Non Standard Finite Difference Method for Singularly Perturbed Singular Two Point Boundary Value Problem using Non Polynomial Spline

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Abstract: - In this paper, a non standard finite difference method, with reference to the solution of a class of singularly perturbed singular boundary value problems on a uniform mesh, is discussed. The non-polynomial spline forms the tool for the solution of the problem. The discretized equation of the problem is developed using the condition of continuity for the first order derivatives of the non polynomial spline, at the interior nodes and it is not valid at the singularity. Hence, at the singularity, the boundary value problem is modified in order to get a three term relation. The tridiagonal scheme of the method is processed using discrete invariant imbedding algorithm. The convergence of the method is analyzed and maximum absolute errors in the solution are tabulated. Root mean square errors in the solution of the examples are presented in comparison with the methods chosen from the literature, to establish the proposed method.

Key-Words: - Singularly perturbed two point singular boundary value problem, Interior nodes, Singular point, Non-polynomial spline, Boundary layer

1 Introduction

We consider a class of singularly perturbed two point singular boundary value problems of the form:

$$\varepsilon y''(x) = \frac{k}{x} y'(x) + q(x)y(x) + r(x),\ \ 0 \le x \le 1, \qquad (1)$$

with boundary conditions

$$y(0) = \gamma_1 \quad \text{and} \quad y(1) = \gamma_2 \qquad (2)$$

where $0 < \varepsilon \ll 1$, $q(x)$, $r(x)$ are bounded continuous functions in (0, 1), and γ_1, γ_2 are finite constants. Let $p(x) = \frac{k}{x}$. If $p(x) \ge \bar{M} > 0$ throughout the domain [0, 1], where $\bar{M}$ is a positive constant, then the boundary layer exist in the neighbourhood of $x = 0$. If $p(x) \le \bar{N} < 0$ throughout the interval [0, 1], where $\bar{N}$ is a negative constant, then the boundary layer will be in the neighbourhood of $x = 1$.

The numerical treatment of these problems gives major computational difficulties due to the presence of boundary and/or interior layers. A wide variety of books and papers have been published, describing various methods for solving singularly perturbed two-point boundary value problems, among these, we mention [1-7, 11]. Kadalbajoo and Aggarwal [4] proposed a fitted mesh B-spline method for singular singularly perturbed boundary value problems. Mohanty et al. [8, 9, 10] established various methods based on tension spline and compression spline methods both on a uniform and non-uniform mesh for singularly perturbed two point singular boundary value problems. J. Rashidinia [12] used Cubic spline solution of singularly perturbed two-point boundary value problems on a uniform mesh.

The paper is organized as follows: In section 2, the non polynomial spline method is defined. In section 3, description of the numerical method is given. In section 4, truncation error and classification of various orders of the proposed method are projected. In section 5, convergence analysis of the method is discussed. Finally, Numerical results and comparison with other methods are presented in section 6.

2 Non Polynomial Spline Method

Decompose the domain of the integration [*a, b*] into *N* equal subintervals with mesh size $h = \frac{1}{N}$, so that $x_i = a + ih,\ i = 0,\ 1,\ \ldots,\ N$ are the nodes with $a = x_0, b = x_N$. Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ by the non polynomial cubic spline $S_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). Here $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, also the continuity of first derivative at the common nodes (x_i, y_i) are fulfilled.

For each i^{th} subinterval, the cubic non-polynomial spline function $S_i(x)$ has the form

$$\begin{aligned} S_i(x) &= a_i + b_i(x-x_i) + c_i \sin\tau(x-x_i) \\ &+ d_i \cos\tau(x-x_i), \quad i = 0,\ 1, \ldots, N-1. \end{aligned} \tag{3}$$

where a_i, b_i, c_i and d_i are constants and τ is a free parameter.

A non-polynomial function $S_i(x)$ of class $C^2[a,\ b]$ interpolating $y(x)$ at the grid points x_i, i = 0, 1, ..., N depends on a parameter τ, and reduce to ordinary cubic spline in [a, b] as $\tau \to 0$.

