjoint approaches, are not sufficiently effective, since calculated sensitivity functions are uninformative and inconclusive [29-32]. In this context, the exploration of sensitivity of nonlinear models of the ECS with respect to variations in their parameters requires special consideration. In this paper, we will discuss the use of sensitivity analysis in climate engineering and controlling the weather and climate problems. Certain critical issues that can arise will be also considered.

This paper is organized as follows. Section 2 considers the Earth's climate system as a dissipative and chaotic dynamical system, as well as its unique properties as physical and control object. This section also presents the formulation of the optimal control problem for controlling the Earth's climate system. Special attention is paid to the problem of stabilizing the trajectory of the system around the reference trajectory in the phase space. Sensitivity analysis methods used in climate research and modelling are discussed in Section 3. Besides conventional methods of sensitivity analysis, this section presents the novel approach that is based on shadowing properties of dynamical systems. Section 4 describes two low order coupled chaotic climate models and examines the results of numerical experiments. Summary and discussions are given in section 5.

2 Controlling the Weather and Climate: Problem Formulation

2.1 The Earth's Climate System as a Dynamical System

Dynamical systems theory represents a very powerful and comprehensive framework for exploring, predicting, explaining and understanding physical processes and phenomena occurring in the ECS. In general, any abstract dynamical system can be considered as a pair $\left(X,\varphi^t\right)$, where X is the system phase (or state) space, and $\varphi^t : X \to X$ is a family of evolution operators parameterized by a real variable $t \in T$, where T is a time set. It is usually assumed that the phase space is a complete metric or Banach space, which can be either finite- or infinite-dimensional. A family of operators forms a semigroup, therefore: $\varphi^{t+\tau} = \varphi^t \circ \varphi^\tau$, $\varphi^0 \equiv I$, $\forall t, \tau \in T$, where I is the identity operator.

Suppose that $t \in \mathbb{R}_+$, then a continuous time dynamical system can be generated by the following set of autonomous ordinary differential equations

$$\dot{x} = f\left(x\right) \qquad (1)$$

with given initial conditions

$$x\left(0\right) = x_0, \qquad (2)$$

where $x \in X$, f is the (nonlinear) vector-valued function, and x_0 is a given initial state of the system. In most cases, mathematical models cannot be solved analytically, requiring a numerical solution. Consequently, the set of infinite-dimensional equations (1) has to be truncated by some means to finite-dimensional approximate model, for which a solution can be sought numerically. Applying either a projection onto a finite set of basic functions or a discretization in time and space, one can derive a discrete model, which approximates the system (1) and can be solved numerically if the initial conditions are specified:

$$x_{k+1} = f\left(x_k\right), \ k \in \mathbb{Z}_+. \qquad (3)$$

Given the system state $x_0 \in X$ at time $t = 0$, we define the trajectory of x_0 under f to be the sequence of points $\left\{x_k \in X : k \in \mathbb{Z}_+\right\}$ such that $x_k = f^k\left(x_0\right)$, where f^k indicates the k-fold composition of f with itself, and $f^0\left(x\right) \equiv x$. Thus, given a map $f : X \to X$ and the initial condition x_0, equation (3) uniquely specifies the orbit of a dynamical system. If x_k is the state of dynamical system at time t_k, then the state at the next time x_{k+1} is given by $f\left(x_k\right)$.

The ECS is a physical continuum and its evolution is mathematically described by the set of partial differential equations of the form:

$$\partial_t \varphi(r,t) = L\left(\varphi(r,t), \lambda(r,t)\right), \ \varphi(r,0) = \varphi_0\left(r\right). \qquad (4)$$

Here $\varphi \in Q\left(\Omega_t\right)$ is the state vector of a system, where $Q\left(\Omega_t\right)$ is the infinite real space of sufficiently smooth state functions satisfying some problem-specific boundary conditions at the boundary $\partial\Omega$ of the space domain Ω; $r \in \Omega \subset \mathbb{R}^3$ is a vector of spatial variables; $\Omega_t = \Omega \times \left[0,\tau\right]$ is a bounded space-time domain, $t \in \left[0,\tau\right]$ is the time; L is a nonlinear operator that describes the dynamics, dissipation and external forcing of the system; $\lambda \in G\left(\Omega_t\right)$ is the system parameter vector, where $G\left(\Omega_t\right)$ is the domain of admissible values of the parameters; and φ_0 is the initial state estimate. Note that the system (4) characterizes a continuous medium for which the state vector φ is infinite-dimensional: $\varphi \in \Phi$, where Φ is the infinite-dimensional Hilbert space. In order to obtain a system with a finite number of degrees of freedom, which is required to solve the problem numerically, the equations (4) can be projected onto the subspace spanned by the orthogonal basis $\left\{\psi_i\right\}_{i=1}^n$ so that φ can be represented as a normally convergent series:

$$\varphi\left(r,t\right) \approx \sum_{i=1}^{n} x_i(t)\psi(r). \qquad (5)$$

Substituting (5) into (4) and using then the Galerkin method, we can obtain the dynamical system that is described by the set of ordinary differential equations (ODEs):

$$\dot{x} = f\left(x,\alpha\right), \ t \in [0,\tau], \ x(0) = x_0, \qquad (6)$$

where $x \in X \subseteq \mathbb{R}^n$ is the state vector the components of which belong to the class of continuously differentiable functions $\mathcal{C}^1\left([0,\tau]\right)$, $\alpha \in P \subseteq \mathbb{R}^m$ is the parameter vector the components of which belong to the class of piecewise continuous functions $\hat{\mathcal{C}}\left([0,\tau]\right)$, $f \in \mathbb{R}^n$ is a nonlinear vector-function defined in the domain $X \times P \times \left[0,\tau\right]$ that is continuous with respect to both x and α, continuously differentiable with respect to x**,** as well as piecewise continuous with respect to t, and x_0 is a given initial conditions. The finite-dimensional dynamical system (6) can be also obtained by discretization in space of the equations (4). For the ECS, the vector-function f is strongly nonlinear. This nonlinearity arises from numerous feedbacks existed in the ECS, a broad spectrum of interactive oscillations, and external forcing caused by natural and anthropogenic processes.

