

Approximations on Convex Star-Type Manifolds

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Abstract: - The construction of approximations on radial ring grids on a circle and on a sphere is considered. In this case, to construct approximations, the values of the function at the grid nodes and the basis functions are required. Particular attention is given to the development of an algorithm for the sufficiently fine refinement of surface triangulation and the implementation of Courant-type approximations on a sphere and on a sphere with a cutout. The proposed refinement algorithm overcomes limitations associated with computer system memory. An algorithm for constructing a triangulation on a surface is described. Courant-type approximations are constructed on a sphere with and without a cutout.

Dedicated to Prof. Yu .K. Dem'yanovich on the occasion of his 88th birthday.

Key-Words: - Radial ring grids, Courant-type approximations, approximation on a sphere, local approximation, triangulation, basis functions

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1 Introduction

Numerous papers are devoted to the construction of approximations on convex star-type manifolds.

In paper [1], the concept of convexity and star-shapedness in the Cartesian product of two complete, simply connected smooth Riemannian manifolds without conjugate points is studied in terms of the same concepts in the components of the product.

Much attention is devoted to the construction of approximations on radial ring grids in paper [2].

An effective evaluation of the speed of convergence for multidimensional approximations of the functions defined on the differential manifold is given in [3].

In paper [4], the author obtains a convergence rate estimate of the approximation of functions defined on a differentiable manifold.

In [5], the authors present a way to automate the process of solving PDEs (partial differential equations) over a manifold (e.g., a sphere) using the software package FEniCS.

In [6], the authors discuss the non-uniform Haar wavelets.

Some of the papers related to this investigation are written by our teacher [7]. In his paper [7], the concept of wavelets on manifolds is investigated. In

[8], the spline wavelet decompositions on manifolds are studied. In [9], this research is continued.

The Haar wavelet collocation method is used in [10] for the numerical solution of 1D and 2D cubic nonlinear Schrodinger equations with initial and Dirichlet boundary conditions. The approximation of functions, given on the manifold, is considered in [11].

A new type of non-linear adaptive phase-locked loop (PLL) is proposed in [12], which is based on spline interpolation.

In paper [13], a specialized interpolation technique is presented for visualizing large-scale, adaptive elevation fields in real time by reconstructing a C-continuous surface using a specialized interpolation technique.

Splines and wavelets are often used to solve problems of approximating solutions to differential equations. An efficient collocation method is proposed in [14] for the numerical solution of second- and fourth-order two-point boundary value problems (B.V.P.) based on a uniform Haar wavelet. Approximations with geometric regularity, curvelets, and bandlets are discussed in [15].

Data recovery is an important issue today. To address data recovery issues, an adaptive method for recovering missing data from a phasor measurement unit (PMU) is proposed in [16].

In paper [17], a new approach to studying approximations by rational points to points on smooth submanifolds of R^n is discussed.

In paper [18], a randomized neural network architecture is used to approximate functions on smooth, compact submanifolds of Euclidean space, providing theoretical guarantees in both the asymptotic and non-asymptotic forms.

In this paper, we consider two approaches to approximating functions in star-shaped convex domains: the first one (Section 2.1) is the approximation on Radial-ring grids; in Section 2.2 (and following), particular attention is given to the development of an algorithm for the sufficiently fine refinement of surface triangulation and the implementation of Courant-type approximations on a sphere and on a sphere with a cutout.

2 Problem Formulation

2.1 Radial-ring Grids

Radial-ring grids are known to be advantageous for circular domains. Using a radial-ring grid in a circle provides virtually the same order of approximation as a square grid in a rectangle.

Let K denote the circle $x_1^2 + x_2^2 \leq 1$, cut along the radius $x_2 = 0$, $0 < x_1 < 1$, or parts thereof. We introduce polar coordinates r, s , centered at the origin of Cartesian coordinates, and construct a radial-ring grid with steps h_{1i} for r and h_{2j} for s . In the simplest case, the steps h_{1i} and h_{2j} can be equal to $h_{1i} = 1/(n_1)$, $h_{2j} = \pi/(2n_2)$, where n_1 and n_2 are integers; in this case, the circle K will be completely filled with curvilinear rectangles of the grid.

Suppose a grid of nodes x_j is constructed on the interval $[a, b]$, $a = 0, b = 2\pi$.

