Computational Engineering: Mathematical Models in Heat Flow Dynamics Study

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Abstract: - According to the latest educational innovation trends in engineering, new teaching paradigms emerge to achieve higher development in intellectual capacities, acquisition of skills, substitution of outdated techniques for more efficient and fast means, and a better integration of knowledge in the teaching and learning processes. In line with these paradigms, we have set up new pedagogic methodologies together with computer resources to work with the development of functions with a complex variable and Laplace transform from a multidisciplinary view in the subject Advanced Calculus of the Mechanical Engineering Programme. The didactic proposal presented is about systems related with heat transfer in a fluid where the concepts used in the models are approached analytically and graphically making the curricular content meaningful and facilitating the interpretation and conceptualization of the theory.

Key-Words: - Modelling, multidisciplinary approach, simulation, complex variable, Laplace Transform, heat flow.

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1 Introduction

The fast technology development and the use of digital tools condition the academic work of faculty in the university fields. In this context, professors have to face continuous challenges, find new teaching approached and agree on curricular contents. The design of learning strategies should comply with the opportunities which available and new computational systems offer.

These new developments in technology have potential to be used in the mathematic training of engineering students. Professors have to distinguish which students' basic needs are to approach the analysis and solution of simple models that can give the opportunity to develop suitable strategies to connect and integrate computational mathematics to applied basic technologies in engineering.

Real systems modelling determines that the mathematical objects dealt with in the classroom must be adapted to students' levels and learning processes. This is why the tasks designed should be adapted to tangible learning situations [1].

Thus, it must be considered that the topics developed in Advanced Mathematics will be further used in applied technologies and in the professional career. These topics often present difficulties on the complexities of mathematical structures meant to be taught. Activities should then rely on didactic resources from symbolic, numeric and graphic Calculus tools.

This work presents two experiences carried out in Advanced Calculus with modelling of the curricular topics in connection to heat Flow systems dealing with functions of complex variables and Laplace Transform. The teaching approach gives relevance to both the match contents meant to be taught and the competencies which students should develop to achieve meaningful learning functional applications of specific software.

2 Methodology Criteria

The contents of the engineering programmes usually have a high levels of abstraction and generalizations, particularly in Advanced Math courses.

Teaching Math through modelling in simple dynamic systems facilitates student's engagement in curriculum topics. This situation fosters learning autonomy and the understanding of concepts in a meaningful way, two important competences for future professional development.

In fact, training suitable professionals requires the articulation of an academic programme to achieve the professional competencies, right contents and activities to develop such competencies together with political and curricular management [2].

Designing curricular activities to foster the construction of concepts aims at a student who can be protagonist of the teaching and learning process relying on a multidisciplinary approach to curricular contents.

The selection of real mathematical models raises students' interest. When the work proposal results engaging and convenient, students can analyse and visualize complex systems and thus assimilate the target topic.

These objects can be focused towards collaborative learning and can be identified as an exercise in determining the association scheme. This tool is positive because it reinforces communication, reflection, teamwork and helps strengthen the educational role that is expected to give today [3].

Collective interdependence was observed in the construction of learning supported by collaborative instruments in which there was no evidence of positions of competence but of group interaction. The interests were shared in the vocational training against their daily practice and work experience.

From the collaborative learning and the area of the computer science was propitiated the construction of significant knowledge in the subject of the mining of texts [3].

The aim is to create a reflexive and critical environment where the students can make decisions and delimit mistakes made during the learning process. This implies students' need to know the object of study and to create the internal conditions for the assimilation of new knowledge in an active and independent way.

In this type of experiences, the professor is a guide and works as a facilitator of the learning process towards the deep understanding of the target subject, making use of questions which can activate a continuing mechanism for further enhancement and research of different lines of the proposed problem.

The task is carried out in the Informatic and Multidisciplinary Lab of the Department of basic Sciences at our College equipped with 25 computers connected to Internet and to a specific software. This modelling analysis interacts with Mathematics and GeoGebra.

In order to carry out the didactic proposal, an activity is designed with two topics from the subject Advanced Calculus: Functions of complex variable and Laplace Transform. The activity includes two cases of heat transfer: the first one presents the balance of heat flow in two concentric pipelines, and the second one the analysis of thermometer when a system is prompted by a sinusoidal perturbation

The methodological design has the following stages: generation of a situation to train on symbolic, numeric and graphic calculus; integration of disciplines with a multidisciplinary perspective; complementation of contents for analytic and simulated methods in the reproduction and analysis of the behaviours of two real simple systems.

