Statistical Design for Monitoring Process Mean on Modified EWMA Control Chart based on Autocorrelated Data

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Abstract: - This research endeavor is focused on establishing explicit formulas for the computation of the average run length (ARL) within the context of a moving average process characterized by exogenous variables, denoted as MAX(q,r), and subjected to exponential white noise. Additionally, we aim to conduct a comparative analysis of their performance against the exponentially weighted moving average (EWMA) and the modified exponentially weighted moving average (modified EWMA) methodologies. The evaluation of their performance will be based on metrics such as the absolute percentage relative error (APRE) and the relative mean index (RMI). Furthermore, we undertake a rigorous assessment of the accuracy of these explicit formulas in relation to ARL by considering CPU time, utilizing the numerical integral equation (NIE) method derived through the application of the Gauss-Legendre quadrature rule. This comparative evaluation is carried out for both control chart methodologies. To ascertain the efficacy of our explicit formulas approach, we apply it to two distinct datasets. The first dataset pertains to the closing price of natural gas, with the crude oil WTI price serving as the exogenous variable. The second dataset encompasses the closing stock price of KTB Public Company Limited, with daily foreign exchange rates for USD/JPY and EUR/USD as the exogenous variables. The results of applying the ARL based on the explicit formulas to these two datasets demonstrate that, under these conditions, the modified EWMA control chart outperforms the EWMA control chart.

Key-Words: - Average Run Length, Moving Average Process, Explanatory Variable, Explicit Formulas, Modified Exponentially Weighted Moving Average.

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1 Introduction

Currently, Statistical Process Control (SPC) stands as a vital methodology employed to monitor and control process variations, ensuring that various industries such as manufacturing, healthcare, medical sciences, finance analysis, and others operate at their maximum potential to produce conforming products. In particular, researchers have conducted reviews on the advantages and limitations of SPC in the context of quality improvement, which has implications for financial systems, healthcare, and the manufacturing industry, [1], [2]. The foundational work in this field began with, [3], who introduced the first control chart, widely used for monitoring and detecting significant process changes, especially when observations follow a normal distribution. The study, [4], subsequently proposed the Cumulative Sum (CUSUM) control chart, while, [5], presented the Exponentially Weighted Moving Average (EWMA) control chart, which is particularly adept at detecting subtle shifts in the process means. The benefits of employing the EWMA control chart have been extensively documented. Building on this foundation, [6], introduced the modified EWMA control chart, a highly effective tool for detecting small, sudden shifts in the process mean. Subsequently, [7], further developed the Modified EWMA control chart by introducing an additional constant factor, denoted as 'k,' and incorporating an exponential smoothing limiting factor. This modified approach outperforms both existing EWMA control charts in terms of Average Run Length (ARL).

The ARL is a measurement method utilized in control charts to assess performance, and it can be categorized into two components. Firstly, ARL_0, also known as in-control ARL, represents the average number of data points before an out-of-control condition is detected. Simultaneously, ARL_1, which pertains to out-of-control situations, denotes the average number of data points that fall outside control limits before the process is recognized as out-of-control. The objective is to keep ARL_1 as small as possible. Various methods can be employed to evaluate ARL, such as explicit formulas, the Markov chain approach (MCA), or numerical integral equations (NIE). Researchers commonly employ these methods to determine ARL in a variety of contexts. For instance, [8], utilized the martingale approach to derive explicit formulas for ARL and average delay time. They compared these results to performance metrics under the Exponentially Weighted Moving Average (EWMA) and other measures in the exponential distribution. The study, [9], employed Fredholm's second-kind integral equations method to resolve ARL in the context of the EWMA procedure for AR(1) processes. The study, [10], considered the numerical integral equation (NIE) method, employing Gauss-Legendre quadrature rules, to analyze the modified exponentially weighted moving average (Modified EWMA) control chart for MA(1) processes with exponential white noise. The study, [11], derived approaches involving Markov chains and integral equations to evaluate ARL in the context of CUSUM and EWMA control charts. The study, [12], employed the Fredholm integral equation approach to establish an explicit formula for calculating the ARL in CUSUM control charts based on the $SAR(P)_L$ with a trend process. Furthermore, researcher conducted a comparative analysis with the NIE approach. Building on this work, [13], derived an explicit formula and extended the NIE method for ARL calculations in CUSUM charts when dealing with observations that follow seasonal autoregressive models with exogenous variables, specifically $SARX(P,r)_L$ with exponential white noise. In another research endeavor, [14] introduced a novel explicit formula for ARL in EWMA control charts, focusing on stationary moving average processes with exogenous variables represented as MAX(q,r). Their approach made innovative use of the Fredholm integral equation technique. The study, [15], conducted an explicit formula was developed for ARL in CUSUM control charts, considering a seasonal autoregressive model with one exogenous variable ($SARX(1,1)_L$). They also compared their results to those obtained through NIE, employing various numerical integration techniques such as the Gaussian rule, the Midpoint rule, and the Trapezoidal rule. Furthermore, [16], introduced a new solution for calculating ARL within the context of the EWMA control chart, specifically when the process adheres to the SMAX(Q,r)L model. This explicit ARL solution for the SMAX(Q,r)L process is analyzed using the Fredholm integral equation method.

The observers typically encounter situations governed by Stochastic processes, which involve accidental time-space or time-series dynamics. Moreover, these processes often originate from econometric models, specifically the autoregressive (AR) model and moving average (MA) model. However, these situations tend to exhibit unpredictability in their movement patterns, and the error factors from discrepancies between actual values and predictions. Consequently, they evolve into seasonal moving average (SMA) models. Subsequently, in cases where the time-series errors follow a white noise pattern and there is auto-correlated due to seasonal factors, it is referred to as exponential white noise, [17], [18], [19]. Exogenous variables are defined as those not directly influenced by other variables, and they find frequent application in econometric models. Furthermore, processes that incorporate exogenous variables, especially multiple exogenous variables simultaneously, often yield enhanced performance. Consequently, the components of the AR, MA, SAR, or SMA models incorporating exogenous variables are designated as ARX, MAX, SARX, and SMAX, respectively. The Numerical Integral Equation (NIE) method is widely adopted for explaining continuous distributions. Additionally, researchers who explore these methods often make comparisons with explicit formulas while assessing the performance of a modified EWMA control chart, [14], [15], [16], [20], [21], [22].