To derive an expression for the coefficients of Eq. (3) in term of y_i, y_{i+1}, M_i and M_{i+1}, define

$$S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1},$$
$$S^{''}(x_i) = M_i, S^{''}(x_{i+1}) = M_{i+1}.$$

Using algebraic manipulation, the following expressions are obtained for the coefficients:

$$a_i = y_i + \frac{M_i}{\tau^2}, \quad b_i = \frac{y_{i+1}-y_i}{h} + \frac{M_{i+1}-M_i}{\tau\theta},$$
$$c_i = \frac{M_i\cos\theta - M_{i+1}}{\tau^2\sin\theta}, \quad d_i = -\frac{M_i}{\tau^2}$$

where $\theta = \tau h$, for i = 0, 1, ... , N-1.

Using the continuity of the first derivative at (x_i, y_i), that is $S'_{i-1}(x_i) = S'_i(x_i)$, we get the following relations for i =1,2,...,N-1.

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{4}$$

where

$$\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta\sin\theta}, \quad \beta = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta\sin\theta},$$
$$M_j = y^{''}(x_j), \quad j = i-1, i, i+1 \text{ and } \theta = \tau h$$

3 Numerical Scheme

At the grid points x_i, Eq. (1) may be discretized by

$\varepsilon y_i'' = p(x_i)y_i' + q(x_i)y_i + r_i$

Using spline's second derivatives, we have

$$\begin{aligned} &\varepsilon\, M_j = p(x_j)y_j'(x) + q(x_j)y(x_j) + r(x_j) \\ &\text{for } j = i-1,\ i,\ i+1 \end{aligned} \tag{5}$$

Using Eq. (5) in Eq. (4) and with the following approximations for the first derivative of y:

$$\begin{aligned} y'_{i+1} &\approx \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h}, \\ y'_{i-1} &\approx \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h}, \\ y'_i &\cong \left(\frac{1 + 2\omega h^2 q_{i+1} + \omega h[3p_{i+1} + p_{i-1}]}{2h}\right) y_{i+1} \\ &- 2\omega\ [p_{i+1} + p_{i-1}] y_i \\ &- \left(\frac{1 + 2\omega h^2 q_{i-1} - \omega h[\ p_{i+1} + 3p_{i-1}]}{2h}\right) y_{i-1} \\ &+ \omega\ h[r_{i+1} - r_{i-1}] \end{aligned} \tag{6}$$

we get the tridiagonal system

$$E_{i-1}y_{i-1} + F_i\ y_i + G_{i+1}y_{i+1} = H_i \tag{7}$$

for $i = 1, 2, \ldots,\ N-1$

where

$$E_{i-1} = -\ \varepsilon - \frac{3}{2}\alpha\ p_{i-1}\ h\ + \beta\ p_i h^2 \omega[p_{i+1} + 3p_{i-1}] -$$
$$2\omega\ p_i \beta\ h^3 q_{i-1} + \frac{\alpha}{2} p_{i+1} h + \alpha\ q_{i-1} h^2 - h\ \beta\ p_i$$

$$F_i = 2\varepsilon + 2\alpha\ p_{i-1}\ h\ - 4\beta\ p_i h^2 \omega[p_{i+1} + p_{i-1}]$$
$$-2\alpha\ p_{i+1} h\ + 2\beta\ q_i h^2$$

$$G_{i+1} = -\varepsilon - \frac{\alpha}{2}\ p_{i-1}\ h\ \ + \beta\ p_i h^2 \omega[3p_{i+1} + p_{i-1}]$$
$$+2\omega h^3 \beta\ p_i q_{i+1} + \frac{3}{2}\alpha\ p_{i+1} h + \alpha\ q_{i+1} h^2 + h\beta\ p_i$$

$$H_i = -h^2 \begin{bmatrix} (\alpha - 2\omega\beta\ p_i h) r_{i-1} + 2\beta\ r_i \\ +(\alpha + 2\omega\beta\ p_i h) r_{i+1} \end{bmatrix}$$

$p(x_i) = p_i$, $q(x_i) = q_i$, $r(x_i) = r_i$ for $i = 1,\ 2, ..., N-1$.