2.2 Essential features of the Optimal Control Problem for the Earth's Climate System

The ECS is extremely difficult to control because it is a very unique physical object, which possesses a number of specific features [14-17]:

- The ECS is a complex interactive system with numerous positive and negative feedback mechanisms. Various physical and chemical interaction processes occur among the different components of the ECS system on a wide range of space and time scales, making the system enormously sophisticated;

- The components of the ECS are very different in their physical and chemical properties, structure and dynamics; they are linked together by fluxes of momentum, mass and energy;

- The ECS is an open system but its impact on the external environment is negligible;

- Time scales of physical processes occurring in the ECS vary over a wide range - from seconds to tens and hundreds of years;

- The ECS is a global system and its spatial spectrum of motions covers molecular to planetary scales;

- Dynamical processes in the ECS oscillate due to both internal factors (natural oscillations) and external forcing (forced oscillations). Natural oscillations are due to internal instability of the atmosphere and ocean with respect to stochastic infinitesimal disturbances. Anthropogenic impacts on the ECS, both intentional and unintentional, belong to the category of external forcing;

- Nonlinear motions in the atmosphere, which is the most fast-oscillating component of the ECS, under certain conditions exhibit the chaotic behavior;

- The ECS is a dissipative dynamical system, which possess a global attractor. This implies that there exists an absorbing set which is bounded set is the phase space that attracts any trajectory of the system. In other words, the norm of the solution of the model equations with arbitrary initial conditions, from a certain moment of time t^*, does not exceed some fixed value: $\|x(t)\| < d_0,\ t \geq t^*$. Usually, in many applications, the ECS evolution is considered on its attractor assuming that the system is ergodic.

Certainly, the ECS has a number of other specific features that make it a unique physical object, which is virtually impossible to study using laboratory simulations (with rare exceptions). Therefore, the main method of studying the ECS is mathematical modelling. It is important to note that the atmosphere is the most unstable and rapidly changing component of the ECS.

With respect to the problem of control of the ECS, it should be emphasized that the application of cybernetic approaches and techniques developed for the study and optimal control of systems in many scientific areas, ranging from engineering and sciences to economics and social sciences, is very difficult. This is due to the following factors:

- Climate processes are not sufficiently well identified as control objects; their mathematical models are insufficiently perfect, accurate and adequate;

- The ECS refers to a class of distributed parameter systems described by partial differential equations, making the mathematical models of climate processes quite complex. Synthesis of control systems of such objects requires the development of control theory, which was created mainly for objects (systems) with lumped parameters.

2.3 Formulation of the Optimal Control Problem for the Earth's Climate System

The formulation of optimal control problems includes mathematical model of the ECS that describes its behavior under the influence of control actions and external forcing (disturbances); specification of the control objectives; control model that imposes constrains on the controls and the state variables of the ECS; and specifications of boundary and initial conditions for the model equations. For this reason, deterministic mathematical models are mainly used for numerical modelling and prediction of the dynamics and evolution of the ECS and its processes.

Let us make the following critical note. Physical processes, those are too small-scale to be explicitly represented in the model due to its discrete spatialtemporal structure, are parameterized, i.e. replaced by simplified parametric schemes generating additional model parameters. Some of them together with external forcing can be considered as control variables. *By varying the control parameters, we can formally control the climate dynamics*.

Let us omit uncontrolled parameters in the equation (6), then we get the following controllable system on the time interval $t \in [0,\tau] \subset \mathbb{R}^+$:

$$\dot{x} = f(x,u), \; x(0) = x_0, \tag{7}$$

where $u \in U \subset \mathbb{R}^m$ is a vector of control variables. Suppose that the dynamical system (7) is controllable, and control parameters belong to a set of admissible controls $u \in \mathcal{U} \subset U$. It is extremely important that the set $\mathcal{U}$ *must be defined on the basis of physical and technical feasibility* taking into account the above-mentioned specific properties of the ECS as a control object. Further, suppose that controls belong to the class of piecewise continuous functions $\hat{\mathcal{C}}([0,\tau])$ with values in U or Lebesgue measurable functions with values in U, then, according to the classical Caratheodory's theorem [33], one can prove that the Cauchy problem (7) has a unique solution defined on a time interval in $\mathbb{R}^+$. However, we cannot a priori determine whether the ECS is controllable or not. Conclusion concerning controllability of the system can only be made by solving a specific problem. If control parameters depend on the state of the system, i.e. $u(t) = g(t, x(t))$, then equations (7) describe a closed-loop control system, representing the ECS.

