Modifying the Kalman Filter for Random Jitter in Sampling Time

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Abstract: It is known that time jitter can vary in nature and magnitude depending on how accurately the time scale is generated and the dynamic process is sampled. We modify the Kalman filter for white Gaussian random jitter and call it jitter Kalman filter (JKF). It is shown that to cope with time jitter the system noise covariance acquires an additional term proportional to the fractional time jitter standard deviation and the process rate. Based on numerical simulations, it is shown that if the process rate grows without limits then the estimation error caused by time jitter will also grow without limits. The conclusions are confirmed experimentally.

Key-Words: Sampling, timing jitter, Kalman filter

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1 Introduction

Before working on optimal recursive filtering, Rudolf Kalman examined in his Ph.D. thesis, [1], linear dynamical systems operation on randomly sampled data. About the same time it was highlighted, [2], that the jitter is highly correlated when timing is derived from the zero crossing. Sampling jitter was later discovered in clocks, [3], [4], [5], and many other digital devices, [6], [7], and learned in signals, [8], [9], [10], [11], [12]. In digital circuits the jitter can reach 50 ps with a total jitter of 100 ps, [13]. In signal tracking using GNSS receivers, the clock jitter with the variance of 2×10^{-24} s provokes a degradation of about 1°, [14]. Timing jitter can also heavily affect estimates of rapidly maneuvering targets, [15], [16].

Errors produced by time jitter are proportional to the derivative of the sampled signal, [17]. Indeed, if we consider some process $x(t)$ and suppose that the discrete time instant $t_k = \bar{t}_k + \tilde{\tau}_k$ is defined with some mean $\bar{t}_k$ and jitter $\tilde{\tau}_k$, then the expansion of $x(t_k)$ to the Taylor series around $\bar{t}_k$ gives

$$x(t_k) = x(\bar{t}_k) + \tilde{\tau}_k \left.\frac{dx(t)}{dt}\right|_{t=\bar{t}_k} + \ldots \tag{1}$$

which suggests that the effect of time jitter $\tilde{\tau}_k$ on $x(t_k)$ is amplified by the rate of the process dynamics, i.e. if the derivative is close to zero, then there should not be any need to worry about jitter. Otherwise, the time jitter may cause big trouble. The questions thus arise: how does the sample time jitter affect the Kalman estimate and accordingly the batch optimal FIR estimate, [18], [19]? and how can we modify the Kalman filter so that it copes with timing jitter and performs better? An analysis of available literature shows that there are no direct answers yet, but some results can be noticed, [20], [21], [22], [23], [24], [25], [26], [27]. It has to be remarked that in these papers, the authors mitigated the effects of time jitter by developing special algorithms. To the best of our knowledge, no modification of the Kalman filter has been proposed so far for random jitter in sampling time. Therefore, the impact of time jitter on Kalman estimates in general also remains unclear, except in some practical cases.

2 Model and Problem Formulation

Sampling jitter cannot be considered apart of the continuous-time state-space model, as far as being a product of discretization. Then consider an LTI stochastic system represented in state space with the following equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{w}\,, \tag{2}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}\,, \tag{3}$$

where the continuous-time noise vectors are white Gaussian, $\mathbf{w} \sim \mathcal{N}(0, \mathcal{C}_w(\theta))$ and $\mathbf{v} \sim \mathcal{N}(0, \mathcal{C}_v(\theta))$, with the covariances

$$\mathrm{cov}(w) = \mathcal{C}_w(\theta) = E\{\mathbf{w}(t)\mathbf{w}^T(t+\theta)\} = \mathcal{S}_w\delta(\theta) \tag{4}$$

$$\mathrm{cov}(v) = \mathcal{C}_v(\theta) = E\{\mathbf{v}(t)\mathbf{v}^T(t+\theta)\} = \mathcal{S}_v\delta(\theta) \tag{5}$$

where $\delta(t)$ is the Dirac delta and $\mathcal{S}_w$ and $\mathcal{S}_v$ are the noise double-sided power spectral density (PSD) matrices. Also, the property $E\{\mathbf{w}(t)\mathbf{v}^T(t+\theta)\} = 0$ holds for all θ. Suppose that the process is measured in discrete time t_k with the sampling time $\tau_k = t_k - t_{k-1}$ and assume that the sampling time is jittering and represent it as $\tau_k = \tau + \tilde{\tau}_k$,

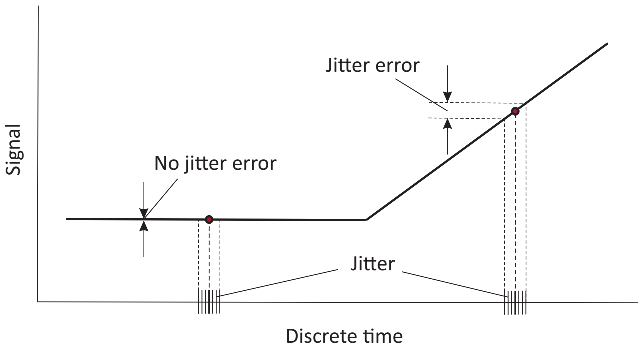

Figure 1: Effect of time jitter on signal discretization. Jitter error is proportional to the derivative of the sampled signal, see (1).