For i = 1, the coefficients y_{i-1}, y_i and y_{i+1} in Eq. (7) are not defined, thus we need to develop a formula for this case. Using L-Hospital rule and Eqs. (4), we get the following three term formula for i = 1:

$$\left[-1+\frac{\alpha h^2 q_0}{\varepsilon - k}\right] y_0 + \left[2+\frac{2\beta h^2 q_1}{\varepsilon - k}\right] y_1 + \left[-1+\frac{\alpha h^2 q_2}{\varepsilon - k}\right] y_2 = -\frac{h^2}{\varepsilon - k}\left[\alpha r_0 + 2\beta r_1 + \alpha r_2\right] \tag{8}$$

We solve the tridiagonal system Eq. (7) together with the Eq. (8) for i = 1, 2,…, N-1 in order to get the approximations $y_1, y_2, ..., y_{N-1}$ of the solution $y(x)$ at $x_1, x_2, ..., x_{N-1}$.

4 Truncation error

The local truncation error associated with the scheme developed in Eq. (7), is

$$T_i(h) = \left[-1+2(\alpha+\beta)\right]\varepsilon h^2 y''(x_i) + \left\{ \begin{array}{l} \left[\left(4\omega\varepsilon + \frac{1}{3}\right)\beta - \frac{2\alpha}{3}\right] p(x_i) y'''(x_i) \\ +\left(-1+12\alpha\right)\frac{\varepsilon}{12} y^{(4)}(x_i) \end{array} \right\} h^4 + O(h^6)$$

Thus for different values of α , β, ω in the scheme Eq. (7), indicates different orders:

(i) for any choice of arbitrary α and β with $\alpha + \beta = \frac{1}{2}$ and for any value of ω , the scheme Eq.(7) gives second order method.

(ii) for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $\omega = -\frac{1}{20\varepsilon}$, from Eq. (7) fourth order method is derived.

3 Convergence Analysis

Incorporating the boundary conditions Eq. (2), the system of Eqs. (7) - (8) can be written in the matrix form as:

$$\left(D+P\right)Y + Q + T(h) = 0 \tag{9}$$

where

$$D = \left[-\varepsilon,\ \ 2\varepsilon,\ \ -\varepsilon\right] = \begin{bmatrix} 2 & -1 & 0 & 0 & .... & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & .... & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & .... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

and

$$P = \left[z_i, v_i, w_i\right] = \begin{bmatrix} v_1^* & w_1^* & 0 & 0 & .... & 0 \\ z_2 & v_2 & w_2 & 0 & .... & 0 \\ 0 & z_3 & v_3 & w_3 & .... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & z_{N-1} & v_{N-1} \end{bmatrix}$$

where $v_1^* = 2 + \frac{2\beta h^2 q_1}{\varepsilon - k}$, $w_1^* = -1 + \frac{\alpha h^2 q_2}{\varepsilon - k}$

$$z_i = -\frac{3}{2}\alpha p_{i-1} h \quad \beta p_i h + \omega p_i^2 \left[p_{i+1} + 3 p_{i-1}\right]$$

$$-2\omega p_i \beta h^3 q_{i-1} + \frac{\alpha}{2} p_{i+1} h + \alpha q_{i-1} h^2 - h\beta p_i$$

$$v_i = 2\alpha p_{i-1} h - 4\beta p_i h \ \omega_i p^2 \left[p_{i+1} + p_{i-1}\right]$$

$$-2\alpha p_{i+1} h + 2\beta q_i h^2$$

$$w_i = -\frac{\alpha}{2} p_{i-1} h - \beta p_i h + \omega p_i^2 \left[3 p_{i+1} + p_{i-1}\right]$$

$$+2\omega h^3 \beta p_i q_{i+1} + \frac{3}{2}\alpha p_{i+1} h + \alpha q_{i+1} h^2 + h\beta p_i$$

for $i = 2, 3, ..., N-1$

and

$$Q = \begin{bmatrix} \dfrac{h^2}{\varepsilon - k}\left(\alpha r_0 + 2\beta r_1 + \alpha r_2\right) \\ +\left(-1 + \dfrac{\alpha h^2 q_0}{\varepsilon - k}\right)\gamma_0, q_2, q_3, ..., q_{N-1} + w_{N-1}\gamma_1 \end{bmatrix}$$

where $q_i = h^2 \begin{bmatrix} \left(\alpha - 2\omega\beta\, p_i h\right) r_{i-1} \\ +2\beta\, r_i + \left(\alpha + 2\omega\beta\, p_i h\right) r_{i+1} \end{bmatrix}$,