The preceding research underscores the utilization of the Numerical Integral Equation (NIE) method to conduct a comparative analysis of explicit Average Run Length (ARL) formulas in various control chart contexts, encompassing CUSUM, EWMA, and modified EWMA control charts, across different models such as AR(1), MA(1), AR(p), and ARX(p,r). Notably, there has been a gap in the exploration of the modified EWMA control chart applied to the MAX(q,r) process with exponential white noise. Consequently, our primary objective revolves around deriving explicit formulas and conducting ARL comparisons using the NIE method. The outcomes of this study reveal that explicit formulas enable more rapid evaluation, particularly when employing the Gauss-Legendre quadrature rule, for a MAX(q,r) process with exponential white noise when implemented within a modified EWMA control chart. To provide empirical validation, real data were employed in the observational process, involving two distinct datasets. The first dataset pertains to the closing price of natural gas, with the crude oil WTI price serving as the exogenous variable, covering the period from July 1st to August 31, 2022. The second dataset involves the closing stock price of KTB Public Company Limited, with daily foreign exchange rates for USD/JPY and EUR/USD as exogenous variables, spanning from August 1st to September 15, 2022. Furthermore, this study encompasses a comparative analysis of the performance of the modified and EWMA control charts based on metrics such as the Absolute Percentage Relative Error (APRE) and the Relative Mean Index (RMI).

2 Materials and Methods

The EWMA control chart used to monitor and detect small changes in the process mean, [3], can be derived by using the recursive equation.

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t,\ t = 1, 2, 3, ... \tag{1}$$

where E_t is the EWMA statistic, $0 < \lambda < 1$ is an exponential smoothing parameter, and Y_t is the sequence of the MAX(q,r) process with exponential white noise. The mean and variance of the EWMA control chart are $E(E_t) = \mu_0$ and $\mathrm{Var}(E_t) = \sigma^2(\frac{\lambda}{2-\lambda})$, respectively. Therefore, the general upper control limit (*UCL*) and lower control limit (*LCL*) to detect the sequence are respectively given by

$$UCL = \mu_0 + L_1\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{2}$$

$$LCL = \mu_0 - L_1\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{3}$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is an appropriate control width limit.

The stopping time for the one-sided EWMA control chart is given by

$$\zeta_h = \inf\{t > 0 : E_t > h\} \tag{4}$$

The study, [7], proposed a new structure for the control statistics of the modified EWMA control chart by using the following recursive equation:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}) \tag{5}$$

where $0 < \lambda < 1$ is an exponential smoothing parameter and k is a constant. The mean and variance of the modified EWMA control chart is $E(Z_t) = \mu_0$ and $Var(Z_t) = \sigma^2(\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda})$, respectively. Therefore, the general *UCL* and *LCL* to detect the sequence are respectively given by

$$UCL = \mu_0 + L_2\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}} \tag{6}$$

$$LCL = \mu_0 - L_2\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}} \tag{7}$$

where μ_0 is the target mean, σ is the process standard deviation, L_2 is an appropriate control width limit, Y_t is the sequence of observations, $Z_0 = u$ and $Y_0 = v$ are the initial values, and $0 < \lambda \le 1$ is an exponential smoothing parameter.

The stopping time for the one-sided modified EWMA control chart is given by

$$\zeta_b = \inf\{t > 0 : Z_t > l\} \tag{8}$$

where ζ_b is the stopping time, a is the *LCL*, and l is the *UCL*.

3 The ARL of Modified EWMA Control Chart

3.1 The Exact Solution of ARL the modified EWMA Control Chart for MAX(q,r) process with Exponential White Noise

A MAX (q,r) process with exponential white noise can be derived as

$$Y_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i x_{it} \quad (9)$$

where μ is a constant, ε_t is the white noise process $\varepsilon_t \sim Exp(\alpha)$, θ is the MA coefficient with an initial value of $\varepsilon_0 = s$, X_t is an exogenous variable, and β is the coefficient of X_t. Therefore, modified EWMA statistics (Z_t) can be written as

$$Z_t = (1-\lambda)Z_{t-1} + (\lambda+k)\varepsilon_t - kY_{t-1} + (\lambda+k)\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)$$

If Y_t signals the out-of-control state for Z_1 when $Z_0 = u$, then

$$Z_1 = (1-\lambda)u + (\lambda+k)\varepsilon_1 - kY_0 + (\lambda+k)\left(\mu - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)$$

If ε_1 is the in-control limit for Z_1, then $0 \le Z_1 \le l$. Consider the following function:

$$F(u) = 1 + \int F(Z_1) f(\varepsilon_1) d(\varepsilon_1) \quad (10)$$

which is a Fredholm integral equation of the second kind, [23]. Moreover, $F(u)$ can be rewritten as

$$F(u) = 1 + \int_0^l F\Big((1-\lambda)u - kY_{t-1} + (\lambda+k)y + (\lambda+k)\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)\Big) f(y)\,dy$$

Let $w = (1-\lambda)u - kY_{t-1} + (\lambda+k)y + (\lambda+k)\Big(\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\Big)$

By changing the integral variable, we can obtain the following integral equation:

$$F(u) = 1 + \frac{1}{\lambda+k}\int_0^l F(w) f\left(\frac{w-(1-\lambda)u}{\lambda+k} + \frac{kY_{t-1}}{\lambda+k} - \left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)\right) dw. \quad (11)$$

If $Y_t \sim Exp(\alpha)$ and $f(y) = \frac{1}{\alpha}e^{\frac{-y}{\alpha}}$; $y \ge 0$, then

$$F(u) = 1 + \frac{1}{\lambda+k}\int_0^l F(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{\lambda+k} + \frac{kY_{t-1}}{\lambda+k} - \left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)\right\}} dw \quad (12)$$

Let $M(u) = e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)}$, then we obtain

$$F(u) = 1 + \frac{M(u)}{\alpha(\lambda+k)}\int_0^l F(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw \quad ; 0 \le u \le l.$$

Let $g = \int_0^l F(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw$, then $F(u) = 1 + \frac{M(u)}{\alpha(\lambda+k)} \cdot g$. Consequently, we obtain

$$F(u) = 1 + \frac{1}{\alpha(\lambda+k)} e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i x_{it}\right)} \cdot g \quad (13)$$