We denote the basis functions of one variable $w_j(x)$, $w_{j+1}(x)$, $j = 0, 1, \dots, n_i$, as follows:

$$w_j(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}}, \quad x \in [x_j, x_{j+1}],$$

$$w_{j+1}(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}}, \quad x \in [x_{j-1}, x_j].$$

From the last relation it follows that the next relation is valid:

$$w_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}, \quad x \in [x_j, x_{j+1}].$$

Thus, it is possible to construct approximations of functions of two variables in each elementary curvilinear rectangle of the form:

$$U(r, s) = u(r_i, s_j)w_i(r)w_j(s) + u(r_{i+1}, s_j)w_{i+1}(r)w_j(s) + u(r_i, s_{j+1})w_i(r)w_{j+1}(s) + u(r_{i+1}, s_{j+1})w_{i+1}(r)w_{j+1}(s).$$

The approximation is constructed on a nested sequence of refining grids of nodes where the number of nodes doubles: 16, 32, 64.

The features of constructing approximations on a multidimensional sphere are considered in [2].

2.2 Courant-type Approximations

2.2.1 On Constructing a Triangulation

In this section, we present a triangulation algorithm for a two-dimensional surface M smoothly embedded in R^3 . This algorithm is convenient for computer implementation and has been used in software packages for local approximation on a sphere and on a sphere with a cutout. Suppose that the surface M is defined by the equation:

$$M: G(\xi) = 0,$$

where $\xi = (x, y, z) \in R^3$, G is a continuously differentiable function of three variables for which the condition is satisfied

$$\nabla G(\xi) \neq 0, \quad \xi \in M.$$

Let T be a triangulation on M , with triangles and vertices numbered, and a table of incidences that can be used to determine which nodes are the vertices of each triangle under consideration. We assume that each triangle T of the triangulation T is assigned a number, which is a natural number from 1 to n_T , where n_T is the number of triangles in T . We further assume that the vertices are numbered from 1 to n_V and their coordinates are indicated (x_j, y_j, z_j) , $j = 1, \dots, n_V$; here n_V is the number of vertices of T . The triangulation T is defined using two tables V_C and T_V with three columns and with the number of rows n_V and n_T , respectively. In the table V_C , in the j -th row, there are coordinates (x_j, y_j, z_j) of the j -th vertex, $j = 1, \dots, n_V$, and in the table T_V (called the incidence table), in the i -th row, there are numbers of the vertices of the i -th triangle.

The geometric refinement of the triangulation adopted here consists of selecting a point on each edge and connecting the selected points with line segments in each triangle, thereby dividing the original triangle into four new triangles. We denote the resulting triangulation as $T^{(1)}$. So, the task of constructing a new triangulation consists of:

- 1) renumbering all triangles to include newly created ones;
- 2) renumbering the vertices (including newly created ones);
- 3) generating a table $T_V^{(1)}$ of incidences for the new subdivision;
- 4) and constructing a table $V_C^{(1)}$ of vertex coordinates for the new subdivision.

Let's consider each point separately.

- 1) If the i -th triangle of the previous triangulation is subdivided, then the resulting four triangles of the new triangulation are numbered $4i - 3$, $4i - 2$, $4i - 1$, $4i$.
- 2) To number the vertices:
 - a) we determine the number k of digits in the number n_V (related to the old triangulation);
 - b) we assign the number $j_1 \cdot 10^k + j_2$ to the new vertex obtained by dividing the edge with numbers j_1 and j_2 (with the old triangulation), $j_1 < j_2$;
 - c) we assign the number $j \cdot 10^k$ to the old vertex with number j .

As a result of executing points a)-c), the numbering of the vertices of the new triangulation has no repetitions and all the vertices in $T^{(1)}$ are renumbered.

- 3) Point 1) and subpoints b) and c) of point 2) are implemented simultaneously, which allows at the same time to fill in the three-column incidence table $T_V^{(1)}$ consisting of $n_T^{(1)}$ rows (here $n_T^{(1)} = 4n_T$), and the row with number $i^{(1)}$ is filled in with the newly formed numbers of the vertices of the new triangle with number $i^{(1)}$.
- 4) Newly appearing vertices can be taken at the midpoint of the side connecting them. If M is a sphere and the edges are the shortest, then implementing the algorithm that determines the coordinates of a new vertex is simple. In the case of a general surface, its implementation may turn out to be a rather complicated matter. Therefore, we take a different approach: we take the newly appearing vertices on M near the aforementioned midpoints (and even, generally speaking, not on the sides of the triangle - see below). Thus, instead of a subdivision $T^{(1)}$ of the old triangulation T , we obtain a new triangulation $\tilde{T}^{(1)}$ which is topologically equivalent to this subdivision.

Now we describe the process of obtaining the coordinates of new vertices in the case of an arbitrary surface.