The main objective of the problem considered is to synthesize the control law that ensures the achievement of the desired results. Since these results are expressed in terms of extremal problem, we are specifically interested in synthesis of an optimal control. Stabilization of the ECS around the reference phase space trajectory in order to weaken the global warming represents one of the most important problems relevant to the optimal control of climate processes. For this particular class of problems, the differential equation (7) is linearized with respect to the natural (reference) trajectory $x^0(t)$ caused by external natural unperturbed forcing $u^0(t)$:

$$\frac{d\delta x(t)}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x^0,u^0} \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{x^0,u^0} \delta u(t), \; \delta x(0) = 0, \tag{8}$$

where δx is the perturbation of natural orbit of the ECS due to anthropogenic disturbances, δu is a control vector to ensure the stabilization of the ECS trajectory, $\partial f / \partial x$ and $\partial f / \partial u$ are the Jacobian matrices. Naturally, we have to assume that

$$u = u^0 + \delta u, \; \|\delta u\| << \|u^0\|; \; x = x^0 + \delta x, \; \|\delta x\| << \|x^0\|$$

The optimal control problem is formulated as follows:

Find the control vector $\delta u^*(t) \in \mathcal{U}$ generating the correction of the natural orbit δx^* such that $x^0 + \delta x^* \in \mathcal{X} \subset X$, and the performance index J is minimized:

$$\delta u^* = \arg\min_{\delta u \in \mathcal{U}} J(\delta x, \delta u), \tag{9}$$

$$\begin{aligned} J = {} & \frac{1}{2}\delta x^{\mathrm{T}}(\tau) G \delta x(\tau) \\ & + \frac{1}{2}\int_0^\tau \left[\delta x^{\mathrm{T}}(t) W \delta x(t) + \delta u^{\mathrm{T}}(t) Q \delta u(t) \right] dt \end{aligned}, \tag{10}$$

where $W(t)$ and G are weighting positive semi-definite $n \times n$ matrices, normalizing the energy of the ECS per unit mass, $Q(t)$ is a weighting positive define $m \times m$ matrix, normalizing the energy of control actions per unit mass.

The stabilization problem is solved, given the fact that the ECS travels along its natural trajectory that is subject to external natural forcing.

The information on the ECS state $x(t)$ is obtained by measurement devices and instruments followed by the processing using data assimilation procedure. The problem (9) includes a set X at which the functional J is defined, and constraints on the model state given by the subset $\mathcal{X}$ of a set X. The dynamic constraints are given by equations (7). There are several methods available for solving the problem (9): classical methods of the variational calculus, dynamical programming, the Pontryagin's maximum principle and other methods.

The formulation of performance index (10) depends on the problem under consideration and there are no universal approaches how it can be specified.

3 Parametric Sensitivity in Controlling the Weather and Climate

3.1 Forward and Adjoint Methods of Sensitivity Analysis

As stated above, in mathematical models of the ECS control actions are described parametrically, i.e. via variations in the parameters. However, control parameters must be *a priori* chosen based on physical and technical feasibility, they must belong to the set of admissible values, and also the system response on variations in the control parameters must be studied. The latter problem is closely related to the parametric sensitivity analysis of dynamical systems.

One of the commonly used measures for estimating the influence of model parameter variations on the system state variables is a sensitivity function, which is the derivative of a certain component of a model state vector x_i with respect to some model parameter α_j [25-27]: $S_{ij} = \partial x_i / \partial \alpha_j \big|_{\alpha_j = \alpha_j^0}$, where α_j^0 is some fixed (nominal) value. Let $\delta\alpha$ be an infinitesimal variation in the parameter vector. Approximating the state vector around $x(\alpha^0)$ by a Taylor expansion, we get:

$$x(\alpha^0 + \delta\alpha) = x(\alpha^0) + S_\alpha \delta\alpha + H.O.T.,$$

where $S_\alpha \equiv \partial x / \partial \alpha \big|_{\alpha = \alpha^\alpha} \in \mathbb{R}^{n \times m}$ is a sensitivity matrix the elements of which are sensitivity functions. Differentiating (6) with respect to α, we obtain the set of non-homogeneous ordinary differential equations, the so-called sensitivity equations, which can be written as

$$\frac{dS_j}{dt} = M \cdot S_j + D_j \quad j = 1, \ldots, m,$$

where $S_j = (\partial x / \partial \alpha_j) = (S_{1,j}, S_{2,j}, \ldots, S_{n,j})^{\mathrm{T}}$ is the sensitivity vector with respect to parameter α_j, M is a Jacobian matrix, and

$$D_j = (\partial f_1 / \partial \alpha_j, \ \partial f_2 / \partial \alpha_j, \ldots, \partial f_n / \partial \alpha_j)^{\mathrm{T}}.$$

Sensitivity equations describe the evolution of sensitivity functions along a given trajectory, and therefore allow tracking the sensitivity dynamics in time. Thus, to analyze the sensitivity of system (6) with respect to the parameter α_j one can solve the following set of differential equations with given initial conditions:

$$\begin{cases} \dot{x} = f(x, \alpha), & x(0) = x_0, \\ \dot{S}_j = M \cdot S_j + D_j, & S_j(0) = S_{j0}. \end{cases}$$

In sensitivity analysis to measure the response of a system to variations in the parameters, a certain generic objective function (performance measure), which characterizes the dynamical system (6), is commonly used [25, 26]:

$$J(x, \alpha) = \int_0^\tau F(t; x, \alpha)\, dt, \qquad (11)$$

where F is a (nonlinear) function of state variables x and parameters α. Let x^0 be the unperturbed state vector that corresponds to the unperturbed parameter vector, i.e. vector x^0 is obtained by solving the equations (6) with $\alpha = \alpha^0$. The impact of parameter variations on the system performance is quantified by the gradient of the response function (11) with respect to α around the unperturbed point (x^0, α^0):

$$\nabla_\alpha J(x^0, \alpha^0) = \left(\frac{dJ}{d\alpha_1}, \ldots, \frac{dJ}{d\alpha_m} \right)^{\mathrm{T}} \Bigg|_{x^0, \alpha^0}. \qquad (12)$$