3 Objectives

The teaching of Advanced Mathematics in engineering should find the balance between the formulation of mathematical models and the skills which students have to use to solve the challenges which they will have to face during applied technologies and their future professional careers.

The general aim of this work is to show where theoretical topics in the subjects Advanced Calculus are used with special emphasis on the simulation and visualization of target systems [4].

The professional competencies in the higher education framework are achieved by mean of a process which can allow the training of competent professionals, not only of their knowledge and skills to carry out their work as engineers, but also of personal and social development.

It is important to educate engineers who can work in a suitable way in the market labour place. In this line, while academic competencies rely on theory and reflection on the role of engineers, labour competences focus on a more pragmatic perspective which demands an efficient performance. Thus, an integrated teaching and learning process is proposed so as to try to give a different experience from dialogue, convergence criteria and active students' participation [5].

The right questions are sequenced in such a way that can guide student's thinking by means of argumentation which accordingly can lead to conclusions, convergent thinking in a dynamic and collaborative experience.

Therefore, the training activities carried out with systems engineering students have revealed new problems that make it necessary to advance the conceptualization of collaborative learning through educational research. One of the first elements to take into account as the basis of each learning, and especially the collaborative learning, is the communicative interaction [3].

The aim is also to foster scientific experimentation to approach problems that require mathematical modelling and application of different methods to solve them and the identification of different tools developed in each method.

All this implies selecting algorithms to solve problems, a skill necessary for a professional who will have to interpret and propose solutions when facing alternatives together with decision making.

4 Learning environment

The following two models show two learning situations set up in the Advanced Calculus class. The problems involve two topics: Functions of complex variable and Laplace Transforms are based on the analysis of the heat flow which undergo the fluid.

4.1 Heat transfer in pipelines

Students have to solve the heat transfer applying the theory of functions of complex variable por a system in stationary state [6].

It is given a cylindrical pipeline with an interior cylindrical cavity which is off-centric through which steam flows at $100°C$. The exterior temperature of the pipeline is $0°C$. The radio of the interior circumference is $\frac{3}{10}$ from the radio of the exterior circumference. Fig.1 shows a chart of the system.

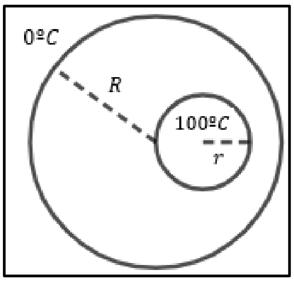

Fig.1 Schematic diagram of the pipeline.

The law of the harmonic function which rules the temperature for fluids in pipelines is

$$T(u, v) = A\, ln(u^2 + v^2) + B \qquad (1)$$

with A and $B \in R$.

Knowing that the law of bilinear mapping which is applied to observe the positions on the plan of the pipeline display, and to transform the problems to one which has axial symmetry is:

$$f(z) = \frac{z-3}{3z-1} \qquad (2)$$

According to given data, student's work is guided by means of the following questions:

a) Verifying if the equation (1) is harmonic in function of *u* and *v*.
b) Making the graphics which transforms the enclosures in Fig. 1 applying the mapping of the equation (2) so as to show the complex plane from the domain (plane *z*) and from the image (plane *w*).
c) Finding the equations of the graphics of the enclosures determined in the item b, which transform the circular regions of the domain into other regions with axial symmetry.
d) What characteristics do the equations of item c with its graphics have?
e) The value of the constants *A* and *B* for the equation (1) which fits this case.
f) The law which rules the distribution of the temperature in function of *x-y*.
g) The graphic of the function *T* is an orthogonal cartesian system *x-y-z*.
h) Showing if *T* is harmonic in function of *x* and *y*.
i) If the potential in a heat transference process follows the law *f*, analysing the graphics of the isotherms and the flow lines.