By solving for constant g, we obtain

$$g = \int_0^l F(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw$$

$$= \int_0^l \left[1 + \frac{g}{\alpha(\lambda+k)} M(w)\right] e^{\frac{-w}{\alpha(\lambda+k)}} dw$$

$$= \int_0^l e^{\frac{-w}{\alpha(\lambda+k)}} dw + \int_0^l \frac{g e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i x_{it}\right)}}{\alpha(\lambda+k)} \times e^{\frac{-w}{\alpha(\lambda+k)}} dw$$

$$g = \frac{-\alpha(\lambda+k)\left(e^{\frac{-l}{\alpha(\lambda+k)}}-1\right)}{1+\frac{e^{\frac{-kY_{t-1}}{\alpha(\lambda+k)}+\frac{1}{\alpha}\left(\mu-\theta_1\varepsilon_{t-1}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i x_{it}\right)}}{\lambda}\left(e^{\frac{-\lambda l}{\alpha(\lambda+k)}}-1\right)}. \quad (14)$$

By substituting constant g Eq. (14) into Eq. (13), we arrive at

$$F(u) = 1+\frac{e^{\frac{(1-\lambda)u-kY_{t-1}}{\alpha(\lambda+k)}+\frac{1}{\alpha}\left(\mu-\theta_1\varepsilon_{t-1}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it}\right)}}{\alpha(\lambda+k)} \times\left(\frac{-\lambda\alpha(\lambda+k)\left[e^{\frac{-l}{\alpha(\lambda+k)}}-1\right]}{\lambda+e^{\frac{-kY_{t-1}}{\alpha(\lambda+k)}+\frac{1}{\alpha}\left(\mu-\theta_1\varepsilon_{t-1}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i x_{it}\right)}\left[e^{\frac{-\lambda l}{\alpha(\lambda+k)}}-1\right]}\right). \quad (15)$$

Using the Fredholm integral equation of the second kind, the explicit one-sided formulations for the ARL of a MAX(q,r) process operating on a modified EWMA control chart can be derived. Fredholm integral equations are encountered in various fields because they are essential for analyzing and solving problems that involve functions and integrals. They offer a robust mathematical framework that facilitates the modeling and comprehension of complex phenomena. When the process is in a state of control with exponential parameters $\alpha=\alpha_0$, we obtain the following explicit solution for ARL_0:

$$ARL_0 = 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_0(\lambda+k)}}\left[e^{\frac{-l}{\alpha_0(\lambda+k)}}-1\right]}{\lambda e^{\frac{kY_{t-1}}{\alpha_0(\lambda+k)}+\frac{1}{\alpha_0}\left(\mu+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\left[e^{\frac{-\lambda l}{\alpha_0(\lambda+k)}}-1\right]}. \quad (16)$$

Similarly, the explicit solution for can be expressed as when the process is in the out-of-control state with an exponential parameter $\alpha=\alpha_1$.

$$ARL_1 = 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_1(\lambda+k)}}\left[e^{\frac{-l}{\alpha_1(\lambda+k)}}-1\right]}{\lambda e^{\frac{kY_{t-1}}{\alpha_1(\lambda+k)}+\frac{1}{\alpha_1}\left(\mu+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i x_{it}\right)}+\left[e^{\frac{-\lambda l}{\alpha_1(\lambda+k)}}-1\right]}. \quad (17)$$

The Existence and Uniqueness of the Explicit Formulas. Here, we show the existence and uniqueness of the solution to the integral equation in Eq. (12). First, we define

$$T(F(u)) = 1+\frac{1}{\lambda+k}\int_0^l F(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k)}+\frac{kY_{t-1}}{(\lambda+k)}-\mu+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right\}}dw \quad (18)$$

Theorem 1.
Banach's fixed-point theorem, [24].
Let $N[0,l]$ be a set of all of the continuous functions on complete metric (X,d), and assume that $T:X\to X$ is a contraction mapping with contraction constant $0\le s<1$; i.e., $\|T(F_1)-T(F_2)\| \le s\|F_1-F_2\|\ \forall F_1,F_2\in X$. Subsequently, $F(.)\in X$ is unique at $T(F(u)) = F(u)$; i.e., it has a unique fixed point in X.

Proof: To show that T defined in (13) is a contraction mapping for $F_1,F_2\in C[0,l]$, we use the inequality $\|T(F_1)-T(F_2)\|\le s\|F_1-F_2\|\ \forall F_1,\ F_2\in N(0,l)$ with $0\le s<1$. Consider Eq.(8) and Eq.(13), then

$$\begin{aligned}\|T(F_1)-T(F_2)\|_\infty &= \sup_{u\in[0,l]}\left|\frac{M(u)}{\alpha(\lambda+k)}\int_0^l (F_1(w)-F_2(w))e^{\frac{-w}{\alpha(\lambda+k)}}dw\right| \\ &\le \sup_{u\in[0,l]}\left|\|F_1-F_2\|_\infty M(u)\left(1-e^{\frac{-l}{\alpha(\lambda+k)}}\right)\right| \\ &= \|F_1-F_2\|_\infty\left|1-e^{\frac{-l}{\alpha(\lambda+k)}}\right|\sup_{u\in[0,l]}|M(u)| \\ &\le s\|F_1-F_2\|_\infty,\end{aligned}$$

where $s = \left|1 - e^{\frac{-l}{\alpha(\lambda+k)}}\right| \sup_{u\in[0,l]} |M(u)|$ and

$$M(u) = e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\mu - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}\right)}; 0 \le s < 1.$$

The solution exists and is unique, as demonstrated by the application of Banach's fixed-point theorem.

3.2 The NIE for the ARL of an MAX(q,r) Process on a Modified EWMA Control Chart

The NIE approach is widely used for evaluating the ARL. It can be based on one of several quadrature rules (midpoint, trapezoidal, Simpson's rule, and Gauss-Legendre), all of which give ARLs that are very close to each other, [25]. In the present study, we use the Gauss-Legendre rule to evaluate the ARL. An integral equation of the second kind for the ARL on the modified EWMA control chart for the MAX(q, r) process in (15) can be approximated by using the quadrature formula. The Gauss-Legendre quadrature rule is applied as follows:

Given that

$$f(a_j) = f\left\{\begin{array}{l} \dfrac{a_j - (1-\lambda)a_i}{(\lambda+k)} \\ + \dfrac{kY_{t-1}}{(\lambda+k)} - \mu + \theta_1\varepsilon_{t-1} + \ldots \\ + \theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i x_{it} \end{array}\right\} \quad (18)$$

The approximation for the integral is in the form

$$\int_0^l F(w) f(w) dw \approx \sum_{j=1}^{m} w_j f(a_j) \quad (19)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}; j = 1, 2, \ldots, m$.