where τ is the mean sampling time. Although the time jitter $\tilde{\tau}_k$ can be random and deterministic, in this paper we are concerned of Kalman filtering. Therefore, we will think that the scalar $\tilde{\tau}_k$ is a random white Gaussian process $\tilde{\tau}_k \sim \mathcal{N}(0, \sigma_\tau^2)$ with known covariance $E\{\tilde{\tau}_k \tilde{\tau}_l\} = \sigma_\tau^2 \delta_{k-l}$. By defining the fractional jitter as

$$\delta_{\tau k} = \frac{\tilde{\tau}_k}{\tau}\,, \tag{6}$$

we represent the jittering sampling time by

$$\tau_k = \tau + \tilde{\tau}_k = \tau(1 + \delta_{\tau k})\,. \tag{7}$$

Note that the Kalman filter treat $\tilde{\tau}_k$ as white Gaussian noise and compute its variance as follows:

$$\sigma_\tau^2 = \int_0^\infty \mathcal{S}_\tau(2\pi \mathrm{f})\, d\mathrm{f}\,, \tag{8}$$

where $\mathcal{S}_\tau(2\pi\mathrm{f})$ is the one-sided (measured) PSD $\mathcal{S}_\tau(2\pi\mathrm{f})$ of the clock oscillator phase noise, which has different slopes in the Fourier frequency domain, [28], [29], [30]. The effect of timing jitter is illustrated in Fig. 1, from which it follows that timing jitter does not affect the constant value. This is used intelligently in event-triggered state estimation, [31], where measurements are transmitted only when the value significantly changes in time, i.e. when the value is noticeably dynamic.

3 Derivation of Noise Covariances under Random Time Jitter

In this section, we first transform the continuous-time state-space model to discrete time in the presence of timing jitter using Euler's backward method and then derive the noise covariances for the transformed model.

3.1 Model Transformation under Random Timing Jitter

With some prior knowledge of sample time jitter, we can integrate the state equation (2) from t_{k-1} to t_k, and convert the continuous-time model to discrete-time as follows, [32]:

$$\mathbf{x}(t_k) = e^{\mathbf{A}\tau_k}\mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k-\theta)}\mathbf{w}(\theta)d\theta \tag{9}$$

$$\mathbf{y}(t_k) = \mathbf{C}\mathbf{x}(t_k) + \mathbf{v}(t_k)\,. \tag{10}$$

By replacing $\mathbf{x}(t_k) = \mathbf{x}_k$, $\mathbf{y}(t_k) = \mathbf{y}_k$, and $\mathbf{v}(t_k) = \mathbf{v}_k$, and introducing

$$\mathbf{F}_k = e^{\mathbf{A}\tau_k}\,, \tag{11}$$

$$\mathbf{w}_k = \int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k-\theta)}\mathbf{w}(\theta)d\theta\,, \tag{12}$$

the equations (9) and (10) become

$$\mathbf{x}_k = \mathbf{F}_k\mathbf{x}_{k-1} + \mathbf{w}_k\,, \tag{13}$$

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k\,, \tag{14}$$

where $\mathbf{H} = \mathbf{C}$ and the discrete-time measurement noise is defined by

$$\mathbf{v}_k = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} \mathbf{v}(t)\, dt\,. \tag{15}$$

Note that the matrix $\mathbf{F}_k$ has a random component in the matrix exponential (11) and is thus also random. Therefore, we make further transformation of the state equation (13) by splitting $\mathbf{F}_k$ into the constant part $\mathbf{F}$ and random part $\tilde{\mathbf{F}}_k$:

$$\mathbf{F}_k = e^{\mathbf{A}(\tau+\tilde{\tau}_k)} \approx \mathbf{I} + \mathbf{A}\tau + \mathbf{A}\tilde{\tau}_k = \mathbf{F} + \tilde{\mathbf{F}}_k\,, \tag{16}$$

where $\mathbf{F} = \mathbf{I} + \mathbf{A}\tau$ and $\tilde{\mathbf{F}}_k = \mathbf{A}\tilde{\tau}_k$. This gives the first order approximation of the state equation (13):

$$\begin{aligned}\mathbf{x}_k &\approx (\mathbf{F} + \tilde{\mathbf{F}}_k)\mathbf{x}_{k-1} + \mathbf{w}_k \\ &= \mathbf{F}\mathbf{x}_{k-1} + \tilde{\tau}_k\mathbf{A}\mathbf{x}_{k-1} + \mathbf{w}_k\,. \end{aligned}\tag{17}$$

The past state $\mathbf{x}_{k-1}$ in the second term of (17) can be replaced by the available past estimate $\hat{\mathbf{x}}_{k-1}$ and the suitable state equation becomes

$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}\mathbf{x}_{k-1} + \tilde{\tau}_k\mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{w}_k \\ &= \mathbf{F}\mathbf{x}_{k-1} + \xi_k \end{aligned}\tag{18}$$

where the noise component

$$\begin{aligned}\xi_k &= \tilde{\tau}_k\mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{w}_k \\ &= \frac{\tilde{\tau}_k}{\tau}(\mathbf{F} - \mathbf{I})\hat{\mathbf{x}}_{k-1} + \mathbf{w}_k \end{aligned}\tag{19}$$

is zero mean and white Gaussian, because $\tilde{\tau}_k$ and $\mathbf{w}_k$ are both zero mean and white Gaussian. This finally gives the modified state space model:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{F}\mathbf{x}_{k-1} + \xi_k & (20) \\ \mathbf{y}_k &= \mathbf{H}\mathbf{x}_k + \mathbf{v}_k & (21) \end{aligned}$$

where $\xi_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_\xi)$ and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ are white Gaussian and for which the covariances, $\mathbf{Q}_\xi$ and $\mathbf{R}$, and the cross-covariance $\mathrm{cov}(\xi_k, \mathbf{v}_l)$ still need to be determined.