The students carry out their work in groups, and once the questions are answered, they agree on the writing a document with the reached conclusions. The answers which determine the following conclusions are cited below:

One function of the variable T is harmonic if $T(u, v)$ verifies

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0 \qquad (3)$$

The function *T* has as second partial derivatives

$$\frac{\partial^2 T}{\partial u^2} = -\frac{4Au^2}{(u^2+v^2)^2} + \frac{2A}{u^2+v^2} \qquad (4)$$

$$\frac{\partial^2 T}{\partial v^2} = \frac{4Au^2}{(u^2+v^2)^2} + \frac{2A}{u^2+v^2} \qquad (5)$$

The equations (4) and (5) verify the condition expressed in the equation (3).

Fig.2 shows the graphic of the systems in Fig.1 in a system of orthogonal axis in a complex plane of the domain of a complex variable function.

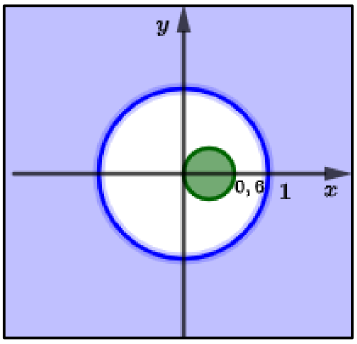

Fig.2 Complex plane of the domain (z).

Fig.3 shows the application of the bilinear mapping of the law f to the domain represented in Fig.2.

As the students know the law of the function expressed by the equation (2), and from previous theoretical knowledge, they consider that it is convenient to work with the inverse function expressed in (2), resulting in:

$$z = \frac{-3 + w}{-1 + 3w} \tag{6}$$

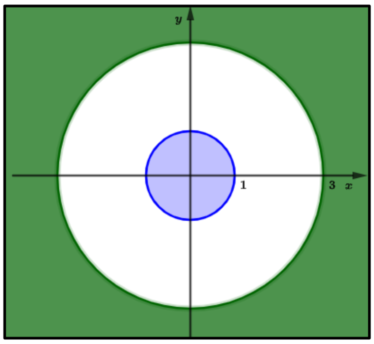

Fig.3 Complex plane of the image (w).

According to equation (6), students determine that the equations of the enclosures of Fig.2 and 3 are the ones which are shown on Table 1 and 2.

The same equations presented in Table 1 are presented in Table 2, but the latter is made in function of the variables z and w.

Table 1. Equations of the relation which exist between the circular enclosures on the complex planes z and w, with function of the variables x-y and u-v.

Complex plane	
Domain (z)	Image (w)
$x^2 + y^2 \geq 1$	$u^2 + v^2 \leq 1$
$(x - 0{,}3)^2 + y^2 \leq 0{,}09$	$u^2 + v^2 \geq 1$

Table 2. Equations of the relation that there is between the circular enclosures on the complex planes z and w, function of the variables z and w.

Complex plane	
Domain (z)	Image (w)
$x^2 + y^2 \geq 1$	$u^2 + v^2 \leq 1$
$(x - 0{,}3)^2 + y^2 \leq 0{,}09$	$u^2 + v^2 \geq 1$

The students conclude that on the complex plane z, the exterior area of the circumference of the unitary radio of the equation $|z| = 1$ is mapped on the complex plane w in the interior area of the circumference of the unitary radio of the equation $|w| = 1$.

While on the complex plane z, the interior area of the circumference of the equation $|z - 0{,}3| = 0{,}3$ is mapped on the complex plane w on the exterior area of the circumference of the equation $|w| = 3$.

The required activity is turned into a problem of axial symmetry on the plane w which consists on finding a harmonic function T so that $T(u, v) = 100$ in $|w| = 1$ and $T\,(u, v) = 0$ in $|w| = 3$.

The harmonic function T with axial symmetry has the general formula of the equation (1).

The system:

$$S)\begin{cases} 100 = A\,ln1 + B \\ 0 = A\,ln9 + 100 \end{cases} \tag{7}$$

(7) is a system of linear equations to true coefficient, a unique solution, so that its solution is:

$$S = \{(-100\,ln9, 100)\}$$

The equation (1) for this case is:

$$T(u, v) = -100\,ln9\,(u^2 + v^2) + 100 \tag{8}$$

From the equation (2), and knowing that the complex variable of the domain $z = x + i\,y$, and the complex variable of the image $w = u + iv$, the module of w is the one indicated in equation (9).

$$|w| = \left|\frac{z - 3}{3z - 1}\right| \tag{9}$$

Squaring both members of the equation (9), and then substituting by $z = x + i\,y$, equation (10) is

$$|w|^2 = \frac{(x - 3)^2 + y^2}{(3x - 1)^2 + 9y^2} \tag{10}$$

Finally, from equation (10) it turns equation (1) which determines the temperature in function of x-y

If now $C(x, y) = (x - 3)^2 + y^2$, and $D(x, y) = (3x - 1)^2 + 9y^2$

$$T(x, y) = 1 - ln[C(x, y)] - ln[D(x, y)] \tag{11}$$

The graphic of the function given by the equation (11) can be observed in Fig.4.

