

Simple Closed Analytic Formulas to approximate the First Two Legendre's Complete Elliptic Integrals by a Fast Converging Recurrent-Iterative Scheme

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Abstract: Two sets of closed *analytic* functions are proposed for the approximate calculus of the complete elliptic integrals $K(k)$ and $E(k)$ in the normal form due to Legendre, their expressions having a remarkable simplicity and accuracy. The special usefulness of the newly proposed formulas consists in they allow performing the *analytic study of variation* of the functions in which they appear, *using derivatives* (they being expressed in terms of elementary functions only, **without any special function; this would mean replacing one difficulty by another of the same kind**). Comparative tables of so found approximate values with the exact ones, reproduced from special functions tables, are given (vs. the elliptic integrals' modulus k). The 1st set of formulas was suggested by *Peano's law on ellipse's perimeter*. The new functions and their derivatives coincide with the exact ones at the left domain's end only. As for their simplicity, the formulas in k/k' do not need mathematical tables (are purely algebraic). As for accuracy, the 2nd set, more intricate, gives more accurate values and extends itself more closely to the right domain's end. An original fast converging recurrent-iterative scheme to get sets of formulas with the desired accuracy is given in appendix.

Key-Words: analytic methods; Legendre complete elliptic integrals $K(k)$ and $E(k)$; elliptic integrals' moduli k and complementary k' ; tables of Legendre complete elliptic integrals; recurrent-iterative scheme; Peano's approximate law for the ellipse perimeter

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1 Introduction – elliptic integrals

There are many interesting domains in pure and applied mathematics where appear both (or, often, only one) complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre. The arc length of a lemniscate, as well as the period of oscillations in a vacuum of the simple pendulum, in the dynamics of a constrained heavy particle, are given by a complete elliptic integral of the 1st kind. The perimeter of an ellipse, as well as the lift coefficient of a thin delta wing with subsonic leading edges, in supersonic aerodynamics (small perturbations theory), are given by a complete elliptic integral of the 2nd kind. In electromagnetic theory, the electric and magnetic fields from a circular coil can be expressed using the complete elliptic integrals. The relations below define the integrals of the 1st and 2nd kind, in *canonical* form, $K(k)$ and $E(k)$, resp.:
 $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-1/2} d\varphi = \int_0^1 [(1 - t^2)(1 - k^2 t^2)]^{-1/2} dt$,
 $E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{1/2} d\varphi = \int_0^1 [(1 - t^2)(1 - k^2 t^2)]^{1/2} dt$,
 $k = \sin \theta \geq 0$ is called *modulus*. $K(k)$, $E(k)$ are typical *elliptic* integrals. They do not admit primitive functions (cannot be expressed in terms of elementary functions), being calculated by expanding the integrands into series, integrating term-by-term, and presented vs. $k \in [0, 1]$, or vs. $\theta \in [0, \pi/2]$, in some mathematical tables [1]–[6]. Modern mathematics defines an elliptic integral as any function f which can be expressed in the form $f(x) = \int_c^x R[t, P(t)]^{1/2} dt$, R is a rational function of its two arguments; P is a polynomial of degree 3 or 4 with no repeated roots; c is a constant. The values given in some special tables allow performing the calculus for a

given case (point), but not the analytic study of variation of the functions in which these integrals appear, using the derivatives. In chapter 2 *two original sets* (subscripts 0; 1) of *closed analytic functions* are given for the approximate calculus of $K(k)$ and $E(k)$. We use an *original purely analytic method* (not some numerical, or sophisticated computer programs, like most authors). There also is a Legendre complete elliptic integral of the 3rd kind. With an appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the *three Legendre canonical forms* (of the 1st, 2nd & 3rd kind).

2 The two sets of newly proposed formulas

The *complementary modulus* is $k' = (1 - k^2)^{1/2} = \cos \theta$. The $E_0(k)$ formula in the 1st set (K_0 , E_0) is suggested by Peano's law.

$$K_0(k) = \frac{\pi}{4\sqrt{1-k^2}} \left(1 - \frac{1}{2\sqrt{2}} \sqrt{\frac{1+\sqrt{1-k^2}}{1-k^2}} \right) = \pi \left(\frac{1}{\sqrt{k'}} - \frac{1}{2\sqrt{2}} \frac{1}{k'^{3/4}} \right),$$

$$K_0(\theta) = \frac{\pi}{\cos^{3/2} \theta} \left(1 - \frac{1}{2} \frac{\cos(\theta/2)}{\cos^{3/4} \theta} \right) = \pi \left(\frac{1}{\cos^{3/2} \theta} - \frac{1}{2} \frac{\cos(\theta/2)}{\cos^{3/4} \theta} \right).$$

$$E_0(k) = \frac{\pi}{4} \sqrt{1-k^2} \left(\frac{3}{2} \frac{1+\sqrt{1-k^2}}{\sqrt{1-k^2}} - 1 \right) = \frac{\pi}{4} \left[\frac{3}{2} (1+k') - \sqrt{k'} \right],$$

$$E_0(\theta) = \frac{\pi}{4} \cos^{3/2} \theta \left(3 \frac{\cos^2(\theta/2)}{\cos^{3/2} \theta} - 1 \right) = \frac{\pi}{4} \left(3 \cos^2 \frac{\theta}{2} - \sqrt{\cos \theta} \right).$$

Similarly, for the 2nd set (K_1 , E_1), are proposed the formulas:

$$K_1(k) = \frac{\pi\sqrt{2}}{\sqrt{(1+k')\sqrt{k'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1+\sqrt{k'}}{\sqrt[4]{(1+k')\sqrt{k'}}} \right),$$

$$K_1(\theta) = \frac{\pi}{\cos(\theta/2)\cos^{3/4}\theta} \left(1 - \frac{1}{4} \frac{1+\cos^{1/2}\theta}{\cos^{1/2}(\theta/2)\cos^{1/8}\theta} \right).$$

$$E_1(k) = \frac{\pi}{4} \left[\frac{3}{2} (1+\sqrt{k'})^2 - \sqrt{2} \sqrt{1+k'} \sqrt[4]{k'} \right] - k' \cdot K_1(k),$$

$$E_1(\theta) = \frac{\pi}{4} \left[\frac{3}{2} (1+\sqrt{\cos\theta})^2 - 2\cos\frac{\theta}{2} \sqrt[4]{\cos\theta} \right] - \cos\theta \cdot K_1(\theta).$$

A 3rd set (K₂, E₂), even more accurate than previous two sets, can be built (a recurrent-iterative scheme) – see appendix.

Table 1. Values of the functions K (part one)