Using the Gauss-Legendre quadrature formula, numerical approximation $\tilde{F}(u)$ for the integral equations can be found as the solution to the following linear equations:

$$\tilde{F}(a_i) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{F}(a_j) f\left\{\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \mu + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i X_{it}\right\}$$

and

$$\tilde{F}(a_m) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{F}(a_j) f\left\{\frac{a_j - (1-\lambda)a_m}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \mu + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i X_{it}\right\}.$$

This set of m equations with m unknowns can be rewritten in matrix form. The column vector of $\tilde{F}(a_i)$ is $\mathbf{L}_{m\times 1} = \left(\tilde{F}(a_1), \tilde{F}(a_2), \ldots, \tilde{F}(a_m)\right)'$. Since $\mathbf{1}_{m\times 1} = (1,1,\ldots,1)'$ is a column vector of ones and $\mathbf{R}_{m\times m}$ is a matrix, we can define m to the m^{th} element of matrix $\mathbf{R}$ as follows:

$$\left[R_{ij}\right] \approx \frac{1}{\lambda+k} w_j f\left\{\begin{array}{l} \dfrac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \dfrac{kY_{t-1}}{(\lambda+k)} - \mu \\ + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i x_{it} \end{array}\right\},$$

and $\mathbf{I}_m = diag(1,1,\ldots,1)$ as a unit matrix of order m. If $(\mathbf{I} - \mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation in terms of the matrix can be written as $\mathbf{G}_{m\times 1} = \left(\mathbf{I}_m - \mathbf{R}_{m\times m}\right)^{-1}\mathbf{1}_{m\times 1}$. Finally, by substituting a_i with u in $\tilde{F}(a_i)$, the numerical integration equation for function $\tilde{F}(u)$ can be derived as

$$\tilde{F}(u) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{F}(a_j) f\left\{\frac{a_j - (1-\lambda)u}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \mu + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i X_{it}\right\}. \quad (20)$$

Here, we compare the results for ARL_0 and ARL_1 derived by using explicit formulas and the NIE method for a MAX(q,r) process with exponential white noise running on a modified EWMA chart. The numerical results were computed by using MATHEMATICA with the number of division points set as 1,000. The performances are reported as the absolute percentage relative error, which is derived as

$$APRE(\%) = \frac{\left|ARL_{Explicit\ Formula} - ARL_{NIE}\right|}{ARL_{Explicit\ Formula}} \times 100$$

For comparison, the performance measure for the ARL of a MAX(q,r) process with exponential white noise on the EWMA and modified EWMA control chart is the *RMI*, which is computed as

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i} - Min[ARL_{shift,i}]}{Min[ARL_{shift,i}]}\right)$$

where $ARL_{shift,i}$ is the ARL of the control chart when a shift in the process mean is detected and $Min[ARL_{shift,i}]$ is the minimum value of the ARL at the same level.

4 Numerical Results

The results from Table 1 (Appendix) and Table 2 (Appendix) allocated for upper control limit (*l*) which is related to the implementation process running on modified EWMA control charts as reported for MAX(2,1) and MAX(3,2) respectively, also compare both Table 1 (Appendix) and Table 2 (Appendix) reveal different results of θ_i. The outcomes for the one-sided ARL when using the explicit formulas and the NIE method to verify Table 3 (Appendix) and Table 4 (Appendix) define ARL_0 = 370, and λ = 0.05, 0.10, 0.15, and 0.20 described in Table 3 (Appendix) was β_1=2.5 and Table 4 (Appendix) was β_1=1 and β_2 =3 are especially in conditions that reveal different results θ_i. Moreover, the CPU time for the explicit formulas was minuscule spending less than 1 second while that for the NIE method was around 9 seconds, and the central processing unit (CPU) time (System: AMD Ryzen 7 5700U with Radeon Graphics@1.8GHz. Processor, 16GB RAM. 64-bit Operating System).

According to a comparison of the ARL values between EWMA and modified EWMA control charts processes for a MAX(2,3) wherever ARL_0 = 370, $\beta_1 = -1$, $\beta_2 = 2$, $\beta_3 = 3$, $X_1 = X_2 = X_3 = 1$, θ_1 =0.10, and θ_2 =0.20 found that $\lambda = 0.05$ the accomplishment of EWMA control chart it's better than modified EWMA control chart for *k*=0.5 when shift size more than or equal to initiative 0.3 Furthermore, *k*=1, *k*=5 and *k*=10 found that performance of EWMA greater than modified EWMA at shift size was 0.4. For $\lambda = 0.10$ modified EWMA reveal that performance was better than EWMA in all of the shift size from Table 5 (Appendix). According to the data presented in Figure 1, the investigation of ARL values for EWMA and modified EWMA control charts shows that the modified EWMA performs more efficiently than the standard EWMA as the parameter *k* increases.

Application*: *Example 1

At this moment, we allocated the closing price of natural gas with the crude oil WTI price as the exogenous variable from 1 July to 31 August 2022, as summarized in Table 6 (Appendix). The performance of the modified EWMA was better than that of the EWMA control chart except for shift size = 0.4 for all *k*. Moreover, the results show that the modified EWMA control chart less than RMI values was under the EWMA control chart. From Figure 2, the ARL of the EWMA and modified EWMA control charts observation research found that the efficiency of modified EWMA was better than EWMA when *k* increased. Finally, the ARL and the RMI values trended to decrease when *k* increases from Figure 3.