To apply Kalman filtering to modified equations (20) and (21), we need to determine the covariances of white Gaussian noise components w_k, ξ_k, and v_k in the presence of time jitter, and find out wether the vectors ξ_k and v_k are correlated. Since we have expanded the matrix exponential in the first order approximation (16), the second order terms in the following transformations should be neglected. However, as we will later see, this should not always be done, since neglecting all second order terms makes the Kalman filter insensitive to time jitter. Therefore, we will first derive the noise covariances taking into account all of the factors related to time jitter. Next, we will neglect second order terms whenever possible.

3.2 Covariances of w_k

Referring to (12), consider the covariance of $\mathbf{w}_k$:

$$\mathrm{cov}(\mathbf{w}_k) = \mathrm{cov}\left(\int_{t_{k-1}}^{t_k} e^{\mathbf{A}(t_k-\theta)}\mathbf{w}(\theta)\,d\theta\right)$$

that, by replacing the lower limit of integration with $t_{k-1} = t_k - \tau - \tilde{\tau}_k$, can be written it as (22) (on the top of next page), and we see that averaging over two independent random variables $w(t)$ and $\tilde{\tau}_k$ is required.

$$\begin{aligned} \mathrm{cov}(\mathbf{w}_k) &= E\left\{\int_{t_{k-1}}^{t_k}\int_{t_{l-1}}^{t_l} e^{\mathbf{A}(t_k-\theta_1)}\mathbf{w}(\theta_1)\mathbf{w}^T(\theta_2)e^{\mathbf{A}(t_l-\theta_2)^T}\,d\theta_1 d\theta_2\right\} \\ &\approx E\left\{\int_{t_k-\tau-\tilde{\tau}_k}^{t_k}\int_{t_l-\tau-\tilde{\tau}_l}^{t_l}[I+\mathbf{A}(t_k-\theta_1)]\mathbf{w}(\theta_1)\mathbf{w}^T(\theta_2)[I+\mathbf{A}^T(t_l-\theta_2)]\,d\theta_1 d\theta_2\right\} \end{aligned} \quad (22)$$

This can be done if we use the law of total expectation, [33], and transform (22) to

$$\begin{aligned} \mathrm{cov}(\mathbf{w}_k) &= \tau\mathcal{S}_w + \frac{\tau^2+\sigma_\tau^2}{2}(\mathcal{S}_w\mathbf{A}^T + \mathbf{A}\mathcal{S}_w) \\ &\quad + \frac{\tau}{3}(\tau^2 + 3\sigma_\tau^2)\mathbf{A}\mathcal{S}_w\mathbf{A}^T\,, \end{aligned} \quad (23)$$

where we took into account that $E\{\tilde{\tau}_k\} = 0$ and $E\{\tilde{\tau}_k^2\} = \sigma_\tau^2$. Finally, neglecting the second order terms gives

$$\mathrm{cov}(\mathbf{w}_k) = \mathbf{Q}\delta_{k-l} = \tau\mathcal{S}_w\,, \quad (24)$$

in which $\mathcal{S}_w$ is the double-sided PSD matrix of continuous-time process noise $\mathbf{w}(t)$. Hence small timing jitter does not affect the system noise covariance very much in the first order approximation associated with Kalman filtering.

3.3 Covariance of ξ_k

Recall that random components $\tilde{\tau}_k$ and $\mathbf{w}_k$ are independent and transform the covariance of ξ_k as follows:

$$\begin{aligned} \mathrm{cov}(\xi_k) &= E\{(\tilde{\tau}_k\mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{w}_k)(\tilde{\tau}_l\mathbf{A}\hat{\mathbf{x}}_{l-1} + \mathbf{w}_l)^T\} \\ &= \left[\frac{\sigma_\tau^2}{\tau^2}(\mathbf{F}-\mathbf{I})\hat{\mathbf{x}}_{k-1}\hat{\mathbf{x}}_{l-1}^T(\mathbf{F}-\mathbf{I})^T + \mathbf{Q}\right]\delta_{k-l} \\ &= \mathbf{Q}_{\xi k}\delta_{k-l} \end{aligned} \quad (25)$$

and we see that the second order term $\frac{\sigma_\tau^2}{\tau^2}$ cannot be neglected, because the product $(\mathbf{F}-\mathbf{I})\hat{\mathbf{x}}_{k-1}\hat{\mathbf{x}}_{k-1}^T(\mathbf{F}-\mathbf{I})^T$ can be large. We will show later that namely this component in (25) yields the effect of time jitter, which is proportional to the process rate, as explained by (1) and Fig. 1.

Example 1 *Given the discrete-time linear time-invariant (LTI) system with the system matrix* $\mathbf{F} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$. *Omitting the time index k, the covariance* (25) *can be transformed to*

$$\begin{aligned} \mathbf{Q}_\xi &= \frac{\sigma_\tau^2}{\tau^2}(\mathbf{F}-\mathbf{I})\hat{\mathbf{x}}\hat{\mathbf{x}}^T(\mathbf{F}-\mathbf{I})^T + \mathbf{Q} \\ &= \frac{\sigma_\tau^2}{\tau^2}\begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}[\hat{x}_1 \quad \hat{x}_2]\begin{bmatrix} 0 & 0 \\ \tau & 0 \end{bmatrix} + \mathbf{Q} \\ &= \begin{bmatrix} \sigma_\tau^2\hat{x}_2^2 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{Q} \end{aligned} \quad (26)$$

and we see that the component $\sigma_\tau^2\hat{x}_2^2$ is proportional to the jitter variance and to the squared estimate of the second state (velocity). If $\hat{x}_2 = 0$ and hence the value is constant, then it follows that timing jitter does not affect the estimate.