Let ξ_i and ξ_j be vertices connected by an edge (in M) in the old triangulation. We connect ξ_i and ξ_j by a rectilinear chord in R^3 , find its midpoint $\xi_{(ij)}$, and construct a ball $B_r(\xi_{(ij)})$ with center at $\xi_{(ij)}$ and radius r . By varying r from zero to $+\infty$, we find the first case of contact of the ball with the surface M , where we take the point ξ_{ij} of contact as a new vertex. In fact, only an approximation of the points of contact is sought, and for this:

- a) on the unit sphere, a correct grid of nodes $\zeta^{(s)}$, $s = 1, \dots, k$; (k is a fixed natural number) is chosen;
- b) on the surface of the ball $B_r(\xi_{(ij)})$ a grid of nodes $\zeta_{rij}^{(s)}$, $\zeta_{rij}^{(s)} = \xi_{(ij)} + r\zeta^{(s)}$, $s = 1, \dots, k$, is constructed;
- c) for each r under consideration, it is checked whether there is a change of sign in the expressions

$$G(\zeta_{rij}^{(s)}), \quad s = 1, \dots, k. \quad (1)$$

A sign change in (1) means that the sphere intersects the surface. The results of the last three steps are used to construct the iteration sequence; these results are stored as the values of three radii and information about the tests. We agree to denote the last three radii as r_1, r_2, r_3 , and store the results of their tests as numbers i_1, i_2, i_3 , where i_q takes a value 0 or 1.

If there is no sign change in the corresponding sequence (1), then $i_q = 0$, and if there is a sign change in this sequence, then $i_q = 1$.

For example, if $i_1 = 0$, $i_1 = i_2 = 1$, then the ball $B_{r_1}(\xi_{(ij)})$ does not intersect M , but the balls $B_{r_2}(\xi_{(ij)})$ and $B_{r_3}(\xi_{(ij)})$ intersect M .

The next assignment of radius r_4 is found as follows. If $i_2 + i_3 = 1$, then we put $r_4 = (r_2 + r_3)/2$, and if $i_1 + i_2 + i_3 = 0$, then we take $r_4 = 2r_3$.

Finally, if $i_1 i_2 i_3 = 1$, then $r_4 = r_3/2$, otherwise, for $i_2 + i_3 = 2$ or for $i_1 = 1$, $i_2 + i_3 = 0$, we take $r_4 = (r_1 + r_3)/2$. It is easy to verify that all eight possible combinations for the values i_1, i_2, i_3 are taken into account in this way.

After finding the radius r_4 and determining r_4 , the values r_1, r_2, r_3 and i_1, i_2, i_3 are shifted as follows: $i_1 := i_2$, $i_2 := i_3$, $i_3 := i_4$, $r_1 := r_2$, $r_2 := r_3$, $r_3 := r_4$. Now, if a specified accuracy is achieved, the process is terminated, and the arithmetic mean of those values $\zeta_{r_3 ij}^{(s)}$, $1 \leq s \leq k$, whose signs of the numbers $G(\zeta_{r_3 ij}^{(s)})$ are opposite to the sign of the number $G(\zeta_{r_3 ij})$, is taken as the approximate value $\tilde{\xi}_{ij}$ of the vector ξ_{ij} .

If a specified accuracy is not achieved, the process of determining the next value of r_4 is repeated.

The coordinates of the new and old vertices are arranged in a three-column table $V_C^{(1)}$ in ascending order of the new vertex numbers; in this case, in the j th row there are coordinates $(x_j^{(1)}, x_j^{(2)}, x_j^{(3)})$ of the vertex with number $i_j^{(1)}$, $j = 1, \dots, n_V^{(1)}$, where $n_V^{(1)}$ is the number of vertices of the new triangulation. For any triangulation of the surface M , the following relation holds:

$$n_V - n_l + n_T = \chi, \quad (2)$$

where n_l is the number of edges, and χ is the Euler characteristic for the surface M . For a sphere, $\chi = 2$; for a sphere with a cutout, $\chi = 1$. With our construction of $T^{(1)}$, it is obvious that $n_T^{(1)} = 4n_T$, $n_V^{(1)} = n_V + n_l$; using (2), we find $n_V^{(1)} = 2n_V + n_T - \chi$. Thanks to this, the number of rows in the tables $T_V^{(1)}$ and $V_C^{(1)}$ are known in advance.

Remark 1. By completing points 1) - 4), the problem of constructing a new triangulation is exhausted. To eliminate the rapid growth of the vertex numbers of the new triangulation, the vertices are renumbered by consecutive numbers of the natural numbers and the table $T_V^{(1)}$ is changed accordingly; the table $V_C^{(1)}$ does not have to be changed.

Remark 2. Calculating the approximation value at a given point on the surface requires finding the curvilinear triangle in which this point is located. The belonging of a point ζ (to a curvilinear triangle) is determined by the belonging of its image during selection χ_ζ to the corresponding flat triangle. For a fine triangulation, the search by enumeration takes a long time. However, if the mentioned triangulation is obtained as a result of several applications of the algorithm just described, then direct access can be organized for a quick search of the desired triangle.

