Fixed-Ratio Approximation Algorithm for the Minimum Cost Cover of a Digraph by Bounded Number of Cycles

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Abstract: We consider polynomial time approximation for the minimum cost cycle cover problem of an edge-weighted digraph, where feasible covers are restricted to have at most k disjoint cycles. In the literature this problem is referred to as Min-k-SCCP. The problem is closely related to classic Traveling Salesman Problem (TSP) and Vehicle Routing Problem (VRP) and has many important applications in algorithms design and operations research. Unlike its unconstrained variant, the Min-k-SCCP is strongly NP-hard even on undirected graphs and remains intractable in very specific settings. For any metric, the problem can be approximated in polynomial time within ratio 2, while in fixed-dimensional Euclidean spaces it admits Polynomial Time Approximation Schemes (PTAS). In the same time, approximation of the more general asymmetric Min-k-SCCP still remains weakly studied. In this paper, we propose the first fixed-ratio approximation algorithm for this problem, which extends the recent breakthrough Svensson-Tarnawski-Végh and Traub-Vygen results for the Asymmetric Traveling Salesman Problem.

Key-Words: approximation algorithms, fixed ratio, asymmetric traveling salesman problem, cycle cover problem

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1 Introduction

The Cycle Cover Problem (CCP) is a well-known combinatorial optimization problem having important applications in operations research and algorithms design for other combinatorial problems, e.g. the Traveling Salesman Problem (TSP) [2, 5], Vehicle Routing Problem (VRP) [11], several versions of the Stacker Crane Problem (SCP) [4, 8], etc.

Most of the studied CCP settings can be considered as extensions of the classic Linear Sum Assignment Problem (LSAP) formulated on subsets of the symmetric group S_n. For each such a setting, the objective is a *permutation cost* coinciding with total weight of the corresponding routes in a given graph, while a set of feasible permutations is constrained in terms of the properties of their *cycle decomposition* including length or amount of the cycles. For instance, the authors of [3] proposed approximation algorithms for minimum cost graph covering problems by cycles of length at least k. Later (see, e.g. [17]) these results were extended to the cheapest covers by cycles whose lengths belong to a given set $L \subset \mathbb{N}$.

In this paper, we are focused on polynomial time approximation of the Minimum-weight k-Size Cycle Cover Problem (Min-k-SCCP), where it is required to construct a minimum cost cover of a given (di)graph by at most k disjoint cycles [7, 13]. Interest to this topic is confirmed by an *intermediate* complexity status of this problem between the strongly NP-hard TSP (which is Min-k-SCCP for $k = 1$) and LSAP (for $k = n$) that can be solved to optimality in polynomial time.

1.1 Related work

As shown in [14, 15], for any fixed $k \geqslant 1$, the symmetric version of the Min-k-SCCP formulated on undirected graphs inherits complexity and main approximation properties of the classic TSP. Thus, the problem is strongly NP-hard in general case and remains intractable even on the Euclidean plane. The metric Min-k-SCCP is APX-complete, while its Euclidean settings formulated in $\mathbb{R}^d$ admit a Polynomial Time Approximation Schemes (PTAS) for an arbitrary fixed dimension d. In addition, we should note asymptotically exact algorithms designed in [7] for the Min-k-SCCP settings on random graphs and for the Euclidean Max-k-SCCP.

On the other hand, polynomial time approximation of the asymmetric Min-k-SCCP (as for many other related combinatorial problems) remain weakly studied for a long time. For instance, while constant-ratio approximation algorithms for the metric TSP, metric Capacitated Vehicle Routing Problem (CVRP) and their numerous modifications have been known since the late 1970s, thanks to the seminal results of N. Christofides [6], A. Serdyukov [20], L. Wolsey [23], and M. Haimovich and A. Rinnooy Kan [10], until 2018 the Asymmetric Traveling Salesman Problem (ATSP) could only be approximated within the factor $O(\log n/ \log\log n)$ [1].

Recently, O. Svensson, J. Tarnawski, and L. Végh [21] and V. Traub and J. Vygen [22] introduced a breakthrough approach to polynomial time approximation of the ATSP with triangle inequality, which led to the first fixed-ratio algorithms for that problem. For the sake of convenience, in the sequel we refer to this approach and the state-of-the-art $(22+\varepsilon)$-approximation algorithm proposed in [22] as $\mathrm{S(TV)}^2$-framework and algorithm, respectively.