$i = 2, 3, ..., N-1$

$T(h) = 0(h^6)$ for $\alpha = \dfrac{1}{12}, \beta = \dfrac{5}{12},\ \omega = -\dfrac{1}{20\varepsilon}$

and $Y = \left[Y_1, Y_2, ..., Y_{N-1}\right]^T$, $T(h) = \left[T_1, T_2, ..., T_{N-1}\right]^T$,

$O = \left[0, 0, ..., 0\right]^T$ are associated vectors of Eq. (8).

Let $y = \left[y_1, y_2, ..., y_{N-1}\right]^T \cong Y$ satisfies the equation

$$\left(D + P\right) y + Q = 0 \tag{10}$$

Let $e_i = y_i - Y_i,\ i = 1, 2, ..., N-1$ be the discretization error so that $E = \left[e_1, e_2, ..., e_{N-1}\right]^T = y - Y$.

Using Eq. (9) from Eq. (10), we get the error equation

$$\left(D + P\right) E = T(h) \tag{11}$$

Let $\left|p(x)\right| \le C_1$ and $\left|q(x)\right| \le C_2$ where C_1, C_2 are positive constants. If $P_{i,j}$ be the $(i, j)^{th}$ element of P, then

$$\left|P_{i,i+1}\right| = 1 + \frac{h^2 \alpha}{\varepsilon - k} C_2 \ne 0 \ \text{ for } \ i = 1$$

$\left|P_{i,i+1}\right| = \left|w_i\right|$

$\le \left(h(\alpha + \beta) C_1 + h^2 \alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3 \beta\omega C_1 C_2\right)$,

$i = 2, 3, ..., N-2$

$\left|P_{i,i-1}\right| = \left|z_i\right|$

$\le \left(h(\alpha + \beta) C_1 + h^2 \alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3 \beta\omega C_1 C_2\right)$,

$i = 2, 3, ..., N-2$

Thus for sufficiently small h

$\left|P_{i,i+1}\right| < \varepsilon,\ \ i = 1, 2, ..., N-2$ and

$$\left|P_{i,i-1}\right| < \varepsilon, \quad i = 2, 3, ..., N-1 \tag{12}$$

Hence, $(D + P)$ is irreducible (see Ref. [13]).

Let S_i be the sum of the elements of the i^{th} row of the matrix $(D + P)$, then we have

$S_i = 1 + \dfrac{h^2}{\varepsilon - k}\left(2\beta q_1 + \alpha q_2\right)$ for $i = 1$

$S_i = h^2\left(\alpha q_{i-1} + 2\beta q_i + \alpha q_{i+1}\right) + 2h^3 \beta p_i \omega\left(q_{i+1} - q_{i-1}\right)$

for $i = 2, 3, ..., N-2$

$S_i = \varepsilon + \dfrac{\alpha h}{2}\left(p_{i-1} - 3p_{i+1}\right) - h\beta p_i + h^2\left(\alpha q_{i-1} + 2\beta q_i\right)$

$-h^2 \beta\omega p_i\left(p_{i+1} + p_{i-1}\right) - 2h^3 \beta\omega p_i q_{i-1}$

for $i = N-1$

Let $C_{1^*} = \min\limits_{1 \le i \le N}\left|p(x)\right|$ and $C_1^* = \max\limits_{1 \le i \le N}\left|p(x)\right|$,

$C_{2^*} = \min\limits_{1 \le i \le N}\left|q(x)\right|$ and $C_2^* = \max\limits_{1 \le i \le N}\left|q(x)\right|$.