If we let the initial and all intermediate triangulations be stored (in the form of tables T_V and V_C) then in this case, for a given point, the number i of the triangle of the initial (large) triangulation is found. In the next (smaller) triangulation, it is sufficient to enumerate triangles with numbers $4i - 3$, $4i - 2$, $4i - 1$, $4i$. This process continues until the last (smallest) triangulation is reached. We need to obtain a sufficiently fine triangulation of the specified surface. To solve this problem, the following approach is proposed: first, the initial large triangulation is defined, and information about this triangulation is stored in a separate file. Each

triangle of the large triangulation is refined according to the algorithm described above, and information about the subdivisions of each such triangle is stored in other separate files. The names of these files can be generated automatically and can have two parameters: the number of the large triangle of the initial triangulation and the number of its subdivisions.

3 Construction of a Courant-type Approximation on a Sphere and on a Sphere with a Cutout

In this section, we will construct a Courant-type approximation on a sphere and on a sphere with a cutout. Recall that, to construct the approximation, it is necessary to have a representation of the surface using an atlas [3]. As a mapping, we consider the central projection onto the tangent plane at point ζ .

For a sphere M :

$$(\xi, \xi) = r^2,$$

we choose the center of the sphere as the center of the projection. We choose the origin in the tangent plane at the point ζ . Then the mapping takes the form:

$$\chi_\zeta: \xi \rightarrow \xi', \quad \xi' = \tau\xi - \zeta, \quad (3)$$

where $\tau = \frac{r^2}{(\xi, \zeta)}$.

We form a sphere with a cutout as follows: we cut off some part of the closed sphere with a plane. We choose a coordinate system in R^3 so that the axes Ox and Oy lie in the plane of the section, and the Oz axis passes through the center of the sphere.

Let d denote the distance from the center of the sphere to the plane of the section. For the boundary points of the sphere with a cutout, the mapping will have the form:

$$\chi_\zeta: U_\zeta \rightarrow \bar{R}_+^2, \text{ where } \bar{R}_+^2 = \{x | x \in R^2, x^{(2)} \geq 0\}$$

and the image of the neighborhood of U_ζ is a semicircle in the tangent plane.

We choose the origin in the tangent plane at the intersection of three planes: the tangent plane, the Oxy plane, and a plane perpendicular to Oxy and passing through the center of the sphere and the point of tangency ζ . As the center of the projection for the sphere with a cutout, we choose the origin in R^3 . We denote this point by $\xi_{0\zeta}$. Then the mapping for the sphere with a cutout is given by the formula:

$$\chi_\zeta: \xi \rightarrow \xi', \quad \xi' = \tau'\xi - \xi_{0\zeta}, \quad (4)$$

where

$$\tau' = \frac{\|\zeta\|^2 - d\zeta^{(3)}}{(\xi, \zeta) - d\xi^{(3)}},$$

$$\xi_{0\zeta} = \frac{(\zeta - \zeta^{(3)}e_3)(r^2 - d^2 + d\zeta^{(3)})}{(r^2 - (\zeta^{(3)} - d)^2)}. \quad (5)$$

Here e_3 is the unit vector of the Cartesian coordinate system in R^3 , directed along the 0z-axis and $\zeta = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})$, $\xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$.

Now we construct an analog of the Courant approximation on the sphere.

Let $U_\zeta = \{\xi | (\xi, \zeta) \geq \varepsilon r^2, \xi \in M\}$, where ε is a fixed number from the interval $(0, 1)$.

Let T be a triangulation of the sphere M , in which $\{\xi_j\}$, $\xi_j \in M$, is the set of vertices of the triangulation, and the edges are segments of geodesics on M . We shall assume that this triangulation is so fine-grained that the closure of the barycentric star of any vertex of a triangle $\bar{T}$ containing the point ζ lies in U_ζ .

The image of the neighborhood U_ζ under the mapping χ_ζ is obviously some ball E_ζ with center at the origin of the tangent plane L_ζ .

Let Φ_ζ be the union of curvilinear triangles lying entirely in U_ζ . We denote the images of the vertices $\xi_i \in \Phi_\zeta$ under the mapping χ_ζ by $\xi'_i \in \Phi'_\zeta$, where Φ'_ζ is the union of rectilinear triangles in the plane L_ζ . By assumption, there exists a vertex $\xi'_j \in \Phi'_\zeta$ which is a closure of the body $|Z'_j|$ of the barycentric star Z'_j that contains the origin of the plane L_ζ . Let T' be a triangle in Z'_j such that $\xi'_j \in T'$. For points $\xi' \in T'$ we can define a flat basis Courant function, which (due to its dependence on ζ) we denote by $\Omega_{\zeta,j}(\xi')$.