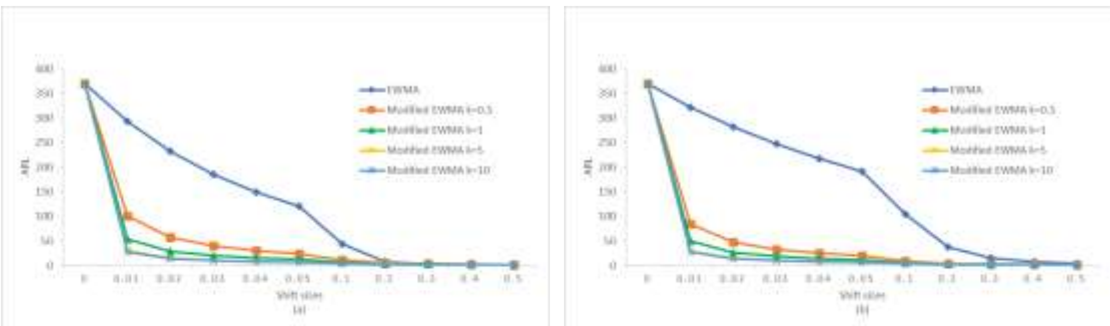


Fig. 1: The ARL of the EWMA and modified EWMA control charts simulation data for (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$

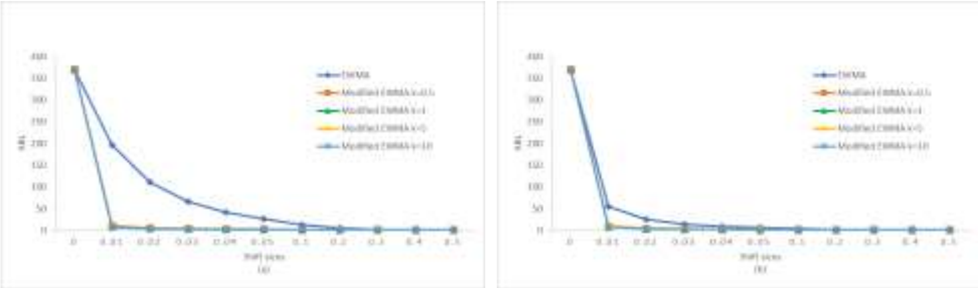
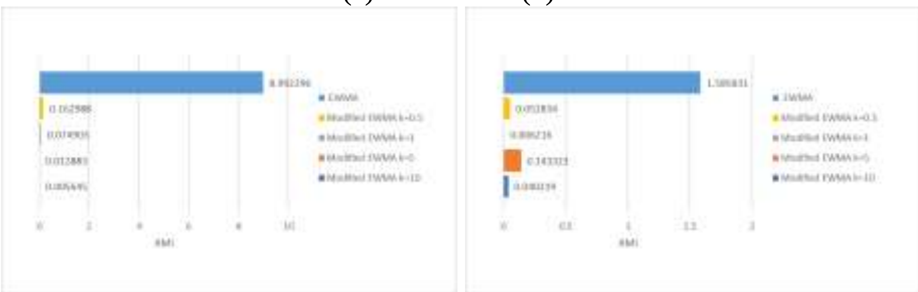


Fig. 2: The ARL of the EWMA and modified EWMA control charts for example 1 when (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$



(a) (b)

Fig. 3: The RMI values of the EWMA and modified EWMA control charts for example 1 when (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$

(a) (b)

Fig. 4: The RMI values of the EWMA and modified EWMA control charts for example 2 when (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$

Application*: *Example 2

At this time, we spend the closing stock price for KTB Public Company Limited with the USD/JPY and EUR/USD are daily foreign exchange rates as the exogenous variable from 1 August to 15 September 2022, as performed in Table 7 (Appendix). The accomplishment

of the EWMA was better than that of the modified EWMA control chart for all shift sizes except for $k=-\lambda/4$. Furthermore, the results show that the modified EWMA control chart revealed RMI larger values than the EWMA control chart except result for $k=-\lambda/4$ from Figure 4.

5 Conclusion

The Explicit formulas were verified for the ARL of a MAX(q,r) process with exponential white noise running on a modified EWMA control chart. The results from notifying the upper control limit which is related to implementing process running on modified EWMA control charts as performed for a MAX(q,r). The precision of the proposed explicit formulas was suggested as the absolute in terms of percentage difference deviation when compared with the NIE method. Acknowledgeable, they were using code and fasting to calculate the way the CPU time to point out was less than the NIE method. The practical applies to real data for the MAX(q,r) process which results show that a modified EWMA control chart is good for notating small shifts in the process mean. Particularly, in some cases could be chosen some specific $k=-3/4$ for modified EWMA control chart performed outcomes greater than the EWMA control chart.

In this specific study, we chose *k* values of 0.5, 1, 5, and 10. It is clear that higher values of k result in improved detection performance. However, in real-world data applications, certain lower *k* values, especially *k*=1, demonstrate efficiency comparable to their larger counterparts. Furthermore, even when λ values vary for different shift sizes, the ARL remains consistent.

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APPENDIX

Table 1. Upper control limit values (l) for a MAX(2,1) process running on modified EWMA control charts when ARL_0 = 370, β_1=2.5 and X =1

Parameter	λ	Modified EWMA			
		k = 0.5	k = 1	k = 5	k = 10
θ_1 = 0.30	0.05	0.24913371	0.500416482	2.50745388	5.0162278
θ_2 = 0.50	0.10	0.25503099	0.507821090	2.53911469	5.0799058
	0.15	0.26185780	0.515851454	2.57146878	5.1452603
	0.20	0.26942969	0.524444580	2.60454198	5.2123718
θ_1 = 0.30	0.05	0.09084219	0.182566560	0.91485655	1.8301810
θ_2 = -0.50	0.10	0.09245095	0.183978770	0.91910885	1.8385789
	0.15	0.09441432	0.185652263	0.92341818	1.8470413
	0.20	0.09665296	0.187550392	0.92778394	1.8555688
θ_1 = 0.20	0.05	0.18389680	0.36945555	1.8512819	3.7035240
θ_2 = 0.30	0.10	0.18779194	0.37383657	1.8685122	3.7380478
	0.15	0.19238343	0.37869642	1.8860113	3.7731850
	0.20	0.19752500	0.38397799	1.9037848	3.8089550
θ_1 = -0.20	0.05	0.12284740	0.24685785	1.2370013	2.4746377
θ_2 = 0.30	0.10	0.12516834	0.24911515	1.2447209	2.4899964
	0.15	0.12796327	0.25171206	1.2525365	2.5055183
	0.20	0.13112825	0.25460318	1.2604481	2.5212067