$\theta(^{\circ})$	$k = \sin\theta$	$K(k)$	$K_0(k)$	$K_1(k)$					
0	0.00000	1.5708	1.5708	1.5708	41	0.65606	1.7992	1.7992	1.7992
1	0.01745	1.5709	1.5709	1.5709	42	0.66913	1.8122	1.8121	1.8122
2	0.03490	1.5713	1.5713	1.5713	43	0.68200	1.8256	1.8256	1.8256
3	0.05234	1.5719	1.5719	1.5719	44	0.69466	1.8396	1.8395	1.8396
4	0.06976	1.5727	1.5727	1.5727	45	0.70711	1.8541	1.8540	1.8541
5	0.08716	1.5738	1.5738	1.5738	46	0.71934	1.8691	1.8691	1.8691
6	0.10453	1.5751	1.5751	1.5751	47	0.73135	1.8848	1.8847	1.8848
7	0.12187	1.5767	1.5767	1.5767	48	0.74314	1.9011	1.9009	1.9011
8	0.13917	1.5785	1.5785	1.5785	49	0.75471	1.9180	1.9178	1.9180
9	0.15643	1.5805	1.5805	1.5805	50	0.76604	1.9356	1.9354	1.9356
10	0.17365	1.5828	1.5828	1.5828	51	0.77715	1.9539	1.9536	1.9539
11	0.19081	1.5854	1.5854	1.5854	52	0.78801	1.9729	1.9726	1.9729
12	0.20791	1.5882	1.5882	1.5882	53	0.79864	1.9927	1.9923	1.9927
13	0.22495	1.5913	1.5913	1.5913	54	0.80902	2.0133	2.0128	2.0133
14	0.24192	1.5946	1.5946	1.5946	55	0.81915	2.0347	2.0341	2.0347
15	0.25882	1.5981	1.5981	1.5981	56	0.82904	2.0571	2.0564	2.0571
16	0.27564	1.6020	1.6020	1.6020	57	0.83867	2.0804	2.0795	2.0804
17	0.29237	1.6061	1.6061	1.6061	58	0.84805	2.1047	2.1037	2.1047
18	0.30902	1.6105	1.6105	1.6105	59	0.85717	2.1300	2.1288	2.1300
19	0.32557	1.6151	1.6151	1.6151	60	0.86603	2.1565	2.1551	2.1565
20	0.34202	1.6200	1.6200	1.6200	61	0.87462	2.1842	2.1825	2.1842
21	0.35837	1.6252	1.6252	1.6252	62	0.88295	2.2132	2.2111	2.2132
22	0.37461	1.6307	1.6307	1.6307	63	0.89101	2.2435	2.2410	2.2435
23	0.39073	1.6365	1.6365	1.6365	64	0.89879	2.2754	2.2723	2.2754
24	0.40674	1.6426	1.6426	1.6426	65	0.90631	2.3088	2.3051	2.3088
25	0.42262	1.6490	1.6490	1.6490	66	0.91355	2.3439	2.3394	2.3439
26	0.43837	1.6557	1.6557	1.6557	67	0.92050	2.3809	2.3754	2.3809
27	0.45399	1.6627	1.6627	1.6627	68	0.92718	2.4198	2.4132	2.4198
28	0.46947	1.6701	1.6701	1.6701	69	0.93358	2.4610	2.4530	2.4610
29	0.48481	1.6777	1.6777	1.6777	70	0.93969	2.5046	2.4948	2.5045
30	0.50000	1.6858	1.6857	1.6858	70.5	0.94264	2.5273	2.5165	2.5273
31	0.51504	1.6941	1.6941	1.6941	71	0.94552	2.5507	2.5389	2.5507
32	0.52992	1.7028	1.7028	1.7028	71.5	0.94832	2.5749		2.5749
33	0.54464	1.7119	1.7119	1.7119	72	0.95106	2.5998		2.5998
34	0.55919	1.7214	1.7214	1.7214	72.5	0.95372	2.6256		2.6255
35	0.57358	1.7312	1.7312	1.7312	73	0.95630	2.6521		2.6521
36	0.58779	1.7415	1.7415	1.7415	73.5	0.95882	2.6796		2.6796
37	0.60182	1.7522	1.7522	1.7522	74	0.96126	2.7081		2.7081
38	0.61566	1.7633	1.7632	1.7633	74.5	0.96363	2.7375		2.7375
39	0.62932	1.7748	1.7748	1.7748	75	0.96593	2.7681		2.7680
40	0.64279	1.7868	1.7867	1.7868	75.5	0.96815	2.7998		2.7997
					76	0.97030	2.8327		2.8326
					76.5	0.97237	2.8669		2.8669
					77	0.97437	2.9026		2.9025
					77.5	0.97630	2.9397		2.9397
					78	0.97815	2.9786		2.9785
					78.5	0.97992	3.0192		3.0191
					79	0.98163	3.0617		3.0616
					79.5	0.98325	3.1064		3.1063
					80	0.98481	3.1534		3.1533
					80.2	0.98541	3.1729		3.1727
					80.4	0.98600	3.1928		3.1927
					80.6	0.98657	3.2132		3.2130
					80.8	0.98714	3.2340		3.2338
					81	0.98769	3.2553		3.2551

Table 1. Values of the functions K (part two)

81.2	0.98823	3.2771	3.2769
81.4	0.98876	3.2995	3.2992
81.6	0.98927	3.3223	3.3221
81.8	0.98978	3.3458	3.3455
82	0.99027	3.3699	3.3696
82.2	0.99075	3.3946	3.3942
82.4	0.99122	3.4199	3.4196
82.6	0.99167	3.4460	3.4456
82.8	0.99211	3.4728	3.4724
83	0.99255	3.5004	3.4999
83.2	0.99297	3.5288	3.5283
83.4	0.99337	3.5581	3.5575
83.6	0.99377	3.5884	3.5877
83.8	0.99415	3.6196	3.6188
84	0.99452	3.6519	3.6510
84.2	0.99488	3.6852	3.6843
84.4	0.99523	3.7198	3.7187
84.6	0.99556	3.7557	3.7545
84.8	0.99588	3.7930	3.7916
85	0.99619	3.8317	3.8302
85.2	0.99649	3.8721	3.8704
85.4	0.99678	3.9142	3.9122
85.6	0.99705	3.9583	3.9560
85.8	0.99731	4.0044	4.0018
86	0.99756	4.0528	4.0498
86.2	0.99780	4.1037	4.1003
86.4	0.99803	4.1574	4.1535
86.6	0.99824	4.2142	4.2097
86.8	0.99844	4.2744	4.2692
87	0.99863	4.3387	4.3325
87.2	0.99881	4.4073	4.4001
87.4	0.99897	4.4811	4.4726
87.6	0.99912	4.5609	4.5507
87.8	0.99926	4.6477	4.6354
88	0.99939	4.7427	4.7277
88.2	0.99951	4.8478	4.8293
88.4	0.99961	4.9654	
88.6	0.99970	5.0988	
88.8	0.99978	5.2527	
89	0.99985	5.4349	
89.1	0.99988	5.5402	
89.2	0.99990	5.6579	
89.3	0.99993	5.7914	
89.4	0.99995	5.9455	
89.5	0.99996	6.1278	
89.6	0.99998	6.3509	
89.7	0.99999	6.6385	
89.8	0.99999	7.0440	
89.9	1.00000	7.7371	
90	1.00000	∞	$-\infty$

The values strings in the last two columns of table 1 were canceled when each of the two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 1st kind $K(k)$ gives too great relative

errors ($\geq 4\%$ – also see chapter 3) for being still accepted in the usual mathematical / technical calculus. The same procedure will be applied in case of the next table (no. 2), for the same reason, concerning the accuracy of the values given by each of the other two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 2nd kind $E(k)$. The accuracy analysis of the two sets of formulas will be performed in the next chapter (no. 3). In chapter 4 some series representations for the exact functions and for both sets of approximation, as well as for their first order derivatives, will be given.

Table 2. Values of the functions E (part one)

$\theta(^{\circ})$	$k = \sin \theta$	$E(k)$	$E_0(k)$	$E_1(k)$
0	0.00000	1.5708	1.5708	1.5708
1	0.01745	1.5707	1.5707	1.5707
2	0.03490	1.5703	1.5703	1.5703
3	0.05234	1.5697	1.5697	1.5697
4	0.06976	1.5689	1.5689	1.5689
5	0.08716	1.5678	1.5678	1.5678
6	0.10453	1.5665	1.5665	1.5665
7	0.12187	1.5649	1.5649	1.5649
8	0.13917	1.5632	1.5632	1.5632
9	0.15643	1.5611	1.5611	1.5611
10	0.17365	1.5589	1.5589	1.5589
11	0.19081	1.5564	1.5564	1.5564
12	0.20791	1.5537	1.5537	1.5537
13	0.22495	1.5507	1.5507	1.5507
14	0.24192	1.5476	1.5476	1.5476
15	0.25882	1.5442	1.5442	1.5442
16	0.27564	1.5405	1.5405	1.5405
17	0.29237	1.5367	1.5367	1.5367
18	0.30902	1.5326	1.5326	1.5326
19	0.32557	1.5283	1.5283	1.5283
20	0.34202	1.5238	1.5238	1.5238
21	0.35837	1.5191	1.5191	1.5191
22	0.37461	1.5141	1.5141	1.5141
23	0.39073	1.5090	1.5090	1.5090
24	0.40674	1.5037	1.5037	1.5037
25	0.42262	1.4981	1.4981	1.4981
26	0.43837	1.4924	1.4924	1.4924
27	0.45399	1.4864	1.4864	1.4864
28	0.46947	1.4803	1.4803	1.4803
29	0.48481	1.4740	1.4740	1.4740
30	0.50000	1.4675	1.4675	1.4675
31	0.51504	1.4608	1.4608	1.4608
32	0.52992	1.4539	1.4539	1.4539
33	0.54464	1.4469	1.4469	1.4469
34	0.55919	1.4397	1.4397	1.4397
35	0.57358	1.4323	1.4323	1.4323
36	0.58779	1.4248	1.4248	1.4248
37	0.60182	1.4171	1.4171	1.4171
38	0.61566	1.4092	1.4093	1.4092
39	0.62932	1.4013	1.4013	1.4013
40	0.64279	1.3931	1.3932	1.3931

Table 2. Values of the functions E (part two)