Relying on this algorithm as a building block, the first proofs of polynomial time approximation within fixed factors for several related asymmetric problems including Steiner Cycle Problem (SCP), Rural Postman Problem (RPP), CVRP with unsplittable demands [16] and Prize Collecting ATSP [18] were obtained.

At the same time, for some routing problems, e.g. for Min-k-SCCP, fixed-ratio approximation algorithms still cannot be designed in a similar way, just on the basis of cost-preserving reduction to single or multiple auxiliary ATSP instances. To this end, one can assume that some of these problems can be approximated within fixed ratios by more deep extension of $\mathrm{S(TV)}^2$-framework.

1.2 Contribution

In this paper

(i) we extend the $\mathrm{S(TV)}^2$-framework to the class of routing problems including Min-k-SCCP for $k \geqslant 1$;

(ii) for an arbitrary $\varepsilon > 0$, we propose the first polynomial time $(24+\varepsilon)$-approximation for the Min-k-SCCP.

The rest of the paper is structured as follows. In Section 2, we recall a mathematical formulation of Min-k-SCCP. In the following sections, we show that the initial task of construction of a fixed-ratio approximation algorithm for the Min-k-SCCP can be successively reduced to the similar tasks for more structured instance of this problem, i.e. Min-k-SCCP_S (in Section 3), and strongly laminar instances (in Section 4). We present our main results: $(24+\varepsilon)$-approximation algorithm for the Min-k-SCCP_S and the proof of its performance guarantee in Section 5. Finally, in Section 6 we summarize our paper.

2 Problem statement

Suppose, we are given by a strongly connected digraph $G=(V,E)$ and weighting function $c\colon E \to \mathbb{R}_+$ that specifies transportation cost $c_e = c(e) = c(v,w)$ for a transition along each arc $e=(v,w) \in E$ of the graph G. Hereinafter, we assume that the *triangle inequality*

$$c(v,w) \leqslant c(v,u) + c(u,w) \qquad (1)$$

holds for any arcs (v,u), (u,w), and (v,w). To any multi-set of arcs F, we assign the incidence vector $x=\chi^F$, $x\colon E \to \mathbb{Z}_+$ and cost $c(F) = \sum_{e\in E} c_e x_e$.

Definition 1. *An instance of the Min-k-SCCP is defined as follows. For an ordered pair (G,c), it is required to compute a spanning Eulerian submultigraph $G_F=(V,F)$ of the digraph G, such that*

(i) G_F has no isolated nodes

(ii) G_F has at most k connected components

(iii) F has the minimum cost $c(F)$.

Although the given statement slightly generalizes the known formulation of the Min-k-SCCP studied in previous papers [7, 15], these formulations coincide to each other for complete graphs. Furthermore, the classic ATSP is Min-k-SCCP for $k=1$.

3 Restricted Min-k-SCCP

In this section, we reduce approximation of the Min-k-SCCP to the same task for the restricted version of this problem, which we call Min-k-SCCP_S.

Definition 2. *An instance of the Min-k-SCCP_S is given by a triple (G,c,S), where $S \subset V$, $|S| = k$. It is required to find a spanning Eulerian submultigraph G_F, which along with conditions (i)–(iii) satisfies an additional constraint: $V(D) \cap S \neq \varnothing$ for any connected component D of G_F.*

We use standard notation $\delta^+(U) = \{(v,w)\colon v \in U, w \in V \setminus U\}$, $\delta^-(U) = \delta^+(V \setminus U)$, and $\delta(U) = \delta^-(U) \cup \delta^+(U)$ for the cuts specified by an arbitrary non-empty subset of nodes $U \subset V$. In addition, we use the abbreviations $\delta(v) = \delta(\{v\})$ and $x(E') = \sum_{e \in E'} x_e$ for any $v \in V$ and subset of arcs $E' \subset E$. By OPT and OPT_S we denote the costs of optimum solutions of the initial Min-k-SCCP instance (G, c) and the corresponding auxiliary Min-k-SCCP_S instances (G, c, S), respectively.