Table 2**.** Upper control limit values (l) for a MAX(3,2) process running on modified EWMA control charts when ARL_0 = 370, β_1=1, β_2 =3, X_1 =1 and X_2 =1

Parameter	λ	Modified EWMA			
		k = 0.5	k = 1	k = 5	k = 10
θ_1 = 0.10	0.05	0.05498546	0.11052074	0.55384312	1.10796860
θ_2 = 0.20	0.10	0.05588707	0.11120248	0.55543597	1.11105809
θ_3 = 0.50	0.15	0.05700636	0.11205043	0.55706280	1.11416699
	0.20	0.05829371	0.11304018	0.55872267	1.11729532
θ_1 = -0.10	0.05	0.011073177	0.022261258	0.11156007	0.22317691
θ_2 = -0.20	0.10	0.011237178	0.022356223	0.11164074	0.22331113
θ_3 = -0.50	0.15	0.011445845	0.022486974	0.11173058	0.22344968
	0.20	0.011688782	0.022648007	0.11182928	0.22359248
θ_1 = 0.70	0.05	0.1110828	0.22322707	1.11859545	2.23776429
θ_2 = 0.30	0.10	0.1131330	0.22515245	1.12491999	2.25031930
θ_3 = 0.50	0.15	0.1156136	0.22738921	1.13132329	2.26299299
	0.20	0.1184295	0.22989500	1.13780499	2.27578710
θ_1 = -0.70	0.05	0.00549699	0.011051296	0.055382745	0.11079367
θ_2 = -0.30	0.10	0.00557731	0.011095784	0.055407725	0.11082969
θ_3 = -0.50	0.15	0.00567985	0.011158195	0.055437504	0.11086808
	0.20	0.00579944	0.011235759	0.055471902	0.11090879

Table 3. The one-sided ARL for a MAX(2,1) process running on the modified EWMA chart when ARL_0 = 370, β_1 =2.5, X =1, and k =1

λ	l	θ_i	Shift size	ARL Explicit (time: <0.001)	ARL NIE	NIE (time: s.)	$APRE$
0.05	0.500416482	θ_1=0.30	0.00	370.00004	370.00003	(9.109)	0.00000138
		θ_2=0.50	0.001	279.55999	279.55998	(9.141)	0.00000127
			0.003	187.77302	187.77302	(9.125)	0.00000116
			0.005	141.36583	141.36583	(9.266)	0.00000110
			0.007	113.35555	113.35555	(9.141)	0.00000106
			0.01	87.38934	87.38934	(9.187)	0.00000102
			0.03	34.61916	34.61916	(9.297)	0.00000090
			0.05	21.62573	21.62573	(9.203)	0.00000084
			0.07	15.75544	15.75544	(9.125)	0.00000079
			0.10	11.23569	11.23569	(9.187)	0.00000072
			0.30	4.08730	4.08730	(9.297)	0.00000043
			0.50	2.70231	2.70231	(9.016)	0.00000027
			0.70	2.13800	2.13800	(9.140)	0.00000017
			1.00	1.73869	1.73869	(9.313)	0.00000010
0.10	0.18397877	θ_1= 0.30	0.00	370.00001	370.00001	(9.226)	0.00000032
		θ_2= –0.50	0.001	248.30778	248.30778	(9.047)	0.00000025
			0.003	149.78596	149.78596	(9.094)	0.00000020
			0.005	107.24114	107.24114	(9.141)	0.00000017
			0.007	83.52199	83.52199	(9.234)	0.00000016
			0.01	62.71985	62.71985	(9.046)	0.00000015
			0.03	23.61055	23.61055	(9.079)	0.00000012
			0.05	14.58250	14.58250	(9.203)	0.00000011
			0.07	10.58087	10.58087	(9.312)	0.00000010
			0.10	7.53611	7.53611	(9.063)	0.00000009
			0.30	2.82179	2.82179	(9.109)	0.00000004
			0.50	1.94746	1.94746	(9.406)	0.00000002
			0.70	1.60449	1.60449	(9.203)	0.00000001
			1.00	1.37145	1.37145	(9.203)	0.00000001
0.15	0.37869642	θ_1= 0.20	0.00	370.00003	370.00157	(8.907)	0.00041399
		θ_2= 0.30	0.001	259.39509	259.39584	(9.219)	0.00029015
			0.003	162.38681	162.38710	(9.046)	0.00018153
			0.005	118.22208	118.22224	(9.109)	0.00013208
			0.007	92.96597	92.96607	(9.141)	0.00010380
			0.01	70.42923	70.42928	(9.188)	0.00007856
			0.03	27.04125	27.04126	(9.328)	0.00002997
			0.05	16.82440	16.82440	(9.093)	0.00001851
			0.07	12.26643	12.26643	(9.094)	0.00001340
			0.10	8.78158	8.78158	(9.125)	0.00000947
			0.30	3.31637	3.31637	(9.250)	0.00000321
			0.50	2.26670	2.26670	(9.219)	0.00000193
			0.70	1.84116	1.84116	(9.047)	0.00000136
			1.00	1.54163	1.54163	(9.282)	0.00000093
0.20	0.2546031792	θ_1= –0.20	0.00	370.00004	370.00003	(9.110)	0.00000173
		θ_2= 0.30	0.001	242.53585	242.53584	(9.093)	0.00000119
			0.003	143.66407	143.66407	(9.141)	0.00000078
			0.005	102.10439	102.10439	(9.250)	0.00000060
			0.007	79.22260	79.22260	(9.140)	0.00000050
			0.01	59.32160	59.32160	(9.094)	0.00000042
			0.03	22.31806	22.31806	(9.203)	0.00000025
			0.05	13.84747	13.84747	(9.234)	0.00000020
			0.07	10.09879	10.09879	(9.109)	0.00000018
			0.10	7.24638	7.24638	(9.360)	0.00000015
			0.30	2.80812	2.80812	(9.125)	0.00000007
			0.50	1.96840	1.96840	(9.062)	0.00000004
			0.70	1.63237	1.63237	(9.156)	0.00000002
			1.00	1.39916	1.39916	(9.282)	0.00000001

Table 4. The one-sided ARL for a MAX(3,2) process running on the modified EWMA chart when ARL_0 = 370, $\beta_1 = 1$, $\beta_2 = 3$, $X_1 = 1$, $X_2 = 1$ and $k = 1$