					81	0.98769	1.0338		1.0339
41	0.65606	1.3849	1.3849	1.3849	81.2	0.98823	1.0326		1.0327
42	0.66913	1.3765	1.3765	1.3765	81.4	0.98876	1.0314		1.0315
43	0.68200	1.3680	1.3680	1.3680	81.6	0.98927	1.0302		1.0303
44	0.69466	1.3594	1.3594	1.3594	81.8	0.98978	1.0290		1.0292
45	0.70711	1.3506	1.3507	1.3506	82	0.99027	1.0278		1.0280
46	0.71934	1.3418	1.3419	1.3418	82.2	0.99075	1.0267		1.0269
47	0.73135	1.3329	1.3330	1.3329	82.4	0.99122	1.0256		1.0258
48	0.74314	1.3238	1.3239	1.3238	82.6	0.99167	1.0245		1.0247
49	0.75471	1.3147	1.3148	1.3147	82.8	0.99211	1.0234		1.0236
50	0.76604	1.3055	1.3057	1.3055	83	0.99255	1.0223		1.0226
51	0.77715	1.2963	1.2964	1.2963	83.2	0.99297	1.0213		1.0215
52	0.78801	1.2870	1.2872	1.2870	83.4	0.99337	1.0202		1.0205
53	0.79864	1.2776	1.2778	1.2776	83.6	0.99377	1.0192		1.0196
54	0.80902	1.2681	1.2684	1.2681	83.8	0.99415	1.0182		1.0186
55	0.81915	1.2587	1.2590	1.2587	84	0.99452	1.0172		1.0176
56	0.82904	1.2492	1.2496	1.2492	84.2	0.99488	1.0163		1.0167
57	0.83867	1.2397	1.2401	1.2397	84.4	0.99523	1.0153		1.0158
58	0.84805	1.2301	1.2307	1.2301	84.6	0.99556	1.0144		1.0150
59	0.85717	1.2206	1.2212	1.2206	84.8	0.99588	1.0135		1.0141
60	0.86603	1.2111	1.2118	1.2111	85	0.99619	1.0127		1.0133
61	0.87462	1.2015	1.2024	1.2015	85.2	0.99649	1.0118		1.0125
62	0.88295	1.1920	1.1930	1.1920	85.4	0.99678	1.0110		1.0118
63	0.89101	1.1826	1.1838	1.1826	85.6	0.99705	1.0102		1.0110
64	0.89879	1.1732	1.1745	1.1732	85.8	0.99731	1.0094		1.0103
65	0.90631	1.1638	1.1654	1.1638	86	0.99756	1.0086		1.0097
66	0.91355	1.1545	1.1564	1.1545	86.2	0.99780	1.0079		1.0091
67	0.92050	1.1453	1.1475	1.1453	86.4	0.99803	1.0072		1.0085
68	0.92718	1.1362	1.1387	1.1362	86.6	0.99824	1.0065		1.0080
69	0.93358	1.1272	1.1301	1.1273	86.8	0.99844	1.0059		1.0075
70	0.93969	1.1184	1.1217	1.1184	87	0.99863	1.0053		1.0071
70.5	0.94264	1.1140	1.1176	1.1140	87.2	0.99881	1.0047		1.0067
71	0.94552	1.1096	1.1135	1.1096	87.4	0.99897	1.0041		1.0064
71.5	0.94832	1.1053		1.1053	87.6	0.99912	1.0036		1.0062
72	0.95106	1.1011		1.1011	87.8	0.99926	1.0031		1.0060
72.5	0.95372	1.0968		1.0968	88	0.99939	1.0026		1.0060
73	0.95630	1.0927		1.0927	88.2	0.99951	1.0021		1.0061
73.5	0.95882	1.0885		1.0885	88.4	0.99961	1.0017		
74	0.96126	1.0844		1.0844	88.6	0.99970	1.0014		
74.5	0.96363	1.0804		1.0804	88.8	0.99978	1.0010		
75	0.96593	1.0764		1.0764	89	0.99985	1.0008		
75.5	0.96815	1.0725		1.0725	89.1	0.99988	1.0006		
76	0.97030	1.0686		1.0686	89.2	0.99990	1.0005		
76.5	0.97237	1.0648		1.0648	89.3	0.99993	1.0004		
77	0.97437	1.0611		1.0611	89.4	0.99995	1.0003		
77.5	0.97630	1.0574		1.0574	89.5	0.99996	1.0002		
78	0.97815	1.0538		1.0538	89.6	0.99998	1.0001		
78.5	0.97992	1.0502		1.0503	89.7	0.99999	1.0001		
79	0.98163	1.0468		1.0468	89.8	0.99999	1.0000		
79.5	0.98325	1.0434		1.0435	89.9	1.00000	1.0000		
80	0.98481	1.0401		1.0402	90	1.00000	1.0000	1.1781	1.1781
80.2	0.98541	1.0388		1.0389	In the comparative tables 1 and 2, the 4D (four decimal digit) exact values of both Legendre complete elliptic integrals reproduced from special functions tables [6] (tab. 29, p. 117), as well as their 4D approximate values obtained by applying the two sets of closed				
80.4	0.98600	1.0375		1.0376					
80.6	0.98657	1.0363		1.0364					
80.8	0.98714	1.0350		1.0351					

analytic formulas were given (all versus the respective elliptic integrals modulus $k = \sin \theta$). It is to be noticed that both sets of approximate formulas are not given by spline or regression functions, but by asymptotic expressions, these ones having a remarkable simplicity (see, e.g.: the 2nd form of $E_0(k)$, suggested by Peano's law on ellipse's perimeter, *all newly found formulas in k/k' do not need any mathematical table*, being purely algebraic) and accuracy (see table 3). The identity with the exact functions is satisfied for the left domain's end $k=0$ ($\theta=0^\circ$). The 2nd set (K_1, E_1), although a bit more intricate, gives more accurate values than the 1st one (K_0, E_0) and arrives more closely to the right domain's end $k=1$ ($\theta=90^\circ$).

3 The accuracy of the two sets of formulas

Let us define the following relative error functions:

$$\varepsilon_{K_0}(k) = K_0(k)/K(k) - 1; \quad \varepsilon_{K_1}(k) = K_1(k)/K(k) - 1,$$

$$\varepsilon_{E_0}(k) = E_0(k)/E(k) - 1; \quad \varepsilon_{E_1}(k) = E_1(k)/E(k) - 1,$$

for both sets of approximation of the 1st and 2nd kind integrals, resp. Their values are given in table 3, expressed in thousandths (%). These errors were calculated for the 1st set (K_0, E_0) only in the field $\theta \in [54^\circ, 71^\circ]$ of the domain, with an increment of 1° , while for the 2nd set (K_1, E_1) only in the field $\theta \in [84^\circ.8, 88^\circ.2]$, with an increment of $0^\circ.2$, like in tables 1 & 2.

Table 3. Relative errors ε distribution

$\theta(^{\circ})$	$k = \sin \theta$	$\varepsilon_{K_0}(\%)$	$\varepsilon_{K_1}(\%)$	$\varepsilon_{E_0}(\%)$	$\varepsilon_{E_1}(\%)$
54	0.80902	-0.250		+0.255	
55	0.81915	-0.272		+0.243	
56	0.82904	-0.353		+0.293	
57	0.83867	-0.420		+0.334	
58	0.84805	-0.497		+0.454	
59	0.85717	-0.558		+0.502	
60	0.86603	-0.669		+0.566	
61	0.87462	-0.799		+0.742	
62	0.88295	-0.961		+0.874	
63	0.89101	-1.118		+0.973	
64	0.89879	-1.366		+1.135	
65	0.90631	-1.619		+1.377	
66	0.91355	-1.918		+1.627	
67	0.92050	-2.299		+1.900	
68	0.92718	-2.709		+2.215	
69	0.93358	-3.253		+2.573	
70	0.93969	-3.907		+2.959	
71	0.94552	-4.642		+3.525	
84.8	0.99588	-	-0.369	-	+0.607
85	0.99619	-	-0.396	-	+0.592
85.2	0.99649	-	-0.451	-	+0.705
85.4	0.99678	-	-0.500	-	+0.748
85.6	0.99705	-	-0.582	-	+0.823
85.8	0.99731	-	-0.652	-	+0.932
86	0.99756	-	-0.737	-	+1.076
86.2	0.99780	-	-0.832	-	+1.160
86.4	0.99803	-	-0.945	-	+1.284
86.6	0.99824	-	-1.077	-	+1.453

86.8	0.99844	-	-1.214	-	+1.571
87	0.99863	-	-1.421	-	+1.743
87.2	0.99881	-	-1.626	-	+1.976
87.4	0.99897	-	-1.894	-	+2.275
87.6	0.99912	-	-2.234	-	+2.553
87.8	0.99926	-	-2.655	-	+2.922
88	0.99939	-	-3.156	-	+3.397
88.2	0.99951	-	-3.808	-	+4.004

The relative errors strings are stopped for values $\geq 4\%$. One can see that both sets given in chapter 2 have a much lesser relative error for $K(k)$ than the well-known asymptotic expression: $K(k) \approx \pi/2 + (\pi/8)[k^2/(1-k^2)] - (\pi/16)[k^4/(1-k^4)]$, with a relative precision of $3 \cdot 10^{-4}$ for $k < 0.5$ ($\theta < 30^\circ$), only.