Lemma 1. *For arbitrary $k \geqslant 1$ and $\alpha > 0$, existence of an α-approximation polynomial algorithm for the Min-k-SCCP$_S$ implies polynomial time approximation for the Min-k-SCCP within the same ratio.*

Proof. Let (G, c) be a Min-k-SCCP instance to be solved and $\mathcal{A}_S$ be the α-approximation algorithm for the Min-k-SCCP_S. Since G is strongly connected, the instance (G, c) and all the auxiliary instances (G, c, S) for $S \subset V$, $|S| = k$ are feasible and can be solved to optimality. By applying algorithm $\mathcal{A}_S$ to any instance (G, c, S) we assign the spanning Eulerian subgraph $G_S = (V, F_S)$, such that, for each connected component D of the graph G_S, $V(D) \cap S \neq \varnothing$ and $\mathrm{OPT}_S \leqslant c(F_S) \leqslant \alpha\, \mathrm{OPT}_S$.

Further, let $H^* = (V, F^*)$ be an optimal solution of the initial instance (G, c) and $S^* \subset V$ be an arbitrary k-element subset, which has a non-empty intersection with any connected component of the graph H^*. Then, for the subgraph $(V, F) = \arg\min\{c(F_S)\colon S \subset V, |S| = k\}$, we have

$$\mathrm{OPT} \leqslant c(F) \leqslant c(F_{S^*}) \leqslant \alpha \cdot \mathrm{OPT}_{S^*} \leqslant \alpha \cdot c(F^*) = \alpha \cdot \mathrm{OPT}\,.$$

Thus, the Min-k-SCCP has an α-approximation polynomial time algorithm, since $\big|\{S \subset V\colon |S| = k\}\big| = O(n^k)$. Lemma is proved. □

4 Strongly laminar instances

We proceed with approximation algorithms for the Min-k-SCCP_S by assignment to this problem the following MILP-model

$$\min \sum_{e \in E} c_e x_e \tag{2}$$

$$\text{s.t.}\quad x(\delta^-(v)) - x(\delta^+(v)) = 0 \qquad (v \in V), \tag{3}$$

$$x(\delta(U)) \geqslant 2 \qquad (U \in \mathcal{V}) \tag{4}$$

$$x_e \in \mathbb{Z}_+ \qquad (e \in E), \tag{5}$$

where $\mathcal{V} = \{U\colon \varnothing \neq U \subset V \setminus S\} \cup \{\{u\}\colon u \in S\}$. Here, equations (3) ensure that any feasible submultigraph will be Eulerian while (4) provide an absence of the isolated nodes and upper bound for the number of its connected components. In the sequel, we consider LP-relaxation $\mathcal{P}$ of problem (2)–(5) and its dual $\mathcal{D}$:

$$\max \sum_{U \in \mathcal{V}} 2y_U$$

s.t.

$$a_w - a_v + \sum_{U \in \mathcal{V}\colon e \in \delta(U)} y_U \leqslant c(e) \qquad (e = (v,w) \in E)$$

$$y_U \geqslant 0 \qquad (U \in \mathcal{V}).$$

Under our assumptions, both problems are solvable and have the same optimal value $\mathcal{P}^* = \mathcal{D}^*$.

In this section, we show that approximation of the Min-k-SCCP_S can be reduced to the case of structured instances of this problem called *strongly laminar*. Recall that a family of subsets $\mathcal{L}$ of the set V is called *laminar*, if for any $L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 \neq \varnothing$ implies either $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$.

Definition 3. *A tuple $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$ is a strongly laminar Min-k-SCCP$_S$ instance, if*

- *$G = (V, E)$ is a strongly connected digraph and $|V| > k$;*
- *S is a subset of V of size k;*
- *$\mathcal{L}$ is a laminar family of subsets of $V \setminus S$, such that for any $L \in \mathcal{L}$, the induced subgraph $G[L]$ is strongly connected;*
- *$x : E \to \mathbb{R}_+$ is a feasible solution for (3)–(4), s.t. $x(\delta(L)) = 2$ for an arbitrary $L \in \mathcal{L}$;*
- *$y\colon \mathcal{L} \to \mathbb{R}_+$.*

Each $\mathcal{I}$ induces the structured instance $(G, \bar{c}, S)$ of the Min-k-SCCP_S with special weighting function

$$\bar{c}_e = \bar{c}(e) = \sum_{L \in \mathcal{L}\colon e \in \delta(L)} y_L \quad (e \in E) \tag{6}$$