Particularly, the influence of parameter α_j is calculated as

$$\frac{dJ}{d\alpha_j} = \sum_{i=1}^n \frac{\partial J}{\partial x_i} \frac{\partial x_i}{\partial \alpha_j} + \frac{\partial J}{\partial \alpha_j} = \sum_{i=1}^n S_{ij} \frac{\partial J}{\partial x_i} + \frac{\partial J}{\partial \alpha_j}.$$

The first order sensitivity estimate for variations in the parameter α_j is given by

$$\frac{dJ(x, \alpha)}{d\alpha_j} \Bigg|_{x^0, \alpha_j^0} \approx \frac{J(x^0 + \delta x; \alpha_1^0, \ldots, \alpha_j^0 + \delta\alpha_j, \ldots, \alpha_m^0) - J(x^0, \alpha^0)}{\delta\alpha_j}.$$

This equation approximates the derivative of the first order; therefore the accuracy of approximation essentially depends on the choice of the parameter variation $\delta\alpha_j$. Generally, this selection is made arbitrarily bearing in mind that the value of $\delta\alpha_j$ is bounded below by the round-off error.

Introducing the Gâteaux differential, the sensitivity analysis problem can be considered in the differential formulation that eliminates the need to set the value of $\delta\alpha$. The Gâteaux differential is defined as [25]

$$\delta J(x^0, \alpha^0) = \int_0^\tau \left(\frac{\partial F}{\partial x} \bigg|_{x^0, \alpha^0} \cdot \delta x + \frac{\partial F}{\partial \alpha} \bigg|_{x^0, \alpha^0} \cdot \delta\alpha \right) dt, \qquad (13)$$

where δx is the variation in the state vector due to the variation in the parameter vector in the direction $\delta\alpha$. Linearizing the model (6) around the unperturbed trajectory $x^0(t)$, we obtain the following system of variational equations for calculating $\delta\alpha$:

$$\frac{\partial \delta x}{\partial t} = \left.\frac{\partial f}{\partial x}\right|_{x^0,\alpha^0} \cdot \delta x + \left.\frac{\partial f}{\partial \alpha}\right|_{x^0,\alpha^0} \cdot \delta\alpha,$$
$$t \in [0,\tau], \ \ \delta x(0) = \delta x_0. \tag{14}$$

The model (14) is known as a tangent linear model. Variations δx obtained from the equation (14) are then used in the equation (13) for evaluating the Gâteaux differential.

Since $\delta J\left(x^0,\alpha^0\right) = \left\langle \nabla_\alpha J, \delta\alpha \right\rangle$, where $\langle \cdot,\cdot \rangle$ is a scalar product, then the model sensitivity with respect to variations in the parameters can be estimated by calculating the components of the gradient $\nabla_\alpha J$. However, this "one-at-a-time" method, in spite of its simplicity, requires significant computational resources if the number of model parameters is large. The use of adjoint equations allows obtaining the required sensitivity estimates within a single computational experiment, since the gradient can be calculated as [26]:

$$\nabla_\alpha J\left(x^0,\alpha^0\right) = \int_0^\tau \left[\left.\frac{\partial F}{\partial \alpha}\right|_{x^0,\alpha^0} - \left(\left.\frac{\partial f}{\partial \alpha}\right|_{x^0,\alpha^0} \right)^{\mathrm{T}} x^* \right] dt, \tag{15}$$

where the vector-valued function x^* is a solution of the adjoint model:

$$-\frac{\partial x^*}{\partial t} - \left(\left.\frac{\partial f}{\partial x}\right|_{x^0,\alpha^0} \right)^{\mathrm{T}} x^* = -\left.\frac{\partial F}{\partial x}\right|_{x^0,\alpha^0},$$
$$t \in [0,\tau], \ \ x^*(\tau) = 0. \tag{16}$$

The adjoint equations (16) are integrated backward in time.

As discussed in [27] and [29], general solutions of sensitivity equations for oscillatory nonlinear dynamical systems grow unbounded as time tends to infinity; therefore, sensitivity functions calculated by conventional approaches are highly uncertain. The reason is that nonlinear dynamical systems that exhibit chaotic behaviour are very sensitive to its initial conditions. Thus, solutions to the linearized Cauchy problem (6) grow exponentially $\|\delta x(t)\| \approx \|\delta x(0)\| e^{\lambda t}$, where $\lambda > 0$ is the leading Lyapunov exponent. Consequently, calculated sensitivity functions include a fairly large error, becoming uninformative and inconclusive [30-33].

3.2 Fluctuation-Dissipation Theorem

To estimate the ensemble-averaged response of the ECS to small external forcing Leith [34] has proposed using the fluctuation-dissipation theorem (FDT). According to the FDT, under certain assumptions, the response of stochastic dynamical system to infinitesimal external perturbations including geoengineering activities is described by the covariance matrix of the unperturbed system:

$$\left\langle \delta x(t) \right\rangle = \int_0^t C(\tau) C^{-1}(0)\, d\tau \cdot \delta\alpha^{ext},$$

where $\delta\alpha^{ext}$ is an external forcing, $\langle \cdot \rangle$ is the symbol means an ensemble average over realizations, and $C(\tau)$ is a τ–lagged covariance matrix of *x*. It is generally assumed that the system is close to thermal equilibrium and the probability density function of the unforced system is Gaussian. However, the climate system is characterized by a strong external forcing and dissipation, making it a system for which the standard assumptions of equilibrium statistical mechanics do not hold.