4 Comparative series representations; Legendre's functional relation

Expanding into power series, one obtains for the complete elliptic integrals the set of representations below ([5] – [7]):

$$K(k) = \frac{\pi}{2} \left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409225}{1073741824}k^{16} + \dots \right)$$

$$= \frac{\pi}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 k^{2n} \right\} = \frac{\pi}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k^{2n} \right\};$$

$$E(k) = \frac{\pi}{2} \left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760615}{1073741824}k^{16} - \dots \right)$$

$$= \frac{\pi}{2} \left\{ 1 - \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 \frac{k^{2n}}{2n-1} \right\} = \frac{\pi}{2} \left\{ 1 - \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^n n!} \right]^2 \frac{k^{2n}}{2n-1} \right\}.$$

Proceeding in the same manner, we get for the 1st set (the most inaccurate) of approximate functions the expansions

$$K_0(k) = \frac{\pi}{2} \left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1222}{16384}k^8 + \dots \right);$$

$$E_0(k) = \frac{\pi}{2} \left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{172}{16384}k^8 - \dots \right),$$

for the 2nd set being *practically identical with the exact ones*

$$K_1(k) = \frac{\pi}{2} \left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409222}{1073741824}k^{16} + \dots \right);$$

$$E_1(k) = \frac{\pi}{2} \left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760606}{1073741824}k^{16} - \dots \right).$$

The difference with respect to the expansions of the exact functions (K, E) begins at the terms in k^8 for the 1st set of approximation (K_0, E_0), and at the terms in k^{16} for the 2nd one (K_1, E_1). For the 1st derivatives of K, E we get

$$\frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} = \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 + \frac{19845}{16384} k^8 + \frac{160083}{131072} k^{10} + \frac{1288287}{1048576} k^{12} + \frac{41409225}{33554432} k^{14} + \dots \right)$$

$$= \frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 n k^{2n-1} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^{n-1} n!} \right]^2 n k^{2n-1};$$

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k} = -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{175}{1024} k^6 + \frac{2205}{16384} k^8 + \frac{14553}{131072} k^{10} + \frac{99099}{1048576} k^{12} + \frac{2760615}{33554432} k^{14} + \dots \right)$$

$$= -\frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} \right]^2 \frac{n k^{2n-1}}{2n-1} = -\frac{\pi}{4} \sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{2^{n-1} n!} \right]^2 \frac{n k^{2n-1}}{2n-1}.$$

Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions:

$$\left[\frac{dK(k)}{dk} \right]_0 = \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225.75}{1024} k^6 + \dots \right);$$

$$\left[\frac{dE(k)}{dk} \right]_0 = -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{174.25}{1024} k^6 + \dots \right),$$

for the 1st set of approximate functions (K₀, E₀), and resp.

$$\left[\frac{dK(k)}{dk} \right]_1 = \frac{\pi}{4} k \left(1 + \frac{9}{8} k^2 + \frac{75}{64} k^4 + \frac{1225}{1024} k^6 + \frac{19845}{16384} k^8 + \frac{160083}{131072} k^{10} + \frac{1288287}{1048576} k^{12} + \frac{41409226.125}{33554432} k^{14} + \dots \right);$$

$$\left[\frac{dE(k)}{dk} \right]_1 = -\frac{\pi}{4} k \left(1 + \frac{3}{8} k^2 + \frac{15}{64} k^4 + \frac{175}{1024} k^6 + \frac{2205}{16384} k^8 + \frac{14553}{131072} k^{10} + \frac{99099}{1048576} k^{12} + \frac{276061425}{33554432} k^{14} + \dots \right),$$

for the 2nd set of approximate functions (K₁, E₁). The difference with respect to the expansions of the 1st derivatives of the exact functions (K, E) begins at the terms in k⁷ for the 1st set, and at the terms in k¹⁵ for the 2nd one, so much lesser than that for the expansions of the respective sets (K_{0,1}, E_{0,1}). One can also easily find the analytic expressions and series representations for the 2nd derivatives of all K, K_{0,1}, E, E_{0,1}, with similar results, but a lesser precision than for K, E, K', E'. Besides the above definitions of the derivatives K' (= dK/dk), E' (= dE/dk), there is a useful functional relation (Legendre's): K(k)·E(k') + E(k)·K(k') - K(k)·K(k') = π/2.

5 Graphic comparison

The variation curves of Legendre complete elliptic integrals, as well as that of the two sets of closed analytic functions are graphically represented in the comparative figures 1 and 2, all vs. θ, in sexagesimal degrees, and given by θ = sin⁻¹k. In both figures the exact functions K(k), E(k) were represented by solid (continuous) black lines, the 1st set of approximation [K₀(k), E₀(k)] by dashed black lines, and the 2nd set of approximation [K₁(k), E₁(k)] by solid red lines. At k = 1 the graphs of all K_{0,1}(k) fall to (-∞); the graphs of all E_{0,1}(k) pass through (1, 3π/8).

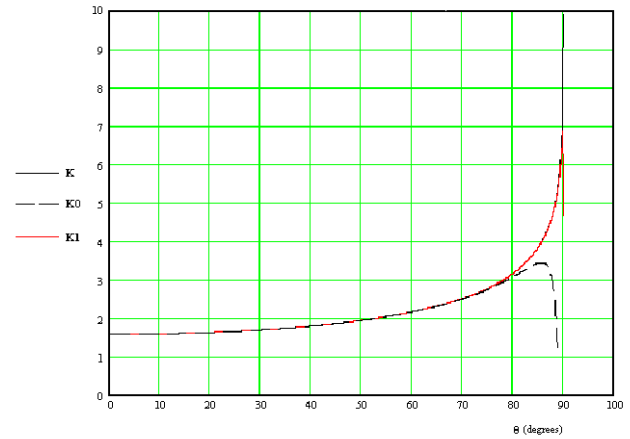


Fig. 1. Comparison of K(k) with the closed analytic functions K₀(k), K₁(k); also see the end of remark 1 in the appendix

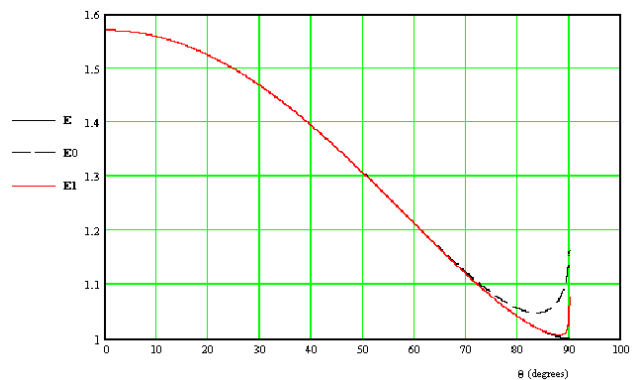


Fig. 2. Comparison of E(k) with the closed analytic functions E₀(k), E₁(k); also see the end of remark 1 in the appendix

6 Conclusions

As for simplicity, the formulas in k/k' do not need mathematical tables (are purely algebraic). As for accuracy, in mathematical/technical applications, it must use the 1st set until θ = 70°.5 (k = 0.94264) only, and (for a better accuracy or a greater upper limit of the validity domain) the 2nd set, until θ = 88°.2 (k = 0.99951).

7 Notes; other methods; future research

Without the comparative tables 1 and 2, the errors table becoming so table 1, this work was published previously in a proceedings volume (scientific bulletin, in Romanian [8]. For the first English version of this work see [9], [10]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [11], are developed in [12], [13] (a mixed numerical-analytic method). Newer formulas (using Γ function—not an elementary, but a special one, like K & E, even if these formulas are the most accurate) are in [14], [15]; as stated in their abstracts, the works [9], [14] do not have the same goal. An original fast converging recurrent-iterative scheme to get a 3rd (and higher) set of closed analytic formulas (seemingly intricate) with desired accuracy is given in article's appendix. This article represents a fully extended version of the paper [9]. Notable special functions suitable for applying such an approximate method of calculation are: Si(x); Ci(x); Ei(x); li(x).