Define $y'\colon \mathcal{V} \to \mathbb{R}_+$ as follows:

$$y'_U = \begin{cases} y_U, & \text{if } U \in \mathcal{L}, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, x and $(0, y')$ are optimal solutions of the corresponding linear programs $\mathcal{P}$ and

$\mathcal{D}$, for $c \equiv \bar{c}$. In the sequel, we call these problems $\mathcal{P}_{ind}$ and $\mathcal{D}_{ind}$) and use the following notation

$$\mathrm{LP}(\mathcal{I}) = \mathcal{P}^*_{ind} = \sum_{e\in E} \bar{c}_e x_e = \sum_{L\in\mathcal{L}} 2y_L = \mathcal{D}^*_{ind}. \quad (7)$$

The concept of strongly laminar Min-k-SCCP$_S$ instance is a natural extension of the known concept of strongly laminar ATSP instance. In [22], those instances were considered for $k = 1$. In our case, we use a more simple notation $\mathcal{I} = (G, \mathcal{L}, x, y)$. Furthermore, as in [21], in the case where $\mathcal{L}$ consists only singletons $\{v\}$, we refer to $\mathcal{I}$ as *singleton instance* of the Min-k-SCCP$_S$.

Lemma 2. *Suppose, for some $\alpha \geqslant 1$, there exists a polynomial time algorithm that that, for any strongly laminar instance $\mathcal{I} = (G, \mathcal{L}, l, S, x, y)$, $l \leqslant k$, computes a feasible solution of cost at most $\alpha \cdot \mathrm{LP}(\mathcal{I})$. Then, there exists a polynomial time algorithm that, for an arbitrary instance of Min-k-SCCP$_S$, finds a feasible solution of cost $c(F_S) \leqslant \alpha \cdot \mathcal{P}^*$.*

Proof. Consider an arbitrary Min-k-SCCP$_S$ instance (G, c, S). Let x^* be an optimal solution of the LP-relaxation $\mathcal{P}$. Although $\mathcal{P}$ has an exponential number of constraints (4), x^* can be found in polynomial time, e.g. by the ellipsoid method augmented with polynomial time separation oracle [9]. In addition, without loss of generality, we can assume that

$$\left|\left\{U \in \mathcal{V} \colon x^*(\delta(U)) = 2\right\}\right| = poly(n). \quad (8)$$

By construction, the graph $G' = (V, E')$, where $E' = \{e \in E \colon x^* > 0\}$ has at most k strongly connected components $W_1, \ldots, W_p$ for $p \leqslant k$. Besides that, there exists a partition $S_1 \cup \ldots \cup S_p$ of the set S, such that $S_i \subset W_i$ for each $i = \overline{1,p}$. By $x'[i]$ we denote the restriction of $x*$ on $E'(W_i)$.

Let us verify that $x'[i]$ is an optimal solution of the LP-relaxation $\mathcal{P}_i$ of the model (2)–(5) for the instance (W_i, k_i, S_i), where $k_i = |S_i|$. Indeed, the equations

$$x'[i](\delta^-(v)) - x'[i](\delta^+(v)) = 0 \qquad (v \in V(W_i))$$
$$x'[i](\delta(U)) \geqslant 2 \qquad (U \in \mathcal{V}_i),$$

for $\mathcal{V}_i = \left\{U \colon \varnothing \neq U \subset V(W_i) \setminus S_i\right\} \cup \left\{\{u\} \colon c \in S_i\right\}$, follows straightforwardly from the choice of $x'[i]$. Next, optimality of $x'[i]$ in problem $\mathcal{P}_i$ follows from the evident decomposition

$$\mathcal{P}^* = \sum_{e\in E} c_e x^*_e = \sum_{i=1}^{p} \sum_{e\in E(W_i)} c_e(x'[i])_e \quad (9)$$

and the optimality of x^* in problem $\mathcal{P}$.

For each $i = \overline{1,p}$ find an optimal solution $(a^*[i], y^*[i])$ of the dual $\mathcal{D}_i$. Due to (8), these computations can also be carried out in polynomial time. Applying Karzanov's algorithm [12] and following the argument of [22, Lemma 3], to any solution $(a^*[i], y^*[i])$ we assign an optimal solution $(a'[i], y'[i])$ of $\mathcal{D}_i$, such that $\mathrm{supp}(y'[i]) = \mathcal{L}_i = \{U \in \mathcal{V}_i \colon (y'[i])_U > 0\}$ is a laminar family and, for any $L \in \mathcal{L}_i$, the subgraph $W_i[L]$ is strongly connected.