Since $0 \in T'$, the value $\Omega_{\zeta,j}(0)$ is defined. Therefore, we set the basis function on the sphere corresponding to the vertex ξ_j to be $\omega_j(\zeta) = \Omega_{\zeta,j}(0)$.

Without loss of generality, we assume that the vertices of the triangle T' are numbered 1, 2, 3. We construct a flat basis function $\Omega_{\zeta,1}(\xi')$ corresponding to the vertex ξ'_1

$$\Omega_{\zeta,1}(\xi') = \frac{\det \begin{pmatrix} \xi'^{(1)}_1 & \xi'^{(1)}_2 & \xi'^{(1)}_3 \\ \xi'^{(2)}_1 & \xi'^{(2)}_2 & \xi'^{(2)}_3 \\ \xi'^{(3)}_1 & \xi'^{(3)}_2 & \xi'^{(3)}_3 \end{pmatrix}}{\det \begin{pmatrix} \xi'^{(1)}_1 & \xi'^{(1)}_2 & \xi'^{(1)}_3 \\ \xi'^{(2)}_1 & \xi'^{(2)}_2 & \xi'^{(2)}_3 \\ \xi'^{(3)}_1 & \xi'^{(3)}_2 & \xi'^{(3)}_3 \end{pmatrix}},$$

where $\xi'_j = \xi'_j(\zeta)$, $j = 1, 2, 3$.

Using the definition of χ_ζ (see formula (3)), taking into account the equality $\chi_\zeta(\zeta) = \zeta$, we obtain

$$\Omega_{\zeta,1}(\zeta) = \frac{(\xi_1, \zeta) \Delta^T_1(\zeta)}{r^2 \Delta^T}, \quad \zeta, \xi_j \in \bar{T},$$

$$j = 1, 2, 3,$$

where $\Delta^T_1(\zeta)$ is obtained from the determinant Δ^T ,

$$\Delta^T \stackrel{\text{def}}{=} \det \begin{pmatrix} \xi^{(1)}_1 & \xi^{(1)}_2 & \xi^{(1)}_3 \\ \xi^{(2)}_1 & \xi^{(2)}_2 & \xi^{(2)}_3 \\ \xi^{(3)}_1 & \xi^{(3)}_2 & \xi^{(3)}_3 \end{pmatrix}, \quad (6)$$

by replacing the first column with the vector $(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})^T$. Similarly, the basis functions $\Omega_{\zeta,2}(\zeta)$, $\Omega_{\zeta,3}(\zeta)$ are found.

Thus, the analog of the Courant approximation on the sphere has the form

$$\omega_j(\zeta) = \begin{cases} \frac{(\xi_j, \zeta) \Delta^T_j(\zeta)}{r^2 \Delta^T}, & \zeta, \xi_j \in \bar{T} \subset |Z_j|, \\ 0, & \zeta \notin |Z_j|, \end{cases} \quad (7)$$

where Δ^T is found by formula (6), and Δ^T_j is obtained from Δ^T by replacing the j th column with the column $(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})^T$.

Now we construct an analogue of the Courant approximation on a sphere with a cutout.

As a chart mapping, we consider mapping (4) and carry out reasoning similar to the above. We confirm that the analogues of the basis functions of Courant on a sphere with a cutout have the form

$$\omega_j(\zeta) = \begin{cases} \frac{(\xi_j, \zeta) - d\xi^{(3)}_j}{\|\zeta\|^2 - d\zeta^{(3)}} \frac{\Delta^T_j(\zeta)}{\Delta^T}, & \zeta, \xi_j \in \bar{T} \subset |Z_j|, \\ 0, & \zeta \notin |Z_j|, \end{cases} \quad (8)$$

where d is the distance from the center of the sphere to the plane of the section, Δ^T is found by formula (6), and Δ^T_j is obtained from Δ^T by replacing the j th column with the column $(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})^T$.

It can be shown that the functions $\omega_j(\zeta)$ are continuous and give an interpolation basis on the grid $\{\xi_j\}_{j \in J}$, $\omega_j(\xi_i) = \delta_{ij}$.

The Courant-type approximation can be represented in the following form:

$$\tilde{u}(\zeta) = \sum_j u(\xi_j) \omega_j(\zeta), \quad (9)$$

where $u(\xi_j)$ are the values of the function being approximated at the vertices ξ_j , and $\omega_j(\zeta)$ are analogs of the Courant basis functions. For an approximation of the form (9), the following estimate is valid, [3]:

$$\max_{\zeta \in M} |\tilde{u}(\zeta) - u(\zeta)| \leq Ch^2 \|u\|_{C^2(M)}, \quad (10)$$

where h is the largest side of the triangles of the triangulation, and C is a constant independent of u and h .