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Appendix – A fast converging recurrent-iterative scheme to get a 3rd (and higher) set of analytic formulas with desired accuracy

The formulas for transforming the modulus (Landen, [16], [17]) are:

$$K(k) = \frac{2}{1 + \sqrt{1 - k^2}} K\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K\left(\frac{1 - k'}{1 + k'}\right),$$

or : $K(\theta) = K[\tan^2(\theta/2)] / \cos^2(\theta/2)$, and, respectively :

$$E(k) = (1 + \sqrt{1 - k^2}) E\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) - \sqrt{1 - k^2} K(k) =$$

$$(1 + k') E\left[\frac{1 - k'}{1 + k'}\right] - k' K(k), \text{ with } k' = (1 - k^2)^{1/2},$$

$$\text{or : } E(\theta) = 2 \cos^2(\theta/2) \cdot E[\tan^2(\theta/2)] - \cos \theta \cdot K(\theta),$$

(passing from k to $k_1 = (1 - k)/(1 + k) \leq k$ and from θ to $\theta_1 = \sin^{-1}[\tan^2(\theta/2)] \leq \theta$; $k_1 = k$ ($\theta_1 = \theta$), for: $k = 0$; 1 ($\theta = 0$; $\pi/2$)), which can be transcribed in recurrent form, as follows:

$$K_2(k) = \frac{2}{1 + \sqrt{1 - k^2}} K_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right),$$

or : $K_2(\theta) = K_1[\tan^2(\theta/2)] / \cos^2(\theta/2)$, and, respectively :

$$E_2(k) = (1 + \sqrt{1 - k^2}) E_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) - \sqrt{1 - k^2} K_2(k) =$$

$$= (1 + k') E_1\left(\frac{1 - k'}{1 + k'}\right) - \frac{2k'}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right), \text{ or : } E_2(\theta) =$$

$$2 \cos^2(\theta/2) E_1[\tan^2(\theta/2)] - [\cos \theta / \cos^2(\theta/2)] K_1[\tan^2(\theta/2)],$$

expressing the 3rd set (K_2, E_2) in function of the 2nd one (K_1, E_1), so starting a recurrent-iterative scheme; it allows writing

$$\text{for the } (n + 1)^{\text{th}} \text{ set : } K_n(k) = \frac{2}{1 + k'} K_{n-1}\left(\frac{1 - k'}{1 + k'}\right), \text{ and}$$

$$E_n(k) = (1 + k') E_{n-1}\left(\frac{1 - k'}{1 + k'}\right) - \frac{2k'}{1 + k'} K_{n-1}\left(\frac{1 - k'}{1 + k'}\right), \text{ resp.}$$

Starting from the newly found closed analytic formulas, which connect the 3rd set (K_2, E_2) with the 2nd one (K_1, E_1), by applying the new recurrent-iterative scheme previously presented, the comparative tables 1 and 2 from chapter 2 were remade, suppressing the column “ $k = \sin \theta$ ”, and inserting the new columns “ $K_2(k)$ ” and “ $E_2(k)$ ” with 4D approximate values, resp., so getting the new tables 4 and 5, given below, resp., keeping for comparison the columns “ $\theta(^{\circ})$ ”, “ $K(k)$ ”, “ $K_0(k)$ ” and “ $K_1(k)$ ” (in table 4), and “ $\theta(^{\circ})$ ”, “ $E(k)$ ”, “ $E_0(k)$ ” and “ $E_1(k)$ ” (in table 5), resp.

Table 4. Values of the functions K (part one)
 (this table completes and replaces table 1)

$\theta(^{\circ})$	$K(k)$	$K_0(k)$	$K_1(k)$	$K_2(k)$
0	1.5708	1.5708	1.5708	1.5708
1	1.5709	1.5709	1.5709	1.5709
2	1.5713	1.5713	1.5713	1.5713
3	1.5719	1.5719	1.5719	1.5719
4	1.5727	1.5727	1.5727	1.5727
5	1.5738	1.5738	1.5738	1.5738
6	1.5751	1.5751	1.5751	1.5751
7	1.5767	1.5767	1.5767	1.5767
8	1.5785	1.5785	1.5785	1.5785
9	1.5805	1.5805	1.5805	1.5805
10	1.5828	1.5828	1.5828	1.5828
11	1.5854	1.5854	1.5854	1.5854
12	1.5882	1.5882	1.5882	1.5882
13	1.5913	1.5913	1.5913	1.5913
14	1.5946	1.5946	1.5946	1.5946
15	1.5981	1.5981	1.5981	1.5981
16	1.6020	1.6020	1.6020	1.6020
17	1.6061	1.6061	1.6061	1.6061
18	1.6105	1.6105	1.6105	1.6105
19	1.6151	1.6151	1.6151	1.6151
20	1.6200	1.6200	1.6200	1.6200
21	1.6252	1.6252	1.6252	1.6252
22	1.6307	1.6307	1.6307	1.6307
23	1.6365	1.6365	1.6365	1.6365
24	1.6426	1.6426	1.6426	1.6426
25	1.6490	1.6490	1.6490	1.6490
26	1.6557	1.6557	1.6557	1.6557
27	1.6627	1.6627	1.6627	1.6627
28	1.6701	1.6701	1.6701	1.6701
29	1.6777	1.6777	1.6777	1.6777
30	1.6858	1.6857	1.6858	1.6858
31	1.6941	1.6941	1.6941	1.6941
32	1.7028	1.7028	1.7028	1.7028
33	1.7119	1.7119	1.7119	1.7119
34	1.7214	1.7214	1.7214	1.7214
35	1.7312	1.7312	1.7312	1.7312
36	1.7415	1.7415	1.7415	1.7415
37	1.7522	1.7522	1.7522	1.7522
38	1.7633	1.7632	1.7633	1.7633
39	1.7748	1.7748	1.7748	1.7748
40	1.7868	1.7867	1.7868	1.7868
41	1.7992	1.7992	1.7992	1.7992
42	1.8122	1.8121	1.8122	1.8122
43	1.8256	1.8256	1.8256	1.8256
44	1.8396	1.8395	1.8396	1.8396
45	1.8541	1.8540	1.8541	1.8541
46	1.8691	1.8691	1.8691	1.8691
47	1.8848	1.8847	1.8848	1.8848
48	1.9011	1.9009	1.9011	1.9011
49	1.9180	1.9178	1.9180	1.9180
50	1.9356	1.9354	1.9356	1.9356
51	1.9539	1.9536	1.9539	1.9539
52	1.9729	1.9726	1.9729	1.9729
53	1.9927	1.9923	1.9927	1.9927
54	2.0133	2.0128	2.0133	2.0133
55	2.0347	2.0341	2.0347	2.0347
56	2.0571	2.0564	2.0571	2.0571
57	2.0804	2.0795	2.0804	2.0804
58	2.1047	2.1037	2.1047	2.1047
59	2.1300	2.1288	2.1300	2.1300
60	2.1565	2.1551	2.1565	2.1565
61	2.1842	2.1825	2.1842	2.1842
62	2.2132	2.2111	2.2132	2.2132
63	2.2435	2.2410	2.2435	2.2435
64	2.2754	2.2723	2.2754	2.2754
65	2.3088	2.3051	2.3088	2.3088
66	2.3439	2.3394	2.3439	2.3439
67	2.3809	2.3754	2.3809	2.3809
68	2.4198	2.4132	2.4198	2.4198
69	2.4610	2.4530	2.4610	2.4610
70	2.5046	2.4948	2.5045	2.5046
70.5	2.5273	2.5165	2.5273	2.5273
71	2.5507	2.5389	2.5507	2.5507
71.5	2.5749		2.5749	2.5749
72	2.5998		2.5998	2.5998
72.5	2.6256		2.6255	2.6256
73	2.6521		2.6521	2.6521
73.5	2.6796		2.6796	2.6796
74	2.7081		2.7081	2.7081
74.5	2.7375		2.7375	2.7375
75	2.7681		2.7680	2.7681
75.5	2.7998		2.7997	2.7998
76	2.8327		2.8326	2.8327
76.5	2.8669		2.8669	2.8669
77	2.9026		2.9025	2.9026
77.5	2.9397		2.9397	2.9397
78	2.9786		2.9785	2.9786
78.5	3.0192		3.0191	3.0192
79	3.0617		3.0616	3.0617
79.5	3.1064		3.1063	3.1064
80	3.1534		3.1533	3.1534
80.2	3.1729		3.1727	3.1729
80.4	3.1928		3.1927	3.1928
80.6	3.2132		3.2130	3.2132
80.8	3.2340		3.2338	3.2340
81	3.2553		3.2551	3.2553

Table 4. Values of the functions K (part two)

81.2	3.2771	3.2769	3.2771
81.4	3.2995	3.2992	3.2995
81.6	3.3223	3.3221	3.3223
81.8	3.3458	3.3455	3.3458
82	3.3699	3.3696	3.3699
82.2	3.3946	3.3942	3.3946
82.4	3.4199	3.4196	3.4199
82.6	3.4460	3.4456	3.4460
82.8	3.4728	3.4724	3.4728
83	3.5004	3.4999	3.5004
83.2	3.5288	3.5283	3.5288
83.4	3.5581	3.5575	3.5581
83.6	3.5884	3.5877	3.5884
83.8	3.6196	3.6188	3.6196
84	3.6519	3.6510	3.6519
84.2	3.6852	3.6843	3.6852
84.4	3.7198	3.7187	3.7198
84.6	3.7557	3.7545	3.7557
84.8	3.7930	3.7916	3.7930
85	3.8317	3.8302	3.8317
85.2	3.8721	3.8704	3.8721
85.4	3.9142	3.9122	3.9142
85.6	3.9583	3.9560	3.9583
85.8	4.0044	4.0018	4.0044
86	4.0528	4.0498	4.0528
86.2	4.1037	4.1003	4.1037
86.4	4.1574	4.1535	4.1574
86.6	4.2142	4.2097	4.2142
86.8	4.2744	4.2692	4.2744
87	4.3387	4.3325	4.3387
87.2	4.4073	4.4001	4.4073
87.4	4.4811	4.4726	4.4811
87.6	4.5609	4.5507	4.5609
87.8	4.6477	4.6354	4.6477
88	4.7427	4.7277	4.7427
88.2	4.8478	4.8293	4.8478
88.4	4.9654		4.9654
88.6	5.0988		5.0987
88.8	5.2527		5.2527
89	5.4349		5.4349
89.1	5.5402		5.5402
89.2	5.6579		5.6579
89.3	5.7914		5.7913
89.4	5.9455		5.9454
89.5	6.1278		6.1276
89.6	6.3509		6.3506
89.7	6.6385		6.6380
89.8	7.0440		7.0428
89.9	7.7371		7.7336
90	∞	$-\infty$	$-\infty$