3.3 Novel Approach of Sensitivity Analysis Based on Shadowing Property of Dynamical Systems

In climate studies, the average values of sensitivity functions $\nabla_\alpha \left\langle J(\alpha) \right\rangle$ over a certain period of time are usually considered as one of the most important measures of sensitivity, where *J* is a generic objective function (11). However, the gradient of *J* with respect to *α* cannot be correctly estimated by using conventional methods of sensitivity analysis since for chaotic systems it is observed [29-32] that $\nabla_\alpha \left\langle J(\alpha) \right\rangle \neq \left\langle \nabla_\alpha J(\alpha) \right\rangle$. This is because the integral

$$\mathcal{I} = \lim_{\tau\to\infty} \int_0^\tau \lim_{\delta\alpha\to 0} \left[\frac{J(\alpha+\delta\alpha) - J(\alpha)}{\delta\alpha} \right] dt \tag{17}$$

does not possess uniform convergence and two limits ($\tau \to \infty$ and $\delta\alpha \to 0$) would not commute.

The "shadowing" approach for estimating the system sensitivity to variations in its parameters suggested in [30] and [31] allows us to calculate correctly the average sensitivities $\left\langle \nabla_\alpha J(\alpha) \right\rangle$ and therefore to make a clear conclusion with respect to the system sensitivity to its parameters. This approach is based on the theory of pseudoorbit shadowing in dynamical systems [35, 36], which is one of the most rapidly developing components of the global theory of dynamical systems and classical theory of structural stability [21]. Naturally, pseudo- (or approximate-) trajectories arise due to the presence of round-off errors, method errors, and other errors in computer simulation of dynamical systems. Consequently, we will not get an exact trajectory of a system, but we can come very close to an exact solution and the resulting approximate solution will be a pseudotrajectory.

The shadowing property (or pseudo orbit tracing property) means that, near an approximate trajectory, there exists the exact trajectory of the system considered, such that it lies uniformly close to a pseudotrajectory. The shadowing theory, which was originated by D.V. Anosov [37] and R. Bowen [38], is well-developed for the hyperbolic dynamics. This dynamics is characterized by the presence of expanding and contracting directions for derivatives.

Let (M, dist) be a compact metric space and let $f: M \to M$ be a homeomorphism (a discrete dynamical system on M). A set of points $X = \{x_k\}_{k\in\mathbb{Z}}$ is a d-pseudotrajectory $(d > 0)$ of f if

$$\operatorname{dist}\left(x_{k+1}, f(x_k)\right) < d,\ k \in \mathbb{Z}.$$

Here the notation $\operatorname{dist}(\ ,\ )$ denotes the distance in the phase space between two geometric objects within the brackets. We say that f has the shadowing property if given $\varepsilon > 0$ there is $d > 0$ such that for any d-pseudotrajectory $X = \{x_k\}_{k\in\mathbb{Z}}$ there exists a corresponding trajectory $Y = \{y_k\}_{k\in\mathbb{Z}}$, which ε-traces X, i.e. $\operatorname{dist}(x_k, y_k) < \varepsilon,\ k \in \mathbb{Z}$.

The shadowing lemma for discrete dynamical systems [35] states that for each $\varepsilon > 0$, there exists $d > 0$ such that each d-pseudotrajectory can be ε-shadowed. The definition of pseudotrajectory and shadowing lemma for flows (continuous dynamical systems) are more complicated than for discrete dynamical systems [35]. Let $\Phi^t : \mathbb{R}\times M \to M$ be a flow of a vector field X on M. A function $g:\mathbb{R} \to M$ is a d-pseudotrajectory of the dynamical system Φ^t if the inequalities

$$\operatorname{dist}\left(\Phi^t\left(t, g(\tau)\right), g(\tau + t)\right) < d$$

hold for any $t \in [-1, 1]$ and $\tau \in \mathbb{R}$. The "continuous" shadowing lemma ensures that for the vector field X generating the flow Φ^t, the shadowing property holds in a small neighborhood of a compact hyperbolic set for dynamical system Φ^t.

However, the shadowing problem for continuous dynamical systems requires reparameterization of shadowing trajectories. This is because if dynamical system is continuous then close points of pseudotrajectory and true trajectory do not correspond to the same moments of time.

To illustrate the applicability of this method, let us consider the continuous one parameter dynamical system $\dot{x} = f(x,\alpha)$ on the time interval $[0,\tau]$. The sensitivity analysis aims to estimate the sensitivity function $S_\alpha = \partial x/\partial\alpha$. Let $x'(t)$ be the pseudo-orbit obtained by integration of the system equations with perturbed parameter $\alpha' = \alpha + \delta\alpha$, where $\delta\alpha$ is the variation in α. Since the pseudotrajectory $x'(t)$ stays uniformly close to the "true" orbit $x(t)$ obtained with unperturbed parameter α, the integral (17) is convergent and the average sensitivities $\langle\nabla_\alpha J(\alpha)\rangle$ can be easily estimated. Let us introduce the following transform $x'(x) = x + \delta x(x)$. It can be shown that $\delta f(x) = A\delta x(x)$, where $A = \lfloor -(\partial f/\partial x) + (d/dt)\rfloor$ is a "shadow" operator [30]. Thus, to find a pseudo-orbit we need to solve the equation $\delta x = A^{-1}\delta f$, i.e. we must numerically invert the operator A for a given δf. To solve this problem, we decompose functions δx and δf into their constituent Lyapunov covariant vectors $\upsilon_1(x),\dots,\upsilon_n(x)$:

$$\delta x(x) = \sum_{i=1}^{n}\psi_i(x)\upsilon_i(x),\ \ \delta f(x) = \sum_{i=1}^{n}\varphi_i(x)\upsilon_i(x),$$

and then compute the Lyapunov exponents λ_i and vectors $\upsilon_1(x),\dots,\upsilon_n(x)$. By executing the spectrum decomposition of δf along the trajectory $x(t)$ we can obtain $\varphi_i(x),\ i = 1,...,n$ and then calculate the expansion coefficients $\psi_i(x),\ i = 1,...,n$ using the equations

$$\frac{d\psi_i(x)}{dt} = \varphi_i(x) + \lambda_i\psi_i(x),\ i = 1,\dots,n,$$

which are derived from the dynamical system equations. The expansion coefficients $\psi_i(x)$ are used to compute δx along the trajectory. By averaging δx over the time interval $[0,\tau]$ we can obtain the sensitivity estimate $S_\alpha = \langle\delta x\rangle/\delta\alpha$.