Example 2 *Given a constant value, which is represented by the one-state equation with $\mathbf{F} = 1$ and $\mathbf{Q} = \sigma_w^2$. The noise covariance* (25) *becomes*

$$Q_\xi = \frac{\sigma_\tau^2}{\tau^2}(1-1)\hat{x}\hat{x}^T(1-1)^T + \sigma_w^2 = \sigma_w^2\,,$$

which confirms that timing jitter does not affect the system noise covariance when the process is constant.

Example 3 *Given the discrete-time conservative harmonic LTI system with the system matrix* $\mathbf{F} =$

$\begin{bmatrix} \cos\omega_0\tau & \frac{1}{\omega_0}\sin\omega_0\tau \\ -\omega_0\sin\omega_0\tau & \cos\omega_0\tau \end{bmatrix}$, *where* ω_0 *is the angular fundamental frequency. By simple routine transformations, the noise covariance* (25) *can be shown to be*

$$\mathbf{Q}_\xi = \sigma_\tau^2 \begin{bmatrix} \hat{x}_2^2 & -\hat{x}_1\hat{x}_2\omega_0^2 \\ -\hat{x}_1\hat{x}_2\omega_0^2 & \hat{x}_1^2\omega_0^4 \end{bmatrix} + \mathbf{Q}\,.$$

In the limiting case of infinite period of repetition, we set $\omega_0^2 \approx 0$ *and arrive at the covariance* (26) *of the two-state polynomial model.*

3.4 Covariances of v_k

Now look at the discrete-time measurement noise $\mathbf{v}_k$, which is defined by (15), and first transform it as follows:

$$\begin{aligned}\mathbf{v}_k &= \frac{1}{\tau+\tilde{\tau}_k}\int_{t_k-\tau-\tilde{\tau}_k}^{t_k}\mathbf{v}(t)\,dt \\ &\cong \frac{1}{\tau}\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\int_{t_k-\tau}^{t_k}\mathbf{v}(t)\,dt \\ &\quad+\frac{1}{\tau}\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\int_{t_k-\tau-\tilde{\tau}_k}^{t_k-\tau}\mathbf{v}(t)\,dt \end{aligned} \tag{27}$$

The covariance $\mathrm{cov}(\mathbf{v}_k) = E\{\mathbf{v}_k\mathbf{v}_l^T\}$ can be transformed to (28) (at the top of the next page), and then averaging and simple transformations give

$$\begin{aligned}\mathrm{cov}(\mathbf{v}_k) = \frac{1}{\tau}\mathcal{S}_v E\left\{\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\right\} + \frac{2}{\tau^2}\mathcal{S}_v E\left\{\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\tilde{\tau}_k\right\} \\ + \frac{1}{\tau^2}\mathcal{S}_v E\left\{\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\left(1-\frac{\tilde{\tau}_k}{\tau}\right)\tilde{\tau}_k\right\}\end{aligned} \tag{28}$$

$$\begin{aligned}\mathrm{cov}(\mathbf{v}_k) &= \frac{1}{\tau}\mathcal{S}_v\left(1-\frac{5\sigma_\tau^2}{\tau^2}\right) \\ &= \mathbf{R}\left(1-\frac{5\sigma_\tau^2}{\tau^2}\right)\,.\end{aligned} \tag{29}$$

By neglecting the second order term, we finally obtain

$$\mathrm{cov}(v_k) = \mathbf{R}\delta_{k-l} = \frac{1}{\tau}\mathcal{S}_v\,, \tag{30}$$

where $\mathcal{S}_v$ is the double-sided PSD matrix of continuous-time noise $\mathbf{v}(t)$. So the covariance of the measurement noise is not affected by time jitter in the first order approximation associated with Kalman filtering. Moreover, no jitter-induced addition is required to the observation equation (21).

3.5 Cross-Covariance of ξ_k and w_k

Finally, look at the cross covariance $\mathrm{cov}(\xi_k, \mathbf{v}_k) = E\{\xi_k\mathbf{v}_l\}$. Because these vectors contain the jitter component $\tilde{\tau}_k$ and $\mathbf{w}_k$ is independent of any of the remaining terms, we neglect $\mathbf{w}_k$, use the law of total expectation, and make more transformations as conclude that $\mathrm{cov}(\xi_k, \mathbf{v}_k) = 0$, which means that there is no correlation between ξ_k and $\mathbf{v}_k$.