Since $0 < \varepsilon \le 1$ it is possible or easy to verify that for a given h, $\left(D + P\right)$ is monotone [13,14].

Hence $\left(D + P\right)^{-1}$ exists and $\left(D + P\right)^{-1} \ge 0$. Thus using Eq.(10), we have

$$\left\|E\right\| \le \left\|\left(D + P\right)^{-1}\right\| \ \left\|T\right\| \tag{13}$$

Let $\left(D + P\right)_{i,k}^{-1}$ be the $(i, k)^{th}$ element of $\left(D + P\right)^{-1}$ and we define

$$\left\|\left(D + P\right)^{-1}\right\| = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1}\left(D + P\right)_{i,k}^{-1},\ \ \left\|T(h)\right\| = \max_{1 \le i \le N-1}\left|T(h)\right|, \tag{14a}$$

Since $\left(D + P\right)_{i,k}^{-1} \ge 0$ and $\sum\limits_{k=1}^{N-1}\left(D + P\right)_{i,k}^{-1} S_k = 1$.

for $i = 1, 2, ..., N-1$

Hence

$$\left(D + P\right)_{i,k}^{-1} \le \frac{1}{S_i} < \frac{\varepsilon - k}{h^2\left(\alpha + 2\beta\right) C_{2^*}}, \quad i = 1 \tag{14b}$$

$$\left(D+P\right)_{i,k}^{-1} \le \frac{1}{S_i} < \frac{1}{h^2\left[\left(\alpha+2\beta\right)C_{2^*} - 4\beta\omega C_{1^*}^2\right]} \quad (14c)$$

for $i = N-1$

Furthermore,

$$\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} \le \frac{1}{\min\limits_{2\le i\le N-2} S_i} < \frac{1}{h^2\left(2\left(\alpha+\beta\right)C_{2^*}\right)}. \quad (14d)$$

Using Eqs. (14a) - (14d), from Eq. (13), we get

$$\|E\| \le O(h^4). \qquad (15)$$

Since $\left(D+P\right)_{i,k}^{-1} \ge 0$ and $\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} S_k = 1$.

for $i = 1, 2, ..., N-1$

Hence the method Eq. (6) is fourth order convergent for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $\omega = -\frac{1}{20\varepsilon}$.

4 Numerical examples

To demonstrate the proposed method computationally, we consider three problems of the type Eq. (1). These problems have been chosen because they have been widely discussed in the literature.

Example 1. Consider the singularly perturbed singular boundary value problem

$$-\varepsilon y'' + (1/x)y' + (1+x^2)y = f(x), \;\; 0 < x < 1.$$

The exact solution is $y(x) = \exp(x^2)$. The maximum absolute errors are tabulated in Table 1 for different values ε and *h*. Comparison of root mean square errors with the existing methods are presented in Table 2.

Example 2. Consider the boundary value problem $-\varepsilon y'' + \frac{1}{x}y' = f(x), \quad 0 < x < 1.$ The exact solution of this problem is $y(x) = x \sinh x$.

The maximum absolute errors are presented in Table 1 for different values ε and *h*. Comparison of root mean square errors the existing methods are presented in Table 3.

7 Discussions and Conclusion

In this paper, non-polynomial spline method is discussed for a class of singularly perturbed singular two-point boundary value problems. The discretized equation is developed for the problem using the condition of continuity especially for the first order derivatives of the non polynomial spline at the interior nodes. It is not valid at the singularity zero. A three term relation is obtained by modifying the boundary value problem at the singularity zero. Using this, the discretized equation of the problem is solved using discrete invariant imbedding algorithm. Convergence of the method is explained. The maximum absolute errors are tabulated for the existing standard examples chosen from the literature with a view to demonstrate the method. Root mean square errors in the solution of the said examples are presented with comparison in order to justify the method. The proposed method is also applicable to non-singular problems and singularly perturbed delay differential equations. Based on the numerical results, it is observed that the method gives good results for smaller values of ε also.