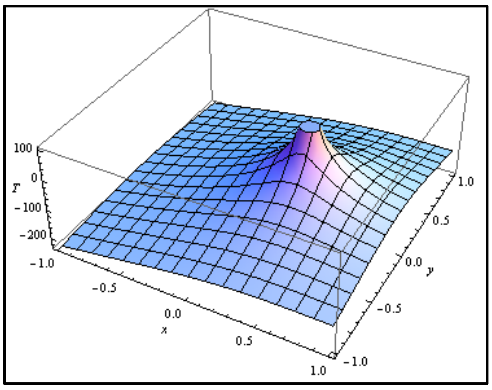

Fig.4 Temperature in the function of x-y.

From Fig.4 students point out that the temperature grows close to the origin of the coordinates, taking significant values, while as the temperature is far from the origin, the value gets lower, with values which decrease smoothly.

A function of a complex variable T is harmonic if $T(x,y)$ verifies

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \qquad (12)$$

If $M(x,y)=4(x-3)^2-2$ and $N(x,y)=36(1-3x)^2-18$, $T(x,y)$ has a second partial derivatives

$$\frac{\partial^2 T}{\partial x^2}=\frac{100}{\ln 9}\left(\frac{M(x,y)}{[C(x,y)]^2}+\frac{N(x,y)}{[D(x,y)]^2}\right) \qquad (13)$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{-100}{\ln 9}\left(\frac{M(x,y)}{[C(x,y)]^2}+\frac{N(x,y)}{[D(x,y)]^2}\right) \qquad (14)$$

Consequently, they verify the equations (13) and (14) and conclude that the function T is harmonic regardless the variable of the domain or the complex image.

The mapping f expressed in the equation (2), presents two types of curves which are themselves orthogonal: the isotherms and the flow curves [7].

The equations of the isotherms for the values k_1(constant) with $k_1 \in Z$ and such that $-10 \leq k_1 \leq 10$ are equation (15)

$$\frac{3-10x+3x^2+3y^2}{(-1+3x)^2+9y^2}=k_1 \qquad (15)$$

The equations of the curves of heat flow for the values k_2(constant) considering $k_2 \in Z$ and such that $-10 \leq k_2 \leq 10$ are equation (16)

$$\frac{8y}{(-1+3x)^2+9y^2}=k_2 \qquad (16)$$

The curves which represent the isotherms and the curves of heat flow are shown in Fig.5.

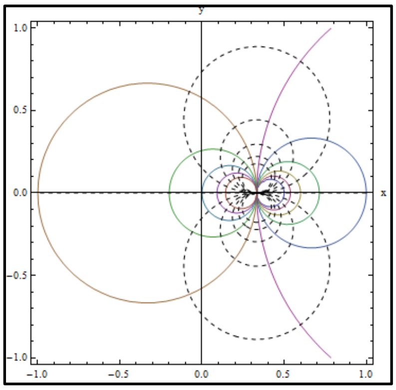

Fig.5 Isotherms and Flow lines for the complex potential of f.

The curves in continuous lines represent the isotherms and the dotted lines represent the curves of heat flow.

4.2 Signal Response with sinusoidal variation from a first order system

Next consider the case of a thermometer submerged in a water bath which varies with a sinusoidal signal between $\pm 20ºC$, with a period of $100s$. And which has a constant of time of $25s$.

A guideline is presented to students in order to look into the characteristics of the system:

a) Which is the law that rules the behaviour of the system when a sinusoidal perturbation is present?
b) What does each term of the solution represent?
c) Which are the characteristics of the system?
d) Which is the highest output answer of the system after applying the effect of the sinusoidal function
e) Which is the function of the output signal of the system?
f) Draw the curve which represents the system output. Identify each component associated with the graphic of the curve.