4 Jitter Kalman Filtering Algorithm

The Kalman filtering algorithm for random jitter in sampling time can now be summarized as follows. Given the LTI continuous-time state space model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{w}(t) && (31)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)+\mathbf{v}(t) && (32)\end{aligned}$$

where the continuous-time noise vectors are white Gaussian, $\mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, \mathcal{C}_w(\theta))$ and $\mathbf{v}(t) \sim \mathcal{N}(\mathbf{0}, \mathcal{C}_v(\theta))$, with the covariances $\mathcal{C}_w(\theta) = \mathcal{S}_w\delta(\theta)$ and $\mathcal{C}_v(\theta) = \mathcal{S}_v\delta(\theta)$, $\mathcal{S}_w$ and $\mathcal{S}_v$ are known double-sided PSD matrices, and the property $E\{\mathbf{w}(t)\mathbf{v}^T(t+\theta)\} = \mathbf{0}$ holds for all θ. In discrete time with the average sampling time τ, the model undergoes random time jitter $\tilde{\tau}_k \sim \mathcal{N}(0, \sigma_\tau^2)$ and becomes:

$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}\mathbf{x}_{k-1}+\xi_k && (33)\\ \mathbf{y}_k &= \mathbf{H}\mathbf{x}_k+\mathbf{v}_k && (34)\end{aligned}$$

where $\xi_k = \mathbf{A}\hat{\mathbf{x}}_{k-1}\tilde{\tau}_k + \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_\xi)$, $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, $E\{\mathbf{w}_k\mathbf{w}_l^T\} = \mathbf{Q}\delta_{k-l}$, $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, $E\{\mathbf{v}_k\mathbf{v}_l^T\} = \mathbf{R}\delta_{k-l}$, and $E\{\mathbf{w}_k\mathbf{v}_l^T\} = \mathbf{0}$ holds for all k and l. The second order statistics of random values are defined as $\mathbf{R} = \frac{1}{\tau}\mathcal{S}_v$, $\mathbf{Q} = \tau\mathcal{S}_w$, $\sigma_\tau^2 = \int_0^\infty \mathcal{S}_\tau(2\pi\mathrm{f})\,d\mathrm{f}$, and

$$\mathbf{Q}_{\xi k} = \frac{\sigma_\tau^2}{\tau^2}(\mathbf{F}-\mathbf{I})\hat{\mathbf{x}}_{k-1}\hat{\mathbf{x}}_{k-1}^T(\mathbf{F}-\mathbf{I})^T + \mathbf{Q}$$

where $\mathcal{S}_\tau(2\pi\mathrm{f})$ is the one-sided PSD of time jitter $\tilde{\tau}_k$. Given the initial $\hat{\mathbf{x}}_0$ and $\mathbf{P}_0$, the JKF algorithm is represented with the pseudo code as Algorithm 1. We see that random timing jitter results in the increase in the system noise covariance, which is proportional to the process velocity. In the next sections we will test the JKF effectiveness numerically and using experimental data.

5 Numerical Simulations

Numerical simulation is now required to learn efficiency of the JKF algorithm. We do it below using polynomial and quasi harmonic state space models with random timing jitter.

5.1 Tracking Polynomial Model

Consider a moving object, whose dynamics is described in discrete time by the model

$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}\mathbf{x}_{k-1}+\mathbf{B}w_k && (35)\\ y_k &= \mathbf{H}\mathbf{x}_k+v_k && (36)\end{aligned}$$

Algorithm 1: Jitter Kalman filtering algorithm

Data: $\mathbf{y}_k$, $\hat{\mathbf{x}}_0$, $\mathbf{P}_0$, $\mathbf{Q}_k$, $\mathbf{R}_k$, σ_τ^2
Result: $\hat{x}_k$, P_k
begin
  for $k = 1, 2, \cdots$ **do**
    $\hat{\mathbf{x}}_k^- = \mathbf{F}\hat{\mathbf{x}}_{k-1}$;
    $\mathbf{Q}_{\xi k} = \frac{\sigma_\tau^2}{\tau^2}(\mathbf{F}-\mathbf{I})\hat{\mathbf{x}}_{k-1}\hat{\mathbf{x}}_{k-1}^T(\mathbf{F}-\mathbf{I})^T + \mathbf{Q}$;
    $\mathbf{P}_k^- = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^T + \mathbf{Q}_{\xi k}$;
    $\mathbf{S}_k = \mathbf{H}\mathbf{P}_k^-\mathbf{H}^T + \mathbf{R}$;
    $\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}^T\mathbf{S}_k^{-1}$;
    $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}\hat{\mathbf{x}}_k^-)$;
    $\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k^-$;
  end for
end

where $\mathbf{F} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \tau \\ 1 \end{bmatrix}$, and $\mathbf{H} = [1 \quad 0]$. The noise components are $w_k \sim \mathcal{N}(0, \sigma_w^2)$ and $v_k \sim \mathcal{N}(0, \sigma_v^2)$ with the standard deviations $\sigma_w = 5\,\mathrm{m/s}$ and $\sigma_v = 15\,\mathrm{m}$ and the covariances are given by $\mathbf{Q} = \mathbf{B}\mathbf{B}^T\sigma_w^2$ and $\mathbf{R} = \sigma_v^2$. Sampling is provided with $\tau = 0.5\,\mathrm{s}$ and the random timing jitter $\tilde{\tau}_k \sim \mathcal{N}(0, \sigma_\tau^2)$ has the standard deviation of $\sigma_\tau = 0.3\tau$.

The standard Kalman filter and the JKF Algorithm 1 are applied for known initial values. The process dynamics generated over 10000 points is shown in Fig. 2a and the estimation errors are sketched in Fig. 2b. We see that the process rate highly affects the Kalman estimate. When the process is relatively slow, both filters produce consistent estimates. But when the process changes faster, the Kalman filter gives errors more than twice as large. The dependence of the estimation RMSE on the jitter standard deviation σ_τ, measured using Monte Carlo simulation, is shown in Fig. 3. Hence we conclude that in this process timing jitter can be neglected when $\sigma_\tau < 0.05\,\mathrm{s}$, and that the RMSE in the Kalman filter becomes about 3 times larger when $\sigma_\tau = 0.3\,\mathrm{s}$. Note that larger values of σ_τ turn both filters to instability.