The values string in the last column is given by:

$$K_2(k) = \frac{2}{1 + \sqrt{1 - k^2}} K_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) = \frac{2}{1 + k'} K_1\left(\frac{1 - k'}{1 + k'}\right),$$

$$\text{with: } K_1(k_1) = \frac{\pi\sqrt{2}}{\sqrt{(1+k'_1)\sqrt{k'_1}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1 + \sqrt{k'_1}}{\sqrt[4]{(1+k'_1)\sqrt{k'_1}}}\right) =$$

$$= \frac{\pi\sqrt{2}}{\sqrt{\left(1 + \frac{2\sqrt{k'}}{1+k'}\right) \sqrt[4]{1+k'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1 + \frac{\sqrt{2\sqrt{k'}}}{\sqrt{1+k'}}}{\sqrt[4]{\left(1 + \frac{2\sqrt{k'}}{1+k'}\right) \sqrt[4]{1+k'}}}\right),$$

and finally the algebraic formula: $K_2(k) = 2K_1(k_1)/(1 + k')$.

Table 5. Values of the functions E (part one)
 (this table completes and replaces table 2)

$\theta(^{\circ})$	$E(k)$	$E_0(k)$	$E_1(k)$	$E_2(k)$
0	1.5708	1.5708	1.5708	1.5708
1	1.5707	1.5707	1.5707	1.5707
2	1.5703	1.5703	1.5703	1.5703
3	1.5697	1.5697	1.5697	1.5697
4	1.5689	1.5689	1.5689	1.5689
5	1.5678	1.5678	1.5678	1.5678
6	1.5665	1.5665	1.5665	1.5665
7	1.5649	1.5649	1.5649	1.5649
8	1.5632	1.5632	1.5632	1.5632
9	1.5611	1.5611	1.5611	1.5611
10	1.5589	1.5589	1.5589	1.5589
11	1.5564	1.5564	1.5564	1.5564
12	1.5537	1.5537	1.5537	1.5537
13	1.5507	1.5507	1.5507	1.5507
14	1.5476	1.5476	1.5476	1.5476
15	1.5442	1.5442	1.5442	1.5442
16	1.5405	1.5405	1.5405	1.5405
17	1.5367	1.5367	1.5367	1.5367
18	1.5326	1.5326	1.5326	1.5326
19	1.5283	1.5283	1.5283	1.5283
20	1.5238	1.5238	1.5238	1.5238
21	1.5191	1.5191	1.5191	1.5191
22	1.5141	1.5141	1.5141	1.5141
23	1.5090	1.5090	1.5090	1.5090
24	1.5037	1.5037	1.5037	1.5037
25	1.4981	1.4981	1.4981	1.4981
26	1.4924	1.4924	1.4924	1.4924
27	1.4864	1.4864	1.4864	1.4864
28	1.4803	1.4803	1.4803	1.4803
29	1.4740	1.4740	1.4740	1.4740
30	1.4675	1.4675	1.4675	1.4675
31	1.4608	1.4608	1.4608	1.4608
32	1.4539	1.4539	1.4539	1.4539
33	1.4469	1.4469	1.4469	1.4469
34	1.4397	1.4397	1.4397	1.4397
35	1.4323	1.4323	1.4323	1.4323
36	1.4248	1.4248	1.4248	1.4248
37	1.4171	1.4171	1.4171	1.4171
38	1.4092	1.4093	1.4092	1.4092
39	1.4013	1.4013	1.4013	1.4013
40	1.3931	1.3932	1.3931	1.3931

Table 5. Values of the functions E (part two)

					81	1.0338		1.0339	1.0338
41	1.3849	1.3849	1.3849	1.3849	81.2	1.0326		1.0327	1.0326
42	1.3765	1.3765	1.3765	1.3765	81.4	1.0314		1.0315	1.0314
43	1.3680	1.3680	1.3680	1.3680	81.6	1.0302		1.0303	1.0302
44	1.3594	1.3594	1.3594	1.3594	81.8	1.0290		1.0292	1.0290
45	1.3506	1.3507	1.3506	1.3506	82	1.0278		1.0280	1.0278
46	1.3418	1.3419	1.3418	1.3418	82.2	1.0267		1.0269	1.0267
47	1.3329	1.3330	1.3329	1.3329	82.4	1.0256		1.0258	1.0256
48	1.3238	1.3239	1.3238	1.3238	82.6	1.0245		1.0247	1.0245
49	1.3147	1.3148	1.3147	1.3147	82.8	1.0234		1.0236	1.0234
50	1.3055	1.3057	1.3055	1.3055	83	1.0223		1.0226	1.0223
51	1.2963	1.2964	1.2963	1.2963	83.2	1.0213		1.0215	1.0213
52	1.2870	1.2872	1.2870	1.2870	83.4	1.0202		1.0205	1.0202
53	1.2776	1.2778	1.2776	1.2776	83.6	1.0192		1.0196	1.0192
54	1.2681	1.2684	1.2681	1.2681	83.8	1.0182		1.0186	1.0182
55	1.2587	1.2590	1.2587	1.2587	84	1.0172		1.0176	1.0172
56	1.2492	1.2496	1.2492	1.2492	84.2	1.0163		1.0167	1.0163
57	1.2397	1.2401	1.2397	1.2397	84.4	1.0153		1.0158	1.0153
58	1.2301	1.2307	1.2301	1.2301	84.6	1.0144		1.0150	1.0144
59	1.2206	1.2212	1.2206	1.2206	84.8	1.0135		1.0141	1.0135
60	1.2111	1.2118	1.2111	1.2111	85	1.0127		1.0133	1.0127
61	1.2015	1.2024	1.2015	1.2015	85.2	1.0118		1.0125	1.0118
62	1.1920	1.1930	1.1920	1.1920	85.4	1.0110		1.0118	1.0110
63	1.1826	1.1838	1.1826	1.1826	85.6	1.0102		1.0110	1.0102
64	1.1732	1.1745	1.1732	1.1732	85.8	1.0094		1.0103	1.0094
65	1.1638	1.1654	1.1638	1.1638	86	1.0086		1.0097	1.0086
66	1.1545	1.1564	1.1545	1.1545	86.2	1.0079		1.0091	1.0079
67	1.1453	1.1475	1.1453	1.1453	86.4	1.0072		1.0085	1.0072
68	1.1362	1.1387	1.1362	1.1362	86.6	1.0065		1.0080	1.0065
69	1.1272	1.1301	1.1273	1.1272	86.8	1.0059		1.0075	1.0059
70	1.1184	1.1217	1.1184	1.1184	87	1.0053		1.0071	1.0053
70.5	1.1140	1.1176	1.1140	1.1140	87.2	1.0047		1.0067	1.0047
71	1.1096	1.1135	1.1096	1.1096	87.4	1.0041		1.0064	1.0041
71.5	1.1053		1.1053	1.1053	87.6	1.0036		1.0062	1.0036
72	1.1011		1.1011	1.1011	87.8	1.0031		1.0060	1.0031
72.5	1.0968		1.0968	1.0968	88	1.0026		1.0060	1.0026
73	1.0927		1.0927	1.0927	88.2	1.0021		1.0061	1.0021
73.5	1.0885		1.0885	1.0885	88.4	1.0017			1.0017
74	1.0844		1.0844	1.0844	88.6	1.0014			1.0014
74.5	1.0804		1.0804	1.0804	88.8	1.0010			1.0011
75	1.0764		1.0764	1.0764	89	1.0008			1.0008
75.5	1.0725		1.0725	1.0725	89.1	1.0006			1.0006
76	1.0686		1.0686	1.0686	89.2	1.0005			1.0005
76.5	1.0648		1.0648	1.0648	89.3	1.0004			1.0004
77	1.0611		1.0611	1.0611	89.4	1.0003			1.0003
77.5	1.0574		1.0574	1.0574	89.5	1.0002			1.0003
78	1.0538		1.0538	1.0538	89.6	1.0001			1.0002
78.5	1.0502		1.0503	1.0502	89.7	1.0001			1.0002
79	1.0468		1.0468	1.0468	89.8	1.0000			1.0003
79.5	1.0434		1.0435	1.0434	89.9	1.0000			1.0007
80	1.0401		1.0402	1.0401	90	1.0000	1.1781	1.1781	1.1781
80.2	1.0388		1.0389	1.0388	The values string in the last column is given by:				
80.4	1.0375		1.0376	1.0375	$E_2(k) = (1 + \sqrt{1 - k^2})E_1\left(\frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}\right) - \sqrt{1 - k^2}K_2(k) =$				
80.6	1.0363		1.0364	1.0363					
80.8	1.0350		1.0351	1.0350					