λ	l	θ_i	Shift size	ARL Explicit (time: <0.001 s.)	ARL NIE	NIE (time: s.)	$APRE$
0.05	0.11052074	θ_1=0.10	0.00	370.00018	370.00018	(9.141)	0.000000064
		θ_2=0.20	0.001	244.71619	244.71619	(9.125)	0.000000058
		θ_3=0.50	0.003	145.86586	145.86586	(9.250)	0.000000053
			0.005	103.87154	103.87154	(9.421)	0.000000050
			0.007	80.63748	80.63748	(9.156)	0.000000049
			0.01	60.36506	60.36506	(9.250)	0.000000047
			0.03	22.51801	22.51801	(9.219)	0.000000043
			0.05	13.83562	13.83562	(9.172)	0.000000039
			0.07	9.99715	9.99715	(9.171)	0.000000037
			0.10	7.08390	7.08390	(9.281)	0.000000032
			0.30	2.61368	2.61368	(9.125)	0.000000019
			0.50	1.80732	1.80732	(9.141)	0.000000006
			0.70	1.49929	1.49929	(9.250)	0.000000007
			1.00	1.29579	1.29579	(9.328)	0.000000000
0.10	0.022356223	θ_1= –0.10	0.00	370.00055	370.00055	(8.969)	0.000000005
		θ_2= –0.20	0.001	204.74900	204.74900	(9.141)	0.000000003
		θ_3= –0.50	0.003	108.12396	108.12396	(9.094)	0.000000002
			0.005	73.44573	73.44573	(9.110)	0.000000002
			0.007	55.60454	55.60454	(9.250)	0.000000002
			0.01	40.75057	40.75057	(9.281)	0.000000002
			0.03	14.65928	14.65928	(9.109)	0.000000002
			0.05	8.96310	8.96310	(9.110)	0.000000001
			0.07	6.48300	6.48300	(9.156)	0.000000000
			0.10	4.62131	4.62131	(9.250)	0.000000002
			0.30	1.84333	1.84333	(9.156)	0.000000000
			0.50	1.37951	1.37951	(9.125)	0.000000000
			0.70	1.21500	1.21500	(9.141)	0.000000000
			1.00	1.11464	1.11464	(9.219)	0.000000000
0.15	0.22738921	θ_1=0.7	0.00	370.00029	370.00028	(9.078)	0.000000862
		θ_2=0.3	0.001	246.28581	246.28581	(9.219)	0.000000626
		θ_3=0.5	0.003	147.62694	147.62694	(9.203)	0.000000437
			0.005	105.43034	105.43034	(9.140)	0.000000355
			0.007	82.01068	82.01068	(9.219)	0.000000309
			0.01	61.53276	61.53276	(9.016)	0.000000269
			0.03	23.18543	23.18543	(9.172)	0.000000185
			0.05	14.35886	14.35886	(9.265)	0.000000159
			0.07	10.44791	10.44791	(9.266)	0.000000143
			0.10	7.47111	7.47111	(9.109)	0.000000126
			0.30	2.84641	2.84641	(9.172)	0.000000063
			0.50	1.97790	1.97790	(9.219)	0.000000035
			0.70	1.63298	1.63298	(9.219)	0.000000024
			1.00	1.39553	1.39553	(9.093)	0.000000014
0.20	0.0112357586	θ_1= –0.7	0.00	370.00057	370.00057	(8.953)	0.000000003
		θ_2= –0.3	0.001	177.84872	177.84872	(9.204)	0.000000002
		θ_3= –0.5	0.003	87.28694	87.28694	(9.187)	0.000000001
			0.005	57.86667	57.86667	(9.156)	0.000000001
			0.007	43.29746	43.29746	(9.125)	0.000000001
			0.01	31.44761	31.44761	(9.156)	0.000000000
			0.03	11.22574	11.22574	(9.047)	0.000000000
			0.05	6.91007	6.91007	(9.141)	0.000000000
			0.07	5.04318	5.04318	(9.187)	0.000000000
			0.10	3.64789	3.64789	(9.250)	0.000000000
			0.30	1.58725	1.58725	(9.047)	0.000000000
			0.50	1.25331	1.25331	(9.141)	0.000000000
			0.70	1.13841	1.13841	(9.125)	0.000000000
			1.00	1.07063	1.07063	(9.266)	0.000000000

Table 5. Comparison of the ARL values for a MAX(2,3) process running on EWMA and modified EWMA control

λ	Shift size	EWMA	Modified EWMA			
			k=0.5	k=1	k=5	k=10
		$h=2.55571\times10^{-9}$	l =0.033308641	l =0.0669569	l =0.335539028	l =0.671248923
0.05	0.00	370	370	370	370	370
	0.01	292.80788	101.16734	54.32554	29.66845	27.35029
	0.02	232.80528	57.67087	29.25695	15.75359	14.52804
	0.03	185.94795	39.90579	20.00313	10.86156	10.04007
	0.04	149.19079	30.27319	15.19340	8.36644	7.75483
	0.05	120.23014	24.24248	12.25001	6.85416	6.37090
	0.1	43.59987	11.68273	6.25939	3.80200	3.57955
	0.2	7.99463	5.38844	3.28070	2.27957	2.18619
	0.3	2.50677	**3.44416**	2.33687	1.78539	1.73250
	0.4	1.40197	**2.56320**	**1.89449**	**1.54697**	**1.51283**
	0.5	1.12728	**2.08603**	**1.64657**	**1.40944**	**1.38565**
RMI		8.422293	1.886133	0.662234	0.081343	0.028025
λ	Shift size	h=0.0001077388	l =0.033828632	l =0.067306441	l =0.336146642	l =0.672394345
0.10	0.00	370	370	370	370	370
	0.01	322.51351	84.86241	50.19911	29.35727	27.24036
	0.02	281.89360	47.30034	26.92598	15.58778	14.46976
	0.03	247.04728	32.50870	18.40651	10.74975	10.00081
	0.04	217.06999	24.61424	13.99326	8.28274	7.72544
	0.05	191.21124	19.71478	11.29728	6.78767	6.34754
	0.1	105.07854	9.60684	5.81994	3.77059	3.56846
	0.2	37.13064	4.57126	3.09828	2.26568	2.18124
	0.3	15.69653	3.00955	2.23433	1.77710	1.72953
	0.4	7.76816	2.29701	1.82841	1.54134	1.51079
	0.5	4.44225	1.90831	1.60038	1.40531	1.38414
RMI		15.285597	1.395811	0.531361	0.048156	0.000000