3.4 Some Important Notes

In order to explore the applicability of geoengineering methods and technologies to climate manipulation and weather modification, we have to choose the system parameters that can be considered as control variables. These can typically be done using detailed analysis of physical and technical feasibility and implementability of geoengineering methods and technologies. Then, sensitivity analysis, discussed in this section, allows establishing the reaction of the dynamical system onto changes in these control variables and defining their range of values.

4 Application of Sensitivity Methods to Low Order Climate Models

A wide spectrum of climate models of various complexities is used in simulation of the ECS. The exploration of the ECS requires considerable computational resources. For simple enough low dimensional models, the computational cost is minor and, for that reason, models of this class are widely applied as simple test instruments to emulate more complex systems such as the ECS. In this paper, we will use two low order models:

(a) multiscale nonlinear dynamical system which is obtained by coupling the fast and slow versions of the original Lorenz model developed in 1963 (L63) [28], and

(b) multiscale coupled nonlinear dynamical system, which is composed of fast (the "atmosphere") and slow (the "ocean") subsystems [39] in which the atmospheric subsystem represents the Lorenz chaotic system developed in 1984 [40].

The equations for the first model can be written as [41, 42]:

$$\begin{aligned}
\dot{x} &= \sigma(y-x) - c(aX+k), \\
\dot{y} &= rx - y - xz + c(aY+k), \\
\dot{z} &= xy - bz + c_z Z, \\
\dot{X} &= \varepsilon\sigma(Y-X) - c(x+k), \\
\dot{Y} &= \varepsilon(rX - Y - aXZ) + c(y+k), \\
\dot{Z} &= \varepsilon(aXY - bZ) - c_z z,
\end{aligned} \tag{18}$$

where lower case letters represent the fast subsystem and capital letters – the slow subsystem, σ, r, b are the parameters of L63 model, c is a coupling strength parameter for the x and y variables, c_z is a coupling strength parameter for z, k is an "uncentering" parameter, ε is the time scale factor, and a is a parameter representing the amplitude scale factor. Thus, the state vector and the parameter vector of the system are, respectively, $\mathbf{x} = (x, y, z, X, Y, Z)^{\mathrm{T}}$ and $\boldsymbol{\alpha} = (\sigma, r, b, a, c, c_z, k, \varepsilon)^{\mathrm{T}}$. The unperturbed parameter values are taken as a=1, k=0, $\sigma^0 = 10$, $c = c_z \in [0.1; 1.0]$, b=8/3, r^0=28, and ε^0=0.1. Chosen values of σ, r and b correspond to chaotic behaviour of the L63 model. Dynamical, correlation and spectral properties of the system (18) were considered in [43].

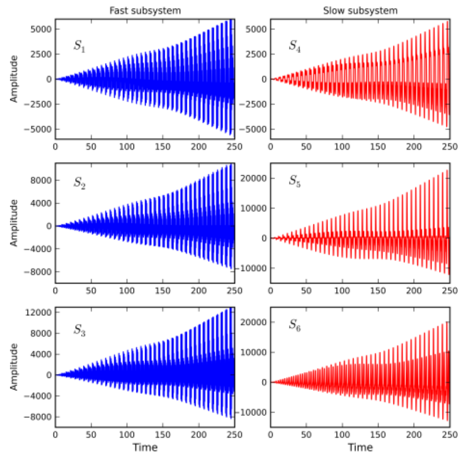

Fig 1. Time dynamics of sensitivity functions with respect to parameter c

The system behaviour strongly depends on the value of parameter c since this parameter controls the synchronization between fast and slow subsystems. Qualitative changes in the dynamical properties of a system can be detected by determining and analysing the corresponding spectrum of Lyapunov exponents. The system (18) has six distinct exponents. If the parameter c tends to zero, then the system (18) has two positive, two zero and two negative Lyapunov exponents. Numerical experiments showed that initially positive two largest conditional Lyapunov exponents decrease monotonically with an increase in the parameter c. At about $c \approx 0.8$ they approach the x-axis and at about $c \approx 0.95$ - negative values. Thus, for $c > 0.95$ the dynamics of both fast and slow subsystems becomes phase synchronous [44]. When $c > 1.0$, a limit circle dynamical regime is observed since all six exponents become negative.

Let us define the following sensitivity functions:

$$S_{1c} = \partial x/\partial c,\ S_{2c} = \partial y/\partial c,\ S_{3c} = \partial z/\partial c,$$
$$S_{4c} = \partial X/\partial c,\ S_{5c} = \partial Y/\partial c,\ S_{6c} = \partial Z/\partial c.$$

Envelopes of these functions calculated by forward approach (see previous section) grow in time and the functions themselves demonstrate the oscillating behaviour (Fig. 1). Thus, these functions are inherently uninformative since it is very difficult to draw a clear conclusion from them about system sensitivity to variations in the parameter *c*.