$$\begin{aligned}
 &= (1+k')E_1\left(\frac{1-k'}{1+k'}\right) - k'K_2(k) = \\
 &= (1+k')E_1\left(\frac{1-k'}{1+k'}\right) - \frac{2k'}{1+k'}K_1\left(\frac{1-k'}{1+k'}\right), \text{ with:} \\
 \frac{1-k'}{1+k'} &= k_1 \text{ (descending Landen transformation), getting} \\
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} (1+\sqrt{k_1}')^2 - \sqrt{2} \sqrt{1+k_1'} \sqrt[4]{k_1'} \right] - k_1' \cdot K_1(k_1), \\
 \text{and: } K_1(k_1) &= \frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right),
 \end{aligned}$$

previously given, thus getting:

$$\begin{aligned}
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} (1+\sqrt{k_1}')^2 - \sqrt{2} \sqrt{(1+k_1')\sqrt{k_1'}} \right] - \\
 &\quad - \frac{\pi k_1' \sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(1 - \frac{\sqrt[4]{2}}{4} \frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right) = \\
 &= \frac{\pi}{4} \left[\frac{3}{2} (1+\sqrt{k_1}')^2 - \sqrt{2} (1+k_1') \sqrt{k_1'} - \right. \\
 &\quad \left. - \frac{k_1' \sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}} \left(4 - \frac{\sqrt[4]{2} (1+\sqrt{k_1}')}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}} \right) \right].
 \end{aligned}$$

Expressing $k_1'(k)$: $k_1' = (1 - k_1^2)^{1/2} = 2(k)^{1/2}/(1+k)$, (ascending Landen transformation), and replacing it:

$$\begin{aligned}
 E_1(k_1) &= \frac{\pi}{4} \left[\frac{3}{2} \left(1 + \frac{\sqrt{2\sqrt[4]{k'}}}{\sqrt{1+k'}} \right)^2 - \sqrt{2} \left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}{\sqrt{1+k'}} \right. \\
 &\quad \left. - \frac{\sqrt{2} \cdot \frac{2\sqrt{k'}}{1+k'}}{\sqrt{\left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}{\sqrt{1+k'}}}} \left(4 - \frac{\sqrt[4]{2} \left(1 + \frac{\sqrt{2\sqrt[4]{k'}}}{\sqrt{1+k'}} \right)}{\sqrt[4]{\left(1 + \frac{2\sqrt{k'}}{1+k'} \right) \frac{\sqrt{2\sqrt[4]{k'}}}{\sqrt{1+k'}}}} \right) \right],
 \end{aligned}$$

and finally: $E_2(k) = (1+k')E_1(k_1) - k'K_2(k)$, where $K_2(k)$ was given just before table 5, so getting another purely algebraic formula (the most accurate, just seemingly intricate), the 3rd set of closed analytic formulas being given by the recurrences: $K_2(k) = 2K_1(k_1)/(1+k)$; $E_2(k) = (1+k')E_1(k_1) - k'K_2(k)$. Noting: $k_1' = x$ and $[(1+x) \cdot x^{1/2}]^{1/2} = y$, one can write:

$$\begin{aligned}
 K_2(k) &= \pi(2/k')^{1/2} \cdot (x/y) \left[1 - (2^{1/4}/4)(1+x^{1/2})/y^{1/2} \right]; \\
 E_2(k) &= \pi(k')^{1/2} / (2x) \cdot \left\{ (3/2)(1+x^{1/2})^2 - 2^{1/2} y - \right. \\
 &\quad \left. - 2^{1/2} (x/y) \left[4 - 2^{1/4} (1+x^{1/2})/y^{1/2} \right] \right\} - k'K_2(k), \text{ resp.,}
 \end{aligned}$$

much simpler than previous ones (for calculation only). The validity of all approximate sets is limited to $k \in [0, k_{\text{extr}}]$; $k_{\text{extr}} \leq 1$, “extr” ≡ extremum (max. for K, and min. for E; $k_{\text{max}} \neq k_{\text{min}}$) (see figs. 1 & 2 – the dashed black lines, and the solid red ones, resp.). The higher the “n” index (of the (K_{n-1}, E_{n-1}) approximation set) value is, the better this approximation is and the closer to the right domain’s end ($k = 1$) the extremum is located. We will cancel the recurrent-iterative scheme (stopping it to a specific “n” index value) when the maximum relative

error (over the whole valid domain of variation $k \in [0, k_{\text{extr}}]$) becomes lesser than the desired (required) accuracy. The first important application of the results obtained in chapter 4 consists in determining the locations of the extrema values k_{extr} (k_{max} for $K_{n-1}(k)$ and k_{min} for $E_{n-1}(k)$), corresponding to the annulling of their first derivatives with respect to k , using the relations: $K'_{n-1}(k) = dK_{n-1}(k)/dk = 0$; $E'_{n-1}(k) = dE_{n-1}(k)/dk = 0$, and adding the recurrent definitions for $K_{n-1}(k)$ and $E_{n-1}(k)$. The 1st ODE above gives the value k_{max} and the 2nd one gives the value k_{min} . Each of these ODEs has really two solutions. Besides the searched for one, both ODEs admit the solution $k = 0$, corresponding to a minimum for $K_{n-1}(k)$ and to a maximum for $E_{n-1}(k)$, both with the value $\pi/2$ (for both approximate and exact functions: $K_{n-1}(0) = E_{n-1}(0) = K(0) = E(0) = \pi/2$, with: $K'_{n-1}(0) = E'_{n-1}(0) = K'(0) = E'(0) = 0$, but with: $K''_{n-1}(0) > 0$ and $K''(0) > 0$ – a minimum, while: $E''_{n-1}(0) < 0$ and $E''(0) < 0$ – a maximum).

Thus one knows now the values k_{max} and k_{min} (the right ends of the validity domains of the approximate functions). In order to evaluate the accuracy of the 3rd set (K_2, E_2), similarly as for the previous two sets, (K_0, E_0) and (K_1, E_1), we will define the following relative error functions: $\varepsilon_{K_2}(k) = K_2(k)/K(k) - 1$, and: $\varepsilon_{E_2}(k) = E_2(k)/E(k) - 1$, for the approximate formulas of 1st & 2nd kind integrals. Their values are given in table 6, expressed in thousandths (%). These errors were calculated for the 3rd set (K_2, E_2) only, with an increment of $0^\circ.2$ in the field $\theta \in [84^\circ, 89^\circ]$ of the domain, and of $0^\circ.1$ beyond 89° . To get table 6, in table 3 were suppressed the columns $\varepsilon_{K_0}(\%)$, $\varepsilon_{E_0}(\%)$ (the most inaccurate) and were inserted the columns $\varepsilon_{K_2}(\%)$, $\varepsilon_{E_2}(\%)$, keeping for comparison the columns “ $\theta(^\circ)$ ”, “ $k = \sin \theta$ ”, “ $\varepsilon_{K_1}(\%)$ ” and “ $\varepsilon_{E_1}(\%)$ ” (from table 3), only.

Table 6. Relative errors ε distribution (this table completes and replaces table 3)

$\theta(^\circ)$	$k = \sin \theta$	$\varepsilon_{K_1}(\%)$	$\varepsilon_{K_2}(\%)$	$\varepsilon_{E_1}(\%)$	$\varepsilon_{E_2}(\%)$
84.8	0.99588	-0.369	0	+0.607	0
85	0.99619	-0.396	0	+0.592	0
85.2	0.99649	-0.451	0	+0.705	0
85.4	0.99678	-0.500	0	+0.748	0
85.6	0.99705	-0.582	0	+0.823	0
85.8	0.99731	-0.652	0	+0.932	0
86	0.99756	-0.737	0	+1.076	0
86.2	0.99780	-0.832	0	+1.160	0
86.4	0.99803	-0.945	0	+1.284	0
86.6	0.99824	-1.077	0	+1.453	0
86.8	0.99844	-1.214	0	+1.571	0
87	0.99863	-1.421	0	+1.743	0
87.2	0.99881	-1.626	0	+1.976	0
87.4	0.99897	-1.894	0	+2.275	0
87.6	0.99912	-2.234	0	+2.553	0
87.8	0.99926	-2.655	0	+2.922	0
88	0.99939	-3.156	0	+3.397	0
88.2	0.99951	-3.808	0	+4.004	0