The proof of formula (10) follows from a more general fact (see [19]). For ease of reading, we present the proofs.

3.1 The Proof of Theorem

In this section, we give the theorem about the error of the approximation for the general case ([19]).

We will need some auxiliary statements.

Let M be some n -dimensional manifold (i.e., a topological space in which every point has a neighborhood homeomorphic to an n -dimensional ball in the Euclidean space R^n), let E_ζ be an open ball in the n -dimensional space R^n , let Z be some set of indices, and let there be a system of differentiable coordinates on the manifold M , i.e., a family of open sets $\{U_\zeta\}_{\zeta \in Z}$ covering M , and homeomorphisms $\psi_\zeta: E_\zeta \rightarrow U_\zeta$ such that the mapping

$$\psi_\zeta^{-1} \psi_{\zeta'}: \psi_\zeta^{-1}(U_\zeta \cap U_{\zeta'}) \rightarrow \psi_{\zeta'}^{-1}(U_\zeta \cap U_{\zeta'}), \quad \zeta, \zeta' \in Z \quad (11)$$

continuously differentiates. Here $\psi_\zeta: E_\zeta \rightarrow U_\zeta$ is called a map, the set $\{\psi_\zeta: E_\zeta \rightarrow U_\zeta \mid \zeta \in Z\}$ is called an atlas, $M \in C^k$.

Definition. A function u is said to be defined on a manifold M if, for each $\zeta \in Z$, a function $F_\zeta(x)$ ($x \in E_\zeta$) is defined such that the relations $F_\zeta(x_\zeta) = F_{\zeta'}(\psi_\zeta^{-1}(x_\zeta))$, $x_\zeta = \psi_\zeta^{-1}(\xi)$, $x_{\zeta'} = \psi_{\zeta'}^{-1}(\xi)$, $\xi \in U_\zeta \cap U_{\zeta'}$ are valid.

Let function $u \in C^{m+1}(M)$ be defined on a manifold M . By the Definition, this means that functions $F_\zeta \in C^{m+1}(E_\zeta)$, $\zeta \in M$ are given, and with these functions F_ζ the function $u = F_\zeta \circ \psi_\zeta^{-1}$ is uniquely determined. We expand the function F_ζ using the Taylor formula in the neighborhood of the point 0 with the remainder $R_\zeta(x)$:

$$F_\zeta(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha F_\zeta(0) x^\alpha + R_\zeta(x), \quad x \in E_\zeta.$$

We represent the remainder in the Taylor formula in integral form:

$$R_\zeta(\xi) = R_m(F_\zeta, \xi) = \int_0^1 (1-\tau)^m \sum_{|\alpha|=m+1} \frac{1}{\alpha!} D^\alpha F_\zeta(\tau \xi') \xi'^\alpha d\tau, \quad \text{where } \xi' = \psi_\zeta^{-1}(\xi), \quad \xi \in \text{supp } f(\zeta).$$

Then we have

$$\begin{aligned} u(\xi) &= F_\zeta(\psi_\zeta^{-1}(\xi)) \\ &= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha F_\zeta(0) [\psi_\zeta^{-1}(\xi)]^\alpha + R_\zeta(\psi_\zeta^{-1}(\xi)), \\ &\quad \xi \in U_\zeta. \end{aligned}$$

Thus, the function u acquires the expansion in the neighborhood U_ζ of the point

$$u(\xi) = (\mathcal{P}_\zeta u)(\xi) + (\mathcal{R}_\zeta u)(\xi), \quad \xi \in U_\zeta, \quad (12)$$

where

$$\begin{aligned} (\mathcal{P}_\zeta u)(\xi) &= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha F_\zeta(0) [\psi_\zeta^{-1}(\xi)]^\alpha, \\ (\mathcal{R}_\zeta u)(\xi) &= R_\zeta(\psi_\zeta^{-1}(\xi)). \end{aligned}$$

Let $f(\eta)$ be an abstract function with values in the space $(C^{m+1}(M))^*$, $\eta \in M$; let the support of the functional $f(\eta)$ (for a fixed η) lie in the set U_η . Using the representation (12) in U_ζ , we obtain $(f(\zeta), u) = (f(\zeta), \mathcal{P}_\zeta u) + (f(\zeta), \mathcal{R}_\zeta u)$, $\zeta \in M$. Assume that

$$(f(\zeta), \mathcal{P}_\zeta u) = u(\zeta) \quad \forall u \in C^{m+1}(M). \quad (13)$$

The function $\tilde{u}(\zeta) = (f(\zeta), u)$ will be called an approximation of the function u . From (13) we easily find

$$u(\zeta) - \tilde{u}(\zeta) = (f(\zeta), \mathcal{R}_\zeta u). \quad (14)$$

Introducing the notation

$$S(\zeta) = \text{supp } f(\zeta), \quad S'(\zeta) = \psi_\zeta^{-1}(S(\zeta)), \quad h(\zeta) = \text{diam } S'(\zeta), \quad \text{for } \xi \in S(\zeta), \text{ we find}$$

$$C_0(h(\zeta))^{m+1} \max_{|\alpha|=m+1} \|\mathcal{R}_\zeta\|_{C(S(\zeta))} \leq \|D^\alpha F_\zeta\|_{C(S'(\zeta))},$$

where the constant C_0 is independent of ζ and F_ζ .