By $y''[i]$ denote the restriction of $y'[i]$ onto $\mathcal{L}_i$ and introduce the strongly laminar instance $\mathcal{I}_i = (W_i, \mathcal{L}_i, k_i, S_i, x'[i], y''[i])$. By construction, the problems $\mathcal{P}_i$ and $(\mathcal{P}_i)_{ind}$ have the same set of feasible solutions. Furthermore, for the corresponding weighting functions c_e and $\bar{c}_e$, we have $c_e = \bar{c}_e + (a'[i])_w - (a'[i])_v$, which follows from (6) and the complementarity conditions. As a consequence, for an arbitrary feasible solution χ of both problems, we have

$$\sum_{e\in E'(W_i)} c_e\chi_e = \sum_{e=(v,w)\in E'(W_i)} (\bar{c}_e + (a'_w[i]) - (a'_v[i]))\chi_e$$
$$= \sum_{e\in E'(W_i)} \bar{c}_e\chi_e + \sum_{u\in V(W_i)} (\chi(\delta^-(u)) - \chi(\delta^+(u)))(a'[i])_u$$
$$= \sum_{e\in E'(W_i)} \bar{c}_e\chi_e. \quad (10)$$

Finally, by the hypothesis of Lemma 2, for each instance $\mathcal{I}_i$ in polynomial time we can find a solution (W_i, F_i), such that $\bar{c}(F_i) \leqslant \alpha\,\mathrm{LP}(()\mathcal{I}_i)$, which implies $c(F_i) \leqslant \alpha\mathcal{P}^*_i$ due to (10). Therefore, the subgraph (V, F), where $F_1 \cup \ldots \cup F_p$, is a desired approximate solution for the initial Min-k-SCCP$_S$ instance (G, c, S) of cost

$$c(F) = \sum_{i=1}^{p} c(F_i) \leqslant \alpha \sum_{i=1}^{p} \mathcal{P}^*_i = \alpha \cdot \mathcal{P}^*,$$

where the last equality follows from (9). Lemma is proved. □

5 Approximation of a strongly laminar instance

In this section, we propose an approximation algorithm for strongly laminar instances of the Min-k-SCCP$_S$. Consider an arbitrary strongly laminar instance $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$ with induced weighting function c.

5.1 Preliminaries

We start with some necessary additional notation and preliminary technical results. Let $W \in$

$\mathcal{L} \cup \{V\}$ be the minimal subset that contains nodes u and v. The u-v-path $P_{u,v}$ in the strongly connected subgraph $G[W]$ (of the graph G) and visiting each $L \in \mathcal{L}$ at most once is called a *nice* path. As shown in [22], for arbitrary u and v the nice path $P_{u,v}$ can be found in polynomial time and its cost is defined by the formula

$$c(E(P_{u,v})) = \sum_{L\in\mathcal{L}_W\colon L\cap V(P_{u,v})\neq\varnothing} 2y_L - \sum_{L\in\mathcal{L}_W\colon u\in L} y_L - \sum_{L\in\mathcal{L}_W\colon v\in L} y_L, \quad (11)$$

where $\mathcal{L}_W = \{L \in \mathcal{L}\colon L \subset W\}$. It is useful to assign to each subset W the value

$$D_W = \max\{D_W(u,v)\colon u, v \in W\}, \quad (12)$$

where

$$D_W(u,v) = c(E(P_{u,v})) + \sum_{L\in\mathcal{L}_W\colon u\in L} y_L + \sum_{L\in\mathcal{L}_W\colon v\in L} y_L. \quad (13)$$

We slightly extend the concept of a *backbone* introduced in [21].