5.2 Quasi-Harmonic Process

When a noisy process varies quasi-periodically, the harmonic model is better suited and we will look into it below, assuming timing jitter. We represent the quasi-harmonic process by equations (35)

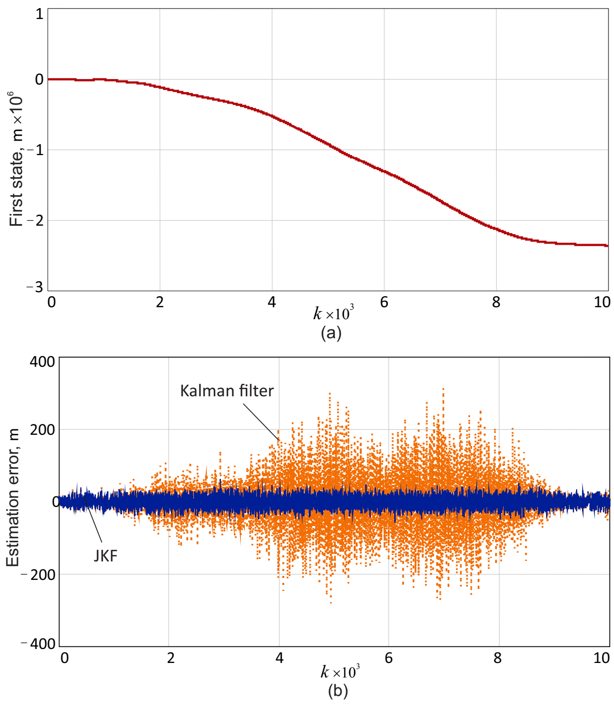

Figure 2: Generated object trajectory sampled with $\tau = 0.5\,\mathrm{s}$ and $\sigma_\tau = 0.3\tau$ over 1000 discrete time points: (a) observed first state and (b) estimation errors produced by the Kalman filter and JKF.

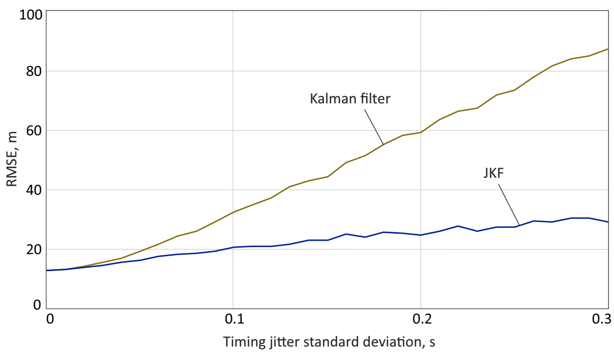

Figure 3: The effect of the standard deviation σ_τ of the random time jitter in the process shown in Fig. 2a on the RMSEs of the Kalman filter and JKF.

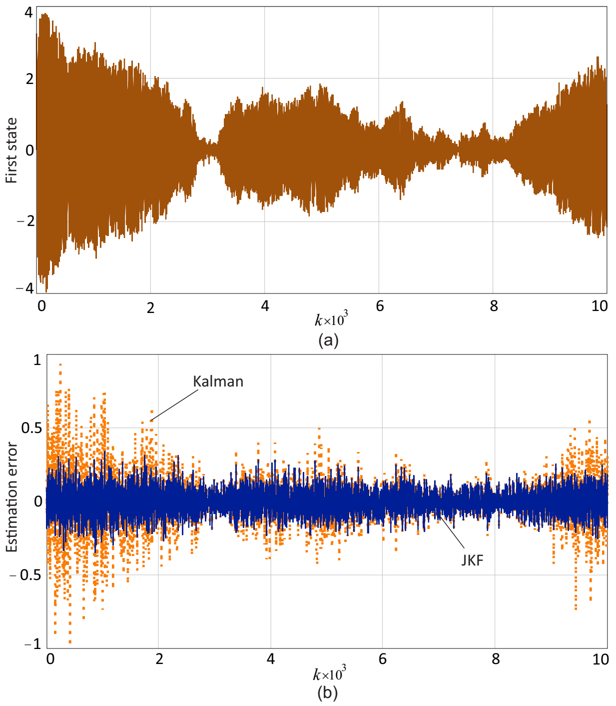

Figure 4: The quasi harmonic process generated over 200 discrete points with time jitter, its estimates produced by the Kalman filter and JKF, and estimation errors: (a) process and estimates (b) estimation errors.

and (36) with matrices $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $H = [1 \quad 0]$, and

$$\mathbf{F}_k = \begin{bmatrix} \cos(\omega_0 \tau_k) & \frac{1}{\omega_0}\sin(\omega_0 \tau_k) \\ -\omega_0 \sin(\omega_0 \tau_k) & \cos(\omega_0 \tau_k) \end{bmatrix}, \qquad (37)$$

where $\omega_0 = 2\pi/T$ is the angular fundamental frequency, $T = 20\,$s is the period of repetition, $\tau_k = \tau + \tilde{\tau}_k$, and $\tau = T/24$. The mutually independent random components are $w_k \sim \mathcal{N}(0, \sigma_w^2)$, $v_k \sim \mathcal{N}(0, \sigma_v^2)$, and $\tilde{\tau}_k \sim \mathcal{N}(0, \sigma_\tau^2)$ with the standard deviations $\sigma_w = 0.01$, $\sigma_v = 0.1$, and $\sigma_\tau = 0.3\tau$. The noise covariances are defined as $\mathbf{Q} = \mathbf{B}\mathbf{B}^T \sigma_w^2$ and $\mathbf{R} = \sigma_v^2$.