We also introduce the notation

$$\begin{aligned} h &= \sup_{\zeta \in M} h(\zeta), \quad K_0 = \sup_{\zeta \in M} \|f(\zeta)\|_{C^*(M)}, \\ \|u\|_{C^k(M)} &= \sup_{\zeta \in M} \|F_\zeta\|_{C^k(E_\zeta)}, \end{aligned}$$

If $f(\zeta)$ is a trajectory in $C^*(M) = (C^{m+1}(M))^*$, then from formula (14) we find

$$|u(\zeta) - \tilde{u}(\zeta)| \leq \|f(\xi)\|_{C^*(M)} \|\mathcal{R}_\zeta\|_{C(S(\zeta))}.$$

We now have the relation

$$|u(\zeta) - \tilde{u}(\zeta)| \leq C_0(h(\zeta))^{m+1} \|f(\zeta)\|_{C^*(M)} \max_{|\alpha|=m+1} \|D^\alpha F_\zeta\|_{C(E_\zeta)}.$$

We also introduce the notation

$$h = \sup_{\zeta \in M} h(\zeta), \quad K_0 = \sup_{\zeta \in M} \|f(\zeta)\|_{C^*(M)},$$

$$\|u\|_{\check{C}^k(M)} = \sup_{\zeta \in M} \|F_\zeta\|_{C^k(E_\zeta)},$$

Now we see that the following theorem is true.

Theorem. If $f(\zeta)$ is a continuous abstract function on a compact manifold M with values in $C^*(M)$, then the following estimate holds:

$$\|u - \tilde{u}\|_{C(M)} \leq C_0 K_0 h^{m+1} \|u\|_{\check{C}^{m+1}(M)}.$$

Corollary. We get formula (10) from the Theorem when $m = 1$.

3.2 Numerical Approximation

The programs created by the author for approximating functions defined on a sphere and on a sphere with a cutout allow us:

- 1) to obtain a sufficiently fine refinement of the original triangulation,
- 2) to calculate the local Courant basis functions at a given point,
- 3) and to obtain an approximation of functions and calculate their error.

The input data of the packages are an array of function values on a sphere and on a sphere with a cutout, as well as the initial triangulation and the number of its refinements. The output data are tables of the basis coefficients and of approximation errors of functions on a control grid. We will give several examples of function approximation using the software packages mentioned. The triangulation generated by the vertices of a tetrahedron inscribed in the sphere was considered as the initial triangulation on the sphere.

The coordinates of the triangulation vertices are as follows:

$$\xi_1 = (1, 0, 0), \quad \xi_2 = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0\right),$$

$$\xi_3 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{\sqrt{3}}\right),$$

$$\xi_4 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{\sqrt{3}}\right).$$

For a sphere with a cutout, the initial triangulation is defined by the following points:

$$\xi_1 = (0, 0, 1 + d), \quad \xi_2 = (\sqrt{1 - d^2}, 0, 0),$$

$$\xi_3 = \left(-\frac{\sqrt{1 - d^2}}{2}, \frac{\sqrt{3(1 - d^2)}}{2}, 0\right),$$

$$\xi_4 = \left(-\frac{\sqrt{1 - d^2}}{2}, -\frac{\sqrt{3(1 - d^2)}}{2}, 0\right),$$

where d is the distance from the center of the sphere to the cutting plane, the parameter d takes the values $1/3, 1/2$.

Each triangle of the initial triangulation is subdivided according to the algorithm described in section 2.2.1, into a number of triangles equal to 4^5 . Thus, the final triangulation on the sphere consists of $4 \cdot 4^5 = 4096$ triangles, and on the sphere with a cutout of $3 \cdot 4^5 = 3072$ triangles.

The functions considered were $\cos 5x, \frac{x^2 + y^2}{2}, x, y, z$, on a control grid consisting of points $\zeta_{k,i}$ with angular coordinates (ψ_k, φ_i) ; here

$$\psi_k = \frac{k\pi}{n} - \frac{\pi}{2}, \quad \varphi_i = \frac{2\pi i}{m}, \quad k = 0, \dots, n,$$

$$i = 1, \dots, m.$$

For a sphere, the parameters m, n took the following values: $n = 4, m = 4$, and for a sphere with a cutout $n = 8, 10, m = 4$.