Definition 4. *A Eulerian submultigraph without isolated nodes B of the graph G is called S-backbone, if*

- $V(B) \cap L \neq \varnothing$ *for any* $L \in \mathcal{L}_{\geqslant 2} = \{L \in \mathcal{L}\colon |L| \geqslant 2\}$
- $S \subset V(B)$
- $V(D) \cap S \neq \varnothing$ *holds for an arbitrary connected component D of B.*

Definition 5. *An ordered pair $(\mathcal{I}, B)$, where $\mathcal{I}$ is a strongly laminar Min-k-SCCP$_S$ instance and B is an S-backbone, is called a vertebrate pair.*

The fixed-ratio approximation algorithm proposed in [22] for the ATSP relies on an efficient building block called (κ, η)-*algorithm for vertebrate pairs.* For given parameters $\kappa, \eta \geqslant 0$ and arbitrary vertebrate pair $(\mathcal{I}, B)$, this algorithm computes in polynomial time a set of arcs F', such that $(V, E(B) \cup F')$ is a spanning Eulerian connected submultigraph of G and

$$c(F') \leqslant \kappa\,\mathrm{LP}(\mathcal{I}) + \eta \cdot \sum_{v\in V\setminus V(B):\{v\}\in L} 2y_{\{v\}}. \quad (14)$$

As follows from [22], for any $\varepsilon > 0$ and $(\mathcal{I}, B)$, where $\mathcal{I}$ as a strongly laminar ATSP instance and B is an arbitrary connected backbone, there exists $(2, 14 + \varepsilon)$-algorithm for vertebrate pairs.

We extended (κ, η)-algorithm to the case of vertebrate pairs, where $\mathcal{I}$ is a strongly laminar Min-k-SCCP$_S$ instance and B is an S-backbone.

Lemma 3. *For any $k > 1$, $\varepsilon > 0$, there exists a polynomial time algorithm, which extends the S-backbone B of an arbitrary vertebrate pair $(\mathcal{I}, B)$ to a feasible solution of the Min-k-SCCP$_S$ instance $\mathcal{I}$ by appending a multiset of arcs F', such that (14) is valid for $\kappa = 2$ and $\eta = 14 + \varepsilon$.*

The proof of Lemma 3 can be obtained in a similar way to the argument of [22, Theorem 35], where existence of (κ, η)-algorithm for $\eta = 4\alpha + \beta + 1 + \varepsilon$ was proved as a consequence of existence of the polynomial time (α, κ, β)-algorithm for an auxiliary Subtour Cover Problem (SCP). In turn, (α, κ, β)-algorithm for the SCP was developed (in [22, Theorem 16]) on top of the well-known flow rerouting and rounding technique (see, e.g. [19]) applicable if

$$x(\delta(U)) \geqslant 2 \quad (\varnothing \neq U \subset V \setminus V(B)). \quad (15)$$

In the ATSP, inequality (15) follows straightforwardly from the problem formulation. In our case, we ensure it by introducing S-backbones. For the sake of brevity, we postpone the full rather technical proof to the forthcoming paper.

5.2 Algorithm: scheme and discussion

The proposed algorithm (Algorithm $\mathcal{A}$) computes a feasible solution $G_F = (V, F)$ of a given strongly laminar Min-k-SCCP instance $\mathcal{I}$, where G_F is a spanning submultigraph of the graph G, each whose connected component D intersects the set S, i.e. $V(D) \cap S \neq \varnothing$. As outer parameters, Algorithm $\mathcal{A}$ takes the (κ, η)-algorithm $\mathcal{A}_{\kappa,\eta}$ for vertebrate pairs (see Def. 4 and Lemma 3) and $(3\kappa + \eta + 2)$-approximation algorithm $\mathcal{A}^*_{ATSP}$ for the ATSP [22].

At step 1, we construct the family $\mathcal{L}_{\bar{S}}$ consisting of high-level non-singleton elements of the laminar family $\mathcal{L}$. By construction, all of them do not intersect the set S. Therefore, the instance $\mathcal{I}'$ obtained at step 2 is a singleton instance. Further, at step 3, the S-backbone B can be computed from the cycle cover provided by $(2, 0)$-light algorithm [21, Th. 4.1] or by a sampling. At step 4, we extend B to a solution for the contracted graph, which is augmented with ATSP tours for each $L \in \mathcal{L}_{\bar{S}}$ at steps 5–8. Finally, at step 9, we combine all the parts of the resulting solution.