The typical quasi-harmonic process generated over 10000 discrete points with timing jitter, the estimates produced by the Kalman filter and JKF, and the estimation errors are sketched in Fig. 4. and the effect of jitter standard deviation on the RMSEs in filtering estimates is exhibited in Fig. 5. Like in the tracking model case (Fig. 3), here errors in the Kalman filter also grow at a higher rate with an increase in σ_τ. A special feature is that the both RMSE functions of σ_τ are oscillating and a much smaller step in σ_τ/τ is required to restore a complete picture.

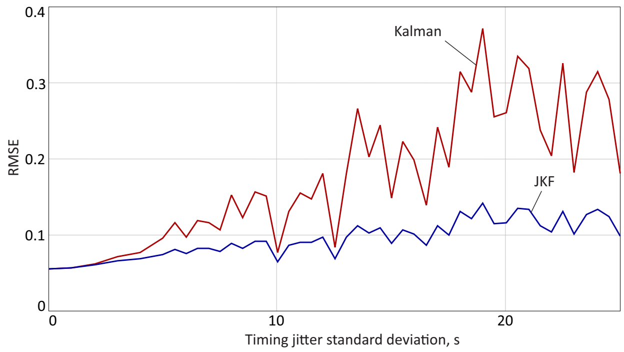

Figure 5: The effect of the standard deviation σ_τ of the random time jitter in the process shown in Fig. 4a on the RMSEs of the Kalman filter and JKF.

The following inferences can be drawn from this simulation: 1) iming jitter significantly effects the Kalman filter only when the process noticeably changes over time and 2) JKF is highly robust to time jitter, as far as being an estimator low sensitive to time errors. It can also be emphasized that JKF is practically insensitive to timing jitter, because the corresponding errors are almost stationary. In the meantime, the Kalman filter exhibits the easily observable nonstationarity and extra errors amplified by the process dynamics.

6 Conclusions

The jitter Kalman filter developed in this paper has demonstrated the ability to cope very well with timing jitter caused by various factors in different practical situations. An important conclusion that has been made is that the extra error ε_τ produced by timing jitter in the sampling time at the estimator output is proportional to the square of the jitter fractional standard deviation σ_τ/τ and the derivative of the process x:

$$\varepsilon_\tau \sim \frac{\sigma_\tau^2}{\tau^2}\frac{dx}{dt},$$

which means that the process dinamics amplifies the effect of fractional jitter at the estimator output. To deal with this error, the JKF has acquired an additional term in the system noise covariance, proportional to ε_τ. It also follows that timing jitter can cause big troubles in estimation of fast sampled processes; that is, if the process rate grows without limits then the Kalman filter error caused by timing jitter will also grow without limits, while for slow processes it can be neglected. We have verified the effectiveness of JKF through simulations and experimental examples. Thus, the main question that should arise in practice is not how large timing jitter is, but how much it is amplified by the process dynamics at the estimator output.

References:

[1] R. Kalman, “Analysis and Synthesis of Linear Dynamical Systems Operating on Randomly Sampled Data,” Ph.D. dissertation, Columbia University, New York, NY, 1957.

[2] P. V. Balakrishnan, “On the problem of time jitter in sampling,” *IRE Trans. Inform. Theory*, vol. 8, no. 3, pp. 226–236, 1962.

[3] Y. S. Shmaliy and L. Arceo-Miguel, “Efficient predictive estimator for holdover in GPS-based clock synchronization,” *IEEE Trans. Ultrason. Ferroel. Freq. Contr.*, vol. 55, no. 10, pp. 2131–2139, 2008.

[4] L. Arceo-Miguel, Y. S. Shmaliy, and O. Ibarra-Manzano, “Optimal synchronization of local clocks by GPS 1PPS signals using predictive FIR filters,” *IEEE Trans. Instrum. Meas.*, vol. 58, no. 6, pp. 1833–1840, 2009.

[5] Y. S. Shmaliy, “General unbiased FIR filter with applications to GPS-based steering of oscillator frequency,” *IEEE Trans. Contr. Syst. Techn.*, vol. 25, no. 3, pp. 1141–1148, 2017.

[6] K. J. Uribe-Murcia and Y. S. Shmaliy, “Robust UFIR observer for WSNs with multistep random delays and multiple packet dropouts,” *IEEE Trans. Automat. Contr.*, vol. 68, no. 10, pp. 6338–6344, 2023.

[7] K. J. Uribe-Murcia, O. G. Ibarra-Manzano, J. A. Andrade-Lucio, and Y. S. Shmaliy, “Moving vehicle tracking under measurement uncertainties, multi-step random delays, and packet dropouts,” *IEEE Access*, vol. 11, pp. 52 381–52 391, 2023.

[8] A. Berkovitz and I. Rusnak, “FFT processing of randomly sampled harmonic signals,” *IEEE Trans. Signal Process.*, vol. 40, no. 11, pp. 2816–2819, Nov. 1992.