TABLE 1. Maximum errors in the solution of Examples

ε/N	2^3	2^4	2^5	2^6	2^7
Example 1					
2^{-3}	2.10(-3)	1.48(-4)	1.05(-5)	7.40(-7)	5.14(-8)
2^{-4}	2.60(-3)	1.96(-4)	1.45(-5)	1.05(-6)	7.58(-8)
2^{-5}	3.50(-3)	2.80(-4)	2.17(-5)	1.64(-6)	1.21(-7)
2^{-7}	5.50(-3)	6.25(-4)	5.45(-5)	4.44(-6)	3.45(-7)

2^{-10}	5.90(-3)	1.20(-3)	2.02(-4)	2.06(-5)	1.78(-6)

Example 2

2^{-3}	7.90(-4)	5.69(-5)	4.03(-6)	2.82(-7)	1.95(-8)
2^{-4}	1.00(-3)	7.98(-5)	5.92(-6)	4.27(-7)	3.03(-8)
2^{-5}	1.40(-3)	1.20(-4)	9.33(-6)	6.95(-7)	5.05(-8)
2^{-7}	1.90(-3)	2.65(-4)	2.50(-5)	2.01(-6)	1.53(-7)
2^{-10}	1.90(-3)	4.07(-4)	8.05(-5)	9.42(-6)	8.32(-7)

TABLE 2. Comparison of Root mean square errors in the solution of Example 1

ε/N	2^3	2^4	2^5	2^6	2^7
R.K. Mohanty, Urvashi Arora method [10]					
2^{-3}	3.94(-2)	1.69(-2)	9.83(-3)	6.75(-3)	4.75(-3)
2^{-4}	6.15(-2)	2.45(-2)	1.36(-2)	9.35(-3)	6.59(-3)
2^{-5}	9.21(-2)	3.29(-2)	1.70(-2)	1.15(-2)	8.10(-3)
2^{-7}	-	4.37(-2)	2.03(-2)	1.49(-2)	1.06(-2)
2^{-10}	-	8.38(-2)	8.01(-2)	7.77(-2)	7.22(-2)
Proposed method					
2^{-3}	1.80(-4)	1.25(-4)	8.95(-6)	6.30(-7)	4.38(-8)
2^{-4}	2.20(-3)	1.66(-4)	1.24(-5)	9.11(-7)	6.55(-8)
2^{-5}	2.80(-3)	2.33(-4)	1.84(-5)	1.40(-6)	1.04(-7)
2^{-7}	4.00(-3)	4.80(-4)	4.43(-5)	3.72(-6)	2.95(-7)
2^{-10}	4.30(-3)	8.39(-4)	1.44(-4)	1.60(-5)	1.45(-6)

TABLE 3. Comparison of Root mean square errors in solution of Example 2

ε/N	2^3	2^4	2^5	2^6	2^7
R.K. Mohanty, Urvashi Arora method [10]					
2^{-3}	3.31(-4)	2.18(-3)	1.55(-4)	1.09(-4)	7.68(-5)
2^{-4}	7.42(-4)	5.30(-4)	3.71(-4)	2.60(-4)	1.83(-4)
2^{-5}	1.22(-3)	8.28(-4)	5.71(-4)	4.01(-4)	2.82(-4)
2^{-7}	1.95(-3)	1.22(-3)	8.33(-4)	5.84(-4)	4.11(-4)
2^{-10}	2.43(-3)	1.44(-3)	9.66(-4)	6.77(-4)	4.77(-4)
Proposed method					
2^{-3}	6.48(-4)	4.73(-5)	3.39(-6)	2.39(-7)	1.67(-8)
2^{-4}	8.22(-4)	6.42(-5)	4.84(-6)	3.56(-7)	2.56(-8)
2^{-5}	1.10(-3)	9.19(-5)	7.34(-6)	5.62(-7)	4.18(-8)
2^{-7}	1.40(-3)	1.88(-4)	1.79(-5)	1.52(-6)	1.20(-7)
2^{-10}	1.40(-3)	2.88(-4)	5.53(-5)	6.50(-6)	6.00(-7)

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