88.4	0.99961	-	0	-	0
88.6	0.99970	-	0	-	0
88.8	0.99978	-	0	-	0
89	0.99985	-	0	-	0
89.1	0.99988	-	0	-	0
89.2	0.99990	-	0	-	0
89.3	0.99993	-	0	-	0
89.4	0.99995	-	0	-	0
89.5	0.99996	-	-	-	-
89.6	0.99998	-	-	-	-
89.7	0.99999	-	-	-	-
89.8	0.99999	-	-	-	-
89.9	1.00000	-	-	-	-
90	1.00000	-2000	-2000	178.097	178.097

The relative errors strings are stopped for values $\geq 4\%$. From the tables 3 and 6 one can see that, for any n^{th} set of approximation and at any value $k = \sin\theta$ of the elliptic integrals' modulus, $\varepsilon_K < 0$ ($K_n < K$) and $\varepsilon_E > 0$ ($E_n > E$), i.e. K is approximated by lack, while E is approximated by excess. Similarly to the 3^{rd} set $[K_2(k), E_2(k)]$, expressed in algebraic functions, one can build the 3^{rd} set $[K_2(\theta), E_2(\theta)]$, expressed in trigonometric functions, replacing k' in $[K_2(k), E_2(k)]$ set by $\cos\theta$ and applying usual trigonometric identities. The comparative series representations and the graphic comparison are superfluous, due to the great accuracy of the approximate values given by the 3^{rd} set (practically identical to the exact ones, which could be already noticed from the analysis of the 2^{nd} set, this showing the fast converging character of this recurrent-iterative scheme). Except for the right domain's end ($k = 1$), the 3^{rd} set of approximation (K_2, E_2), even more accurate than the 2^{nd} one (K_1, E_1), may be considered and successfully used instead of the exact values of $K(k)$ and $E(k)$ from mathematical tables (the 3^{rd} set (K_2, E_2) can be already used until $\theta = 89^\circ.7$ ($k = 0.99999$) – a false minimum, for the $E_2(k)$ function; for the 2^{nd} set this takes place at $\theta = 88^\circ$ ($k = 0.99939$); the graphs of all $E_n(k)$ pass through the point $(1, 3\pi/8)$; for k tending to unity, the graphs of all $K_n(k)$ go toward $(-\infty)$; the higher n^{th} sets ($n \geq 4$) give a much better accuracy). Unlike these tables (and in addition to them), all these approximation sets (the 1^{st} , 2^{nd} , 3^{rd} and the higher n^{th} ($n \geq 4$) ones) allow performing the *analytic study of variation* of the functions in which $K(k)$ and / or $E(k)$ appear /s, using the derivatives of the 1^{st} and 2^{nd} order (with respect to k).

Remarks: 1. As a first step in applying the new recurrent-iterative scheme, just the obtaining of the 2^{nd} set (K_1, E_1) as a function of the 1^{st} one (K_0, E_0) (in chapter 2) may be considered, i.e. *this scheme starts really at the 2^{nd} set*. It is to be highlighted that the used method is a *purely analytic* one (neither numerical procedures nor sophisticated computer programs). Its *simplicity, accuracy and fast convergence*, as well as its *limitations* depend *exclusively* on the *correct choice of its starting point* (approximation set) (K_0, E_0); it must be quite precise, and especially, *as simple as possible*.

The starting approximate formula-definition giving $E_0(k)$ was suggested to the author by an old approximate formula (Peano, [18], [19]) for the perimeter L of an ellipse of semi-axes a and b ($\leq a$): $L \approx \pi[1.5(a+b) - (ab)^{1/2}]$ – a good (& simple) approximation with the best accuracy for $b = a$ (circle): $L = 2\pi a$, and the worst one for $b = 0$ (plane plate): $L = 1.5\pi a$, instead of $L = 4a$ (or optimized Peano's law: $L_1 \approx \pi[1.32(a+b) - 0.64(ab)^{1/2}]$, with the smallest overall error [20](about 7 times smaller than that of the original law); for $b = a$: $L_1 = L = 2\pi a$, and for $b = 0$: $L_1 = 1.32\pi a$, much closer to the exact value $L = 4a$). For its behaviour at low b/a ratios, this formula is not found on the list of the very accurate (but not simple) approximations [20] (Padé, Jacobsen, Ramanujan (2 expressions), Rackauckas), all expressed in terms of the particular ratio $h = [(a-b)/(a+b)]^2$. Thus a reliable approximate (by excess) formula-definition was obtained (see chapter 2) for the Legendre complete elliptic integral of the 2^{nd} kind (in the 1^{st} set of approximation): $E_0(k) = (\pi/4)[1.5(1+k') - (k')^{0.5}]$, with $k' = (1-k^2)^{0.5}$. It can be seen that the error committed if in the expansion in series of powers (of k) we stopped at the term of rank 5 (see chapter 4), is $(3/16384)k^8$ only, i.e. small enough. As for the pair approximate formula-definition giving $K_0(k)$, this was obtained using the previous one for $E_0(k)$ and applying the definition of the first derivative of $E(k)$ with respect to k : $dE(k)/dk = [E(k) - K(k)]/k$ (see chapter 4), thus getting: $K(k) = E(k) - k[dE(k)/dk]$, and replacing $K(k)$ and $E(k)$ by their 1^{st} approximations: $K_0(k)$ and the previously given $E_0(k)$, getting: $K_0(k) = \pi[1/(k')^{0.5} - (1/2^{1.5})(1+k')^{0.5}/(k')^{0.75}]$ (see ch. 2). Being practically generated by the same mathematical source, $K_0(k)$ and $E_0(k)$ vary (ordinates, slopes, asymptote, extrema, concavities, convexities, inflections) in perfectly correlated way. So, at the value k_{extr} corresponding to a false minimum for $E_0(k)$, $K_0(k)$ must equate $E_0(k)$, to satisfy the annulment of $dE_0(k)/dk$. In order to prepare this action, $K_0(k)$ must stop its vertiginous ascension to ∞ , making a false inflection, followed by a false maximum at $k_{\text{extr}} < k_{\text{extr}}$ and a vertiginous (with $k = 1$ – vertical asymptote) fall toward $(-\infty)$; so $K_0 = E_0$ at $k = 0$ and $k = k_{\text{extr}}$. To minimise all these unwished events, limiting them to a very thin region in the neighbourhood of the right domain's end, it was necessary to apply the descending Landen transformation, passing from k to $k_1 = (1-k)/(1+k) \leq k$, where all goes well, maintaining all advantages of the asymptotic behaviour of the new approximate correlated functions (K_n, E_n), i.e. applying a higher n^{th} ($n \geq 2$) set of approximation (repeating this scheme until the desired (required) accuracy for (K_n, E_n) is obtained; fortunately, this scheme is fast converging); though it keeps the limitation at $k = 1$, Peano's optimized law accelerates the scheme.

2. Besides the two formulas for transforming the modulus k (using the descending Landen transformation), there are other two formulas (using the ascending Landen transformation), which are not of interest in the recurrent-iterative scheme:

$$K(k) = \frac{1}{1+k} K\left(\frac{2\sqrt{k}}{1+k}\right); E(k) = \frac{1+k}{2} E\left(\frac{2\sqrt{k}}{1+k}\right) + \frac{k'^2}{2} K(k),$$

passing from k to $k_2 = 2k^{1/2}/(1+k) \geq k$, $k_2 = k$ for $k = 0$; 1.

Appendix' references:

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<https://doi.org/doi:10.1098/rstl.1775.0028>
- [18] Peano, G.: *Applicazioni geometriche del calcolo infinitesimale*, Fratelli Bocca Editori, Torino, 1887; (in Italian), p. 233; the approximate formula for the ellipse perimeter was: $L \approx \pi(a + b) + (\pi/2)(a^{1/2} - b^{1/2})^2$.
- [19] Peano, G.: *VIII An approximation formula for the perimeter of the ellipse*, 1889, pp. 135 – 136 in *Selected Works of Giuseppe Peano*, Translated and edited, with a biographical sketch and bibliography, by Hubert C. Kennedy; Series: Heritage; Copyright Date: 1973; Published by: University of Toronto Press; Pages: 262; the approximate formula for ellipse perimeter (due to J. Boussinesq in the *Comptes rendus*, Académie des Sciences, Paris, 1889, p. 695) was given in the well-known equivalent form: $L \approx \pi[3(a + b)/2 - (ab)^{1/2}]$; <https://www.jstor.org/stable/10.3138/j.ctt1vxmd8x>
- [20] Sýkora, St.: *Approximations of Ellipse Perimeters and of the Complete Elliptic Integral E(x)*. Review of known formulae, Review by Stanislav Sýkora, Extra Byte, Via R.Sanzio 22C, Castano Primo, Italy 20022, in Stan's Library, Ed. S. Sýkora, Vol.I; First release: Dec. 27, 2005. Permalink via DOI: 10.3247/SL1Math05.004; <http://www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html>

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