[9] I. Sharfer and H. Messer, “The bispectrum of sampled data: Part 1—Detection of the sampling jitter,” *IEEE Trans. Signal Process.*, vol. 41, no. 1, pp. 296–312, Jan. 1993.

[10] ——, “The bispectrum of sampled data: Part 2—Monte Carlo simulations of detection and estimation of the sampling jitter,” *IEEE Trans. Signal Process.*, vol. 42, no. 10, pp. 2706–2714, Oct. 1994.

[11] M. R. Morelande and D. R. Iskander, “Formulation and comparison of two detectors of independent timing jitter in a complex harmonic,” *IEEE Trans. Signal Process.*, vol. 51, no. 12, pp. 3043–3052, Dec. 2003.

[12] E. Säckinger, *Analysis and Design of Transimpedance Amplifiers for Optical Receivers*. New York: Wiley, 2017.

[13] “Jitter Specifications for Timing Signals: Renesas Electr. Corp. Application Note AN-840,” Renesas Electr. Corp., Tech. Rep., 2019.

[14] A. Blais, C. Macabiau, and O. Julien, “Effect of sampling jitter on signal tracking in a direct sampling dual band gnss receiver for civil aviation,” in *Toulouse Space Show*. Toulouse, France, 2012, pp. 1–11.

[15] Y. Xu, J. Cao, Y. S. Shmaliy, and Y. Zhuang, “Distributed Kalman filter for UWB/INS integrated pedestrian localization under colored measurement noise,” *Satellite Navigation*, vol. 2, p. 22, 2021.

[16] Y. Xu, Y. S. Shmaliy, W. Ma, X. Jiang, T. Shen, S. Bi, and H. Guo, “Improving tightly LiDAR-Compass-integrated mobile robot localization with uncertain sampling period utilizing EFIR filter,” *Mobile Net. Appl.*, vol. 26, pp. 440–448, 2021.

[17] C. Azeredo-Leme, “Clock jitter effects on sampling: A tutorial,” *IEEE Cir. Syst. Mag.*, vol. 11, no. 3, pp. 26–37, 2011.

[18] Y. S. Shmaliy and O. Ibarra-Manzano, “Time-variant linear optimal finite impulse response estimator for discrete state-space models,” *Int. J. Adapt. Contr. Signal Process.*, vol. 26, no. 2, pp. 95–104, 2012.

[19] ——, “Optimal finite impulse response estimation of linear models in receiver channels with imbedded digital signal processing units,” *IET Signal Process.*, vol. 6, no. 4, pp. 281–287, 2012.

[20] W. M. Brown, “Sampling with random jitter,” *J. Soc. Indust. Appl. Math.*, vol. 11, no. 2, pp. 460–473, 1963.

[21] T. M. Souders, D. R. Flach, C. Hagwood, and Y. G. L., “The effects of timing jitter in sampling systems,” *IEEE Trans. Inst. Meas.*, vol. 39, no. 1, pp. 80–85, 1990.

[22] F. Eng and F. Gustafsson, "Identification with stochastic sampling time jitter," *Automatica*, vol. 44, no. 3, pp. 637–646, Mar. 2008.

[23] V. S. El Hakim and M. J. G. Bekooij, "Sampling jitter mitigation in latency-critical state-estimation applications using particle filters," in *2017 SICE Int. Symp. Contr. Syst. (ISCS)*. Okayama University, Okayama, Japan, 2017, pp. 54–61.

[24] J. Skaf and S. Boyd, "Analysis and synthesis of state-feedback controllers with timing jitter," *IEEE Trans. Autom. Contr.*, vol. 54, no. 3, pp. 652–657, Mar. 2009.

[25] W. Yu, D. I. Wilson, and B. R. Young, "Control performance assessment in the presence of sampling jitter," *Chem. Eng. Res. Des.*, vol. 90, no. 1, pp. 129–137, Jan. 2012.

[26] S. Ahmed and T. Chen, "Minimizing the effect of sampling jitters in wireless sensor networks," *IEEE Signal Proess. Lett.*, vol. 18, no. 4, pp. 219–222, Apr. 2011.

[27] E. G. Pale-Ramon, J. A. Ortega-Contreras, K. J. Uribe-Murcia, and Y. S. Shmaliy, "Effect of sampling time jitter on robust H_2 filtering estimates," *Signal Process.*, vol. 198, p. 108597, Sep. 2022.

[28] Y. S. Shmaliy, *GPS-based Optimal FIR Filtering of Clock Models.* New York: Nova Science Publ., 2009.

[29] ——, "Conversion of 1/f fluctuations in crystal resonator within an inter resonance gap," *IEEE Trans. Ultrason. Ferroel. Freq. Contr.*, vol. 46, no. 1, pp. 61–71, 1999.

[30] ——, "The noise conversion method for oscillatory systems," *IEEE Trans. Ultrason. Ferroel. Freq. Contr.*, vol. 51, no. 8, pp. 948–956, 2004.

[31] M. Miskowicz, "Send-on-delta concept: An event-based data reporting," *Sensors*, vol. 6, pp. 49–63, 2006.

[32] Y. S. Shmaliy and S. Zhao, *Optimal and Robust State Estimation: Finite Impulse Response (FIR) and Kalman Approaches.* New York: Wiley & Sons, 2022.

[33] M. D. Springer, *The Algebra of Random Variables.* New York: John Wiley & Sons, 1979.

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