Table 1 presents the errors of the Courant approximations on a sphere. The rows of these tables correspond to the levels of grid refinement (n is the number of triangles in the triangulation), and the columns correspond to the points on the sphere at which the error is calculated.

Table 1. Errors in the Courant-type approximation of the function $f(x, y, z) = x$ defined on the unit sphere

n	$\zeta_{1,1}$	$\zeta_{1,2}$	$\zeta_{1,3}$	$\zeta_{1,4}$
4^2	$-1.4 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$3.0 \cdot 10^{-2}$	$-1.4 \cdot 10^{-1}$
4^3	0	$4.8 \cdot 10^{-2}$	$8.4 \cdot 10^{-3}$	$-5.4 \cdot 10^{-2}$
4^4	0	$8.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$	$-1.1 \cdot 10^{-2}$
4^5	0	$2.8 \cdot 10^{-3}$	$6.6 \cdot 10^{-5}$	$-4.8 \cdot 10^{-3}$
4^6	0	$8.0 \cdot 10^{-4}$	$3.4 \cdot 10^{-6}$	$-8.3 \cdot 10^{-4}$

Source: created by the authors

Table 2 presents the errors of the Courant approximations on a sphere for some functions $f(x, y, z)$ at point $\zeta_{1,3}$.

Table 2. The errors of the Courant approximations on a sphere for some functions $f(x, y, z)$ at point $\zeta_{1,3}$.

n	$\cos x^2$	$x^3 + y^3$	$\exp(z - y^2)$	x^5
$2 \cdot 4^2$	$-1.2 \cdot 10^{-3}$	$3.2 \cdot 10^{-2}$	$3.510 \cdot 10^{-3}$	$2.1 \cdot 10^{-2}$
$2 \cdot 4^3$	$7.1 \cdot 10^{-4}$	$1.5 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$	$7.5 \cdot 10^{-3}$
$2 \cdot 4^4$	$2.2 \cdot 10^{-4}$	$3.7 \cdot 10^{-3}$	$1.4 \cdot 10^{-4}$	$1.4 \cdot 10^{-3}$

Source: created by the authors

Table 3 presents the errors of the Courant approximations on a sphere with a cutout.

Table 3. Errors (R) in the Courant-type approximation of the function $f(x, y, z) = x^2 + y^2$, $n = 3 \cdot 4^2$, defined on the unit sphere with a cutout ($d = 1/3$)

$\zeta_{i,j}$	R
$\zeta_{4,1}$	$-6.8 \cdot 10^{-3}$
$\zeta_{4,2}$	$2.9 \cdot 10^{-1}$
$\zeta_{4,3}$	$-6.8 \cdot 10^{-3}$
$\zeta_{4,4}$	$9.5 \cdot 10^{-2}$
$\zeta_{5,1}$	$-1.5 \cdot 10^{-1}$
$\zeta_{5,2}$	$3.6 \cdot 10^{-1}$
$\zeta_{5,3}$	$-1.5 \cdot 10^{-1}$
$\zeta_{5,4}$	$-7.8 \cdot 10^{-2}$
$\zeta_{6,1}$	$3.6 \cdot 10^{-1}$
$\zeta_{6,2}$	$2.6 \cdot 10^{-1}$
$\zeta_{6,3}$	$-1.5 \cdot 10^{-1}$
$\zeta_{6,4}$	$-8.2 \cdot 10^{-3}$
$\zeta_{6,1}$	$-4.6 \cdot 10^{-2}$
$\zeta_{6,2}$	$9.1 \cdot 10^{-2}$
$\zeta_{6,3}$	$-4.6 \cdot 10^{-2}$
$\zeta_{6,4}$	$-4.5 \cdot 10^{-2}$

Source: created by the authors

The plot of the triangle on the sphere and on the plane is given in Figure 1.

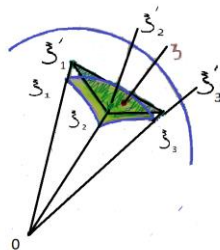


Fig. 1: The plot of the triangle on the sphere and on the plane

Source: created by the authors

4 Conclusion

This paper discusses approximation on a sphere and on a sphere with a cutout. In the future, the authors will consider approximation on other manifolds. Argyris, Zlamal, and other approximations will be discussed.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and they have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- N.A. Lebedinskaya, carried out theoretical computations of the Courant-type approximation on sphere with a cutout.
- D.M. Lebedinski organized and executed the numerical experiments.
- I.G.Burova was responsible for the study's direction and scientific supervision.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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