Analogous results (Fig. 2) were obtained when we considered the system sensitivity with respect to the parameter *r* by introducing the following sensitivity functions:

$$S_{1r} = \partial x/\partial r\,,\ S_{2r} = \partial y/\partial r\,,\ S_{3r} = \partial z/\partial r\,,$$
$$S_{4r} = \partial X/\partial r\,,\ S_{5r} = \partial Y/\partial r\,,\ S_{6r} = \partial Z/\partial r\,.$$

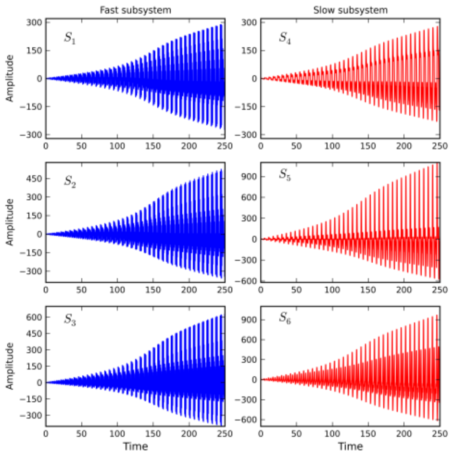

Fig 2. Time dynamics of sensitivity functions with respect to parameter *r*

The parameter *r* represents the temperature difference between the equator and pole, and plays a critical role on the formation of system's dynamical structure and transition to the chaotic behaviour. Obtained sensitivity functions with respect to other system parameters are also uninformative and inconclusive. This is because the average values of sensitivity functions (the components of $\nabla_{\alpha}\langle J(\alpha)\rangle$) over a certain period of time cannot be correctly estimated using conventional sensitivity analysis methods (see the previous section).

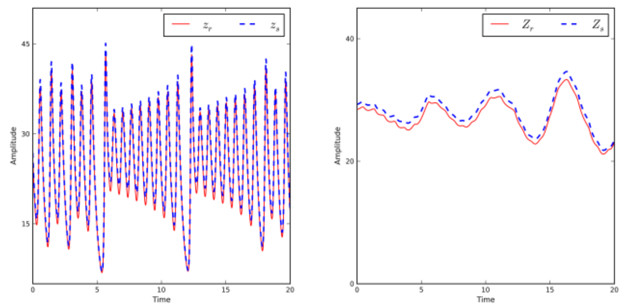

Fig 3. Original (in red) and pseudo (in blue) orbits for the fast *z* (left) and slow *Z* variables (right) for $c = 0.015$

The use of the shadowing method allows calculating both the original and pseudo orbits and then correctly estimating the sensitivities. It is important that the pseudo orbit is calculated by variations in the parameter under consideration.

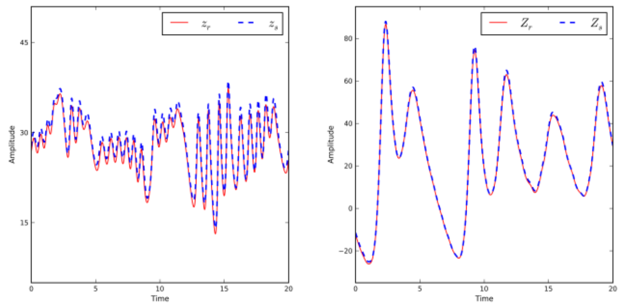

Fig 4. Original (in red) and pseudo (in blue) orbits for the fast *z* (left) and slow *Z* variables (right) for $c = 0.8$

As an example, the original and pseudo orbits for the fast *z* and slow *Z* variables are shown in Fig. 3 (for coupling strength parameter c=0.015) and Fig 4 (for c=0.8). The pseudo orbit was obtained by variations in the parameter *r*. The original orbit lies uniformly close to a pseudotrajectory. Thus, we can estimate the sensitivity of dynamical system (18) with respect to parameters by averaging the calculated sensitivity functions along the trajectory. Table 1 shows the estimates of sensitivity functions with respect to parameter *r* for different values of coupling strength parameter *c*: variables *z* and *Z* are the most sensitive, and the sensitivity of variables *x*, *y*, *X* and *Y* with respect to *r* is significantly less than sensitivity of variables *z* and *Z*.

Table 1. Sensitivity estimates for the fast and slow variables with respect to the parameter *r*

c	$\partial x/\partial r$	$\partial y/\partial r$	$\partial z/\partial r$	$\partial X/\partial r$	$\partial Y/\partial r$	$\partial Z/\partial r$
10^{-4}	-0.01	-0.01	1.02	-0.03	-0.02	1.04
0.15	-0.01	-0.01	1.01	-0.09	-0.08	0.91
0.4	0.09	0.09	1.03	-0.01	0.03	0.95
0.8	0.07	0.07	1.02	0.03	0.08	0.69
1.0	0.03	0.04	1.08	0.01	0.05	1.10

Let us consider the next coupled dynamical system:

$$\begin{aligned}
\dot{x} &= -y^2 - z^2 - ax + aF,\\
\dot{y} &= xy - cy - bxz + G + \alpha X,\\
\dot{z} &= xz - cz + bxy + \alpha Y,\\
\dot{X} &= -\omega Y - \beta y,\\
\dot{Y} &= \omega X - \beta z,
\end{aligned} \tag{19}$$

where *x* is the intensity of the symmetric, globally averaged westerly wind current (the equivalent to meridional temperature gradient); y and z are the amplitudes of cosine and sine phases of a series of superposed large scale eddies, which transport heat poleward; F and G represent the thermal forcing terms due to the average north-south temperature contrast and the earth-sea temperature contrast, respectively. The term b represents displacement of the waves due to interaction with the westerly wind. The coefficient a, if less than 1, allows the westerly wind current to damp less rapidly than the waves. The time unit of t is estimated to be ten days. The model (19) allows mimicking the atmosphere-ocean system and therefore may serve as a key element of a theoretical and computational framework for the study of various aspects of the ECS including geoengineering. Note that the atmospheric system described by equations (19) represents a chaotic Lorenz system [42], while the ocean system is a simple harmonic oscillator. The time dynamics of model variables is shown in Fig. 5.