The students develop the given model from the function of transference for a first order system which has the equation (17)

$$G(s)=\frac{1}{Ps+1} \qquad (17)$$

Where, G: function of transference, P: time constant and s: transformed variable of the time variable t.

When the system is excited with a sinusoidal signal, the prompting power $T_e(t)$ of the input system is:

$$T_e(t)=T_o senwt \qquad (18)$$

Given, w: the angular frequency, and if it's simbolized with H to the period,

$$w=\frac{2\pi}{H} \qquad (19)$$

The Laplace Transform of the equation (18) is θ_e:

$$\theta_e(s)=T_o\frac{w}{w^2+s^2} \qquad (20)$$

The transformed output signal of the system θ_f

$$\theta_f(s)=\theta_e(s)G(s) \qquad (21)$$

$$\theta_f(s)=\frac{1}{Ps+1}\,\frac{T_o w}{w^2+s^2} \qquad (22)$$

In order to find the inverse Laplace Transform of equation (22).

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They apply the convulsion, where the symbol * doesn't represent the product of algebraic expressions but the product of the convulsion [8]

$$\theta_f(s) = \mathcal{L}^{-1}\left[\frac{1}{Ps+1}\right] * \mathcal{L}^{-1}\left[\frac{T_o w}{w^2+s^2}\right] \qquad (23)$$

$$T(t) = \frac{1}{P}e^{-\frac{t}{P}} * T_o senwt \qquad (24)$$

$$T(t) = \int_0^t \frac{T_o}{P}e^{-\frac{(t-z)}{P}} * senwz\, dz \qquad (25)$$

The solution of the integral (25) is the one expressed in equation (26)

$$T(t) = \frac{-T_o\, sen\beta\, e^{-\frac{t}{P}}}{\sqrt{P^2w^2+1}} + \frac{T_o\, sen(wt+\beta)}{\sqrt{P^2w^2+1}} \qquad (26)$$

From where $\beta = -arctg(Pw)$.

In equation (26) in the second member, the first member represents the transitory answer from the system because as t grows, the value of that term goes down until it finally disappears, the second term represents the permanent answer of the system which is also a sinusoidal function of narrower amplitude and moved away angle β as regards original value [9].

The law of the function which represents the input signal, according to the given data to students in the original problem is

$$T_e(t) = 20\, sen\frac{2\pi}{100}t \qquad (27)$$

The output signal of the final temperature informed by students is:

$$T(t) = R\left[\frac{-sen\, k_1}{e^{0.04t}} + sen(k_2 t + k_1)\right] \qquad (28)$$

from where $R = 10{,}74$; $k_1 = -1{,}004$ and $k_2 = 0.0628$

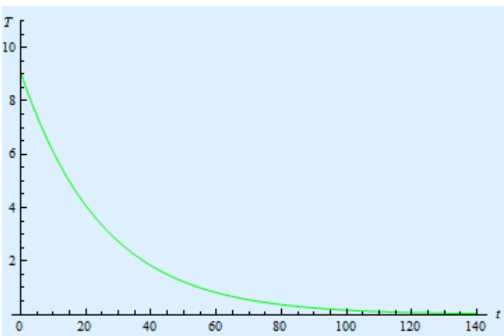

Fig.6 Transitory answer to the system

Students make the graphic of Fig.6 with the transitory answer of the system, probably called this way because its action extinguishes in a short period of time, although its impact is never null, with a negligible time value of about 70s.

The maximum amplitude of the temperature reading is: $T_{máx} = 10{,}74ºC$. Students can observe the value of the maximum temperature does not match with the value of the maximum temperature of the fluid as they are mismatched in $16s$.

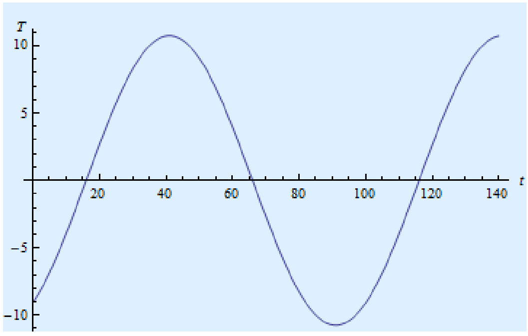

Fig.7 Permanent answer to the system

Fig.7 shows the permanent answer to the system, which results in a sinusoidal function of less amplitude and mismatched by an angle β as regards the original value. It is called permanent because it is the one which stands out in the final values of the temperature.