Table 6. Comparison of the ARL values for a MAX(2,1) process running on EWMA and modified EWMA control charts when ARL_0 = 370, $\beta_1 = 0.097$, $X_1 = 1$, $\theta_1 = -1.223$ and $\theta_2 = -0.699$

λ	Shift size	EWMA	Modified EWMA			
			k=0.5	k=1	k=5	k=10
		h=0.000001869	l=0.000928768	l =0.005070046	l =0.0300315284	l =0.06096242
0.05	0.00	370	370	370	370	370
	0.01	196.318682	**9.879934**	**7.893869**	**6.525167**	**6.367563**
	0.02	111.368549	**5.156209**	**4.258706**	**3.635864**	**3.563844**
	0.03	66.583664	**3.580090**	**3.045727**	**2.671272**	**2.627740**
	0.04	41.597636	**2.803836**	**2.446131**	**2.192959**	**2.163366**
	0.05	27.011222	**2.348224**	**2.092422**	**1.909602**	**1.888120**
	0.10	12.631543	**1.847012**	**1.700065**	**1.593096**	**1.580402**
	0.20	5.120406	**1.498224**	**1.422805**	**1.366535**	**1.359769**
	0.30	1.334549	**1.158702**	**1.143858**	**1.132074**	**1.130610**
	0.40	1.063653	1.077259	1.073194	1.069828	1.069400
	0.50	1.019385	1.046045	1.045018	1.044139	1.044025
RMI		8.992296	0.162988	0.074903	0.012883	0.005645
λ	Shift size	h=0.000014193	l =0.001181441	l =0.0046159374	l =0.0292793555	l =0.0601950977
0.10	0.00	370	370	370	370	370
	0.01	55.41693832	**8.767793**	**7.475860**	**10.501405**	**8.069057**
	0.02	24.81047391	**4.621807**	**4.053835**	**5.469681**	**4.351070**
	0.03	14.16833753	**3.241983**	**2.913299**	**3.787001**	**3.109271**
	0.04	9.131643368	**2.563354**	**2.349964**	**2.956700**	**2.494832**
	0.05	6.367520148	**2.165516**	**2.017906**	**2.468416**	**2.131976**
	0.10	3.654008003	**1.728601**	**1.650002**	**1.929678**	**1.728795**
	0.20	2.124412983	**1.425515**	**1.390603**	**1.552740**	**1.442986**
	0.30	1.153776272	**1.132785**	**1.131058**	**1.181233**	**1.153324**
	0.40	1.042066983	1.063647	1.065998	1.090083	1.078967
	0.50	1.016593125	1.037492	1.040280	1.054521	1.049006
RMI		1.585831	0.052834	0.006216	0.143323	0.040219

Table 7. Comparison of the ARL values for a MAX(2,2) process running on EWMA and modified EWMA control charts when $ARL_0 = 370$, $\beta_1 = 0.078$, $\beta_2 = 5.663$ $X_1 = 35$, $X_2 = 1$, $\theta_1 = -0.831$ and $\theta_2 = -0.837$

λ	Shift size	EWMA	Modified EWMA				
			$k = -\frac{3}{4}$	k=0.5	k=1	k=5	k=10
		h = 1.395702×10^{-12}	l = 8.28088×10^{-14}	l = 1.55245×10^{-8}	l = 4.12034×10^{-8}	l = 2.64061×10^{-7}	l = 5.45583×10^{-7}
0.05	0.000	370	370	370	370	370	370
	0.0001	55.918932	**53.406181**	67.259655	68.038452	68.740983	68.833544
	0.0002	30.439245	**28.922156**	37.217006	37.689487	38.116837	38.173492
	0.0003	21.007965	**19.919012**	25.833354	26.171546	26.477718	26.518395
	0.0004	16.090669	**15.264063**	19.844924	20.108433	20.347094	20.378836
	0.0005	13.075530	**12.391242**	16.151187	16.367230	16.562943	16.588986
	0.0010	6.892126	**6.522190**	8.527914	8.642525	8.746367	8.760201
	0.0020	3.732005	**3.532678**	4.599306	4.659564	4.714135	4.721406
	0.0030	2.677330	**2.537740**	3.277312	3.318731	3.356223	3.361219
	0.0040	2.155613	**2.047267**	2.617535	2.649298	2.678041	2.681870
	0.0050	1.847935	**1.759412**	2.224267	2.250101	2.273474	2.276588
RMI		0.048731	0.001852	0.262755	0.277648	0.291120	0.292905
λ	Shift size	h= 2.79206×10^{-12}	l = 1.65666×10^{-13}	l = 9.51527×10^{-9}	l = 3.10497×10^{-8}	l = 2.47686×10^{-7}	l = 5.28124×10^{-7}
0.10	0.000	370	370	370	370	370	370
	0.0001	52.47239	**48.914789**	65.291415	66.884843	68.478415	68.702274
	0.0002	28.45630	**26.394065**	36.031611	36.992373	37.956722	38.092651
	0.0003	19.62371	**18.191344**	24.988924	25.674280	26.363133	26.460350
	0.0004	15.03460	**13.935765**	19.189626	19.722242	20.257953	20.333604
	0.0005	12.22358	**11.324846**	15.615910	16.051605	16.490011	16.551944
	0.0010	6.47125	**6.002218**	8.248586	8.477580	8.708160	8.740756
	0.0020	3.53672	**3.293741**	4.456621	4.575165	4.694545	4.711425
	0.0030	2.55796	**2.393403**	3.181705	3.262097	3.343061	3.354509
	0.0040	2.07385	**1.949563**	2.545875	2.606789	2.668148	2.676826
	0.0050	1.78832	**1.689018**	2.167173	2.216185	2.265572	2.272558
RMI		0.066618	0.000442	0.315924	0.347874	0.380027	0.384565

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Sittikorn Khamrod carried out the simulation of Section 4.
- Yupaporn Areepong has organized the conceptualization and validation
- Saowanit Sukparungsee has implemented the methodology and software.
- Rapin Sunthornwat was responsible for Mathematics and Statistics.

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Conflicts of Interest

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