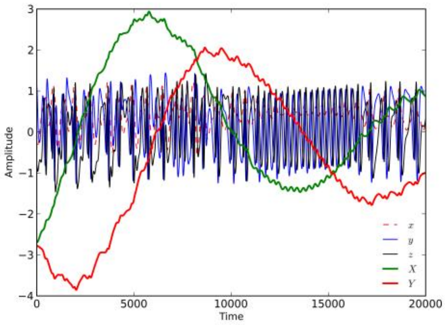

Fig 5. Time dynamics of model variables

Application of conventional sensitivity analysis methods to the system (19) shows that sensitivity functions also contain fairly large errors, similar to the previous model: the envelopes of calculated sensitivity functions grow in time while these functions exhibit oscillating behaviour. Thus, obtained sensitivity functions are essentially uninformative and misleading, and we cannot make a clear conclusion from them about system sensitivity to variations in the model parameters.

The FDT also cannot provide clear information about the system sensitivity with respect to its parameters. In Fig. 6 autocorrelation functions (ACFs) are presented for realizations of all dynamic variables of the model (19). Using ACFs we can easily calculate the system response functions. However, for oscillatory ACFs, the calculated response functions are uninformative. The use of the shadowing method allows us to calculate the average sensitivity functions that can be easily interpreted. However, the shadowing property of dynamical systems is a fundamental attribute of hyperbolic systems, but most real physical systems are non-hyperbolic. Despite the fact that much of shadowing theory has been developed for hyperbolic systems, there is evidence that non-hyperbolic attractors also have the shadowing property. In theory, this property should be verified for each particular dynamical system, but this is more easily said than done.

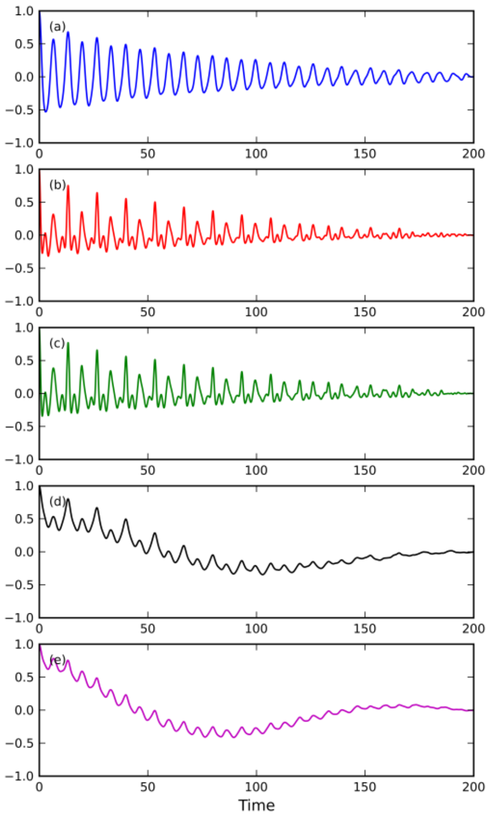

Fig. 6 Autocorrelation functions for variables x (a), y (b), z (c), X (d) and Y (e)

5 Concluding Remarks

Methods and technologies for climate engineering and weather modification represent a potential response measure to the observed climate change and, in particular, to the global warming. So far, however, geoengineering and weather modification are explored outside the framework of the control theory. Meanwhile geophysical cybernetics provides a conceptual and unified theoretical framework for developing and synthesizing the optimal control systems for natural ambient phenomena and processes occurring in the components of the ECS. Since the main method to study the ECS is mathematical modelling, the applicability of geoengineering methods and technologies to climate manipulation and weather modification suggests the use of sensitivity analysis of models to be used because control actions are expressed via variations in the model parameters. Note that climate models contain numerous input parameters that can be interpreted as control variables.

Mathematical models applied in climate system simulation are, in essence, nonlinear and chaotic. There is evidence that conventional sensitivity analysis methods fail when used to chaotic dynamics. For such models, the conventional methods of sensitivity analysis are not sufficiently effective since calculated sensitivity functions are uninformative and inconclusive. The use of shadowing method for estimating the model sensitivity with respect to variations in the parameters allows calculating correctly the average sensitivity functions and, therefore, making a clear conclusion with regard to parametric sensitivity of the model under consideration. This method represents a novel application of the theory of pseudo-orbit shadowing in dynamical system. However, the applicability of shadowing method for sensitivity analysis of modern climate models is a very complicated problem due to the extreme complexity of mathematical models to be applied in climate research. In contrast to conventional sensitivity analysis, which suggests using the ensemble approach for calculating the sensitivities, the shadowing method allows estimating sensitivity functions using a system single orbit of relatively small length. Thus, main advantage of shadowing approach is that it allows us to obtain sensitivity functions, which can be easily interpreted and analysed based on "one-at-a-time" numerical experiment instead of ensemble simulation. However, the shadowing property is a fundamental attribute of hyperbolic dynamical systems, but most real physical systems are non-hyperbolic. Therefore, further research needs to be done to study the pseudo-orbit tracing property of climate models, which is a focus of our future work.

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