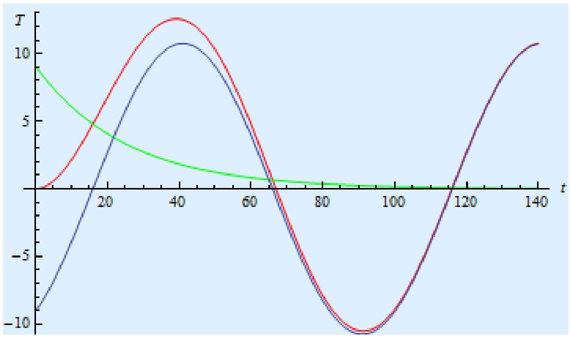

Fig.8 Overall answer to the system

Fig.8 shows the graphic of the temperature in the function of time and in both components (or the graphics of both terms in equation (28). After 60 s., the total answer has the same value on images as the permanent answer.

From the analysis of the equations (26) y (28) students conclude:

- The lower is the time constant of the termometer (P), more T_o and T get closer to the values of maximum amplitude and less is the mismatch in time.
- If the period (H) decreases, the angular frequency (w) grows, and a fluctuation between minimum and maximum values in the graphic can be observed.

- As a consequence, the thermometer has a very low amplitude as an answer (tending to 0) but with the same frequency as the input signal.
- On the other hand, the mismatch grows. It sould be considered that the transitory period is not significant, except for the overall answer in the first cycle, and then that period is no longer considered.

4 Conclusion

The most effect way to face the methodological challenges when teaching Advanced Mathematics in engineering is to design a curricular map that can consider a multidisciplinary perspective in teaching.

The experience enriches as the traditional classroom is turned into a workshop – laboratory where the student can integrate the knowledge of Functions of complex variables and the Laplace Transform with dynamic models into real models.

The learning environment generated fosters motivation when there is interaction between computational mathematics and applied physics.

Applying this type of experiences and adopting it as a teaching match approach has an impact on the students' acquisition of skills. The purpose is to emulate the engineering labour environment.

With the use of virtual environments as a means to facilitate knowledge acquisition and improve education efficiency through the development of an adequate language, collaborative tasks to solve wide and complex problematic situations where research work is central, respecting students' learning time and fostering students' teamwork and active participation.

References:

[1] R. Posada Álvarez, Formación Superior Basada en Competencias, Interdisciplinariedad y Trabajo Autónomo del Estudiante, *Revista Iberoamericana de Educación*, Vol XXXV, No. 1, 2011, pp. 1-33.

[2] G. Bischof, E. Bratschitsch, A. Casey, D. Rubesa, Facilitating Engineering Mathematics Education by Multidisciplinary Projects, *Journal of American Society for Engineering Education*, Vol.I, No.1, 2007, pp. 55-74.

[3] J. Padilla Beltrán, Y. Caviativa Castro, M. Mantilla, Educational Resources with Digital Contents for Pedagogical and Research Formation in Technologies of Knowledge and Learning, *International Journal of Education and Learning Systems*, Vol.IV, No.1, 2019, pp. 36-41.

[4] K. Sumithra, A. Dharani, M. Vijayalakshmi, Transformation in Engineering Education: An Analysis of Challenges and Learning Outcomes, *Proc. of the 14th International Conference on Education and Educational Technology*, Vol.I, No.1, 2015, pp. 33-36.

[5] B. Prepelita-Raileanu, Social Software Technologies and Solutions for Higher Education, *Proceedings of the 8th WSEAS International Conference on Education and educational Technology*, Vol.I, No.1, 2009.

[6] G. James, *Matemáticas Avanzadas para Ingeniería*, Prentice Hall, 2002.

[7] R. Churchill, *Variable compleja y Aplicaciones*, Mc Graw Hill, 2004.

[8] P. O´Neil, *Matemáticas Avanzadas para Ingeniería*, Math Learning, 5ª Ed., 2004.

[9] M. Spiegel, S. Lipschutz, J. Schiller, D. Spellman, *Variable Compleja*, Mc Graw Hill, 2011.