Lemma 4. *For any $k > 1$ and $\kappa, \eta \geqslant 0$, existence of a polynomial time (κ, η)-algorithm for*

Algorithm $\mathcal{A}$

Input: strongly laminar Min-k-SCCP$_S$ instance $\mathcal{I} = (G, \mathcal{L}, k, S, x, y)$
Parameters: (κ, η)-algorithm $\mathcal{A}_{\kappa,\eta}$ for vertebrate pairs; algorithm $\mathcal{A}^*_{ATSP}$;
Output: a feasible solution $G_F = (V, F)$ of the instance $\mathcal{I}$.

set up a sub-family

$$\mathcal{L}_{\bar{S}} = \big\{L \in \mathcal{L}_{\geqslant 2} : (V(S) \cap L = \varnothing) \wedge (\forall U \in \mathcal{L} : L \subset U)(V(S) \cap U \neq \varnothing)\big\}$$

construct a singleton Min-k-SCCP$_S$ instance $\mathcal{I}' = (G', \mathcal{L}', k, S, x', y')$, where
a: multigraph $G' = (V', E')$ is obtained by contracting each $L \in \mathcal{L}_{\bar{S}}$ into the corresponding node v_L,
b: laminar family

$$\mathcal{L}' = \mathcal{L} \setminus \big\{L \in \mathcal{L} : (\exists U \in \mathcal{L}_{\bar{S}})\,(L \subseteq U)\big\} \bigcup_{L \in \mathcal{L}_{\bar{S}}} \{v_L\}$$

c: vector x' is a restriction of x on E' and $y' : \mathcal{L}' \to \mathbb{R}_+$ is defined by:

$$y'_U = \begin{cases} y_U, & \text{if } U \in \mathcal{L} \cap \mathcal{L}' \\ y_L + D_L/2, & \text{otherwise} \end{cases} \qquad (16)$$

construct an S-backbone B
apply algorithm $\mathcal{A}_{\kappa,\eta}$ to the vertebrate pair $(\mathcal{I}', B)$ and compute the Eulerian multiset of arcs F' (in the graph G')
for each $L \in \mathcal{L}_{\bar{S}}$ **do**
  obtain a traveling salesman tour F_L in L by applying $\mathcal{A}^*_{ATSP}$ to $(\mathcal{I}, L)$
  extend F_l by a nice path P_L in $G[L]$ to ensure that the resulting solution remains a Eulerian graph
end for
set $F = \Big(\bigcup_{L \in \mathcal{L}_{\bar{S}}} (F_L \cup P_L)\Big) \dot{\cup} E(B) \dot{\cup} F'$
return (V, F).

vertebrate pairs (Def. 5) implies the existence of an algorithm which, for an arbitrary strongly laminar Min-k-SCCP$_S$ instance $\mathcal{I}$, computes in polynomial time a feasible solution (V, F) of cost

$$cost(F) \leqslant (3\kappa + \eta + 4)\,\mathrm{LP}(\mathcal{I}). \qquad (17)$$

Proof. Since $\mathcal{A}_{\kappa,\eta}$ and $\mathcal{A}^*_{ATSP}$ are polynomial time algorithms, all the steps of Algorithm $\mathcal{A}$ can be carried out in polynomial time as well.

To prove (17), obtain upper bounds for the costs $C(F')$, $c(E(B))$ and $c(F_L)$ separately. By (14) and (16), for $c(F')$ we have

$$\begin{aligned} c(F') &\leqslant \kappa \cdot \mathrm{LP}(\mathcal{I}') + \eta \cdot \Big(\sum_{L \in \mathcal{L}_{\bar{S}}} 2y'_{\{v_L\}} + \sum_{v \notin V(B) :\, \{v\} \in \mathcal{L} \cap \mathcal{L}'} 2y'_{\{v\}}\Big) \\ &= \kappa \cdot \Big(\sum_{L \in \mathcal{L} \cap \mathcal{L}'} 2y_L + \sum_{L \in \mathcal{L}_{\bar{S}}} (2y_L + D_L)\Big) + \eta \cdot \Big(\sum_{v \notin V(B) :\, \{v\} \in \mathcal{L} \cap \mathcal{L}'} 2y_{\{v\}} + \sum_{L \in \mathcal{L}_{\bar{S}}} (2y_L + D_L)\Big) \\ &= \kappa \cdot \sum_{L \in \mathcal{L} \cap \mathcal{L}'} 2y_L + (\kappa + \eta) \cdot \sum_{L \in \mathcal{L}_{\bar{S}}} (2y_L + D_L) + \eta \cdot \sum_{v \notin V(B) :\, \{v\} \in \mathcal{L} \cap \mathcal{L}'} 2y_{\{v\}}. \end{aligned}$$

Next, for each F_L due to [22, Lemma 12],

$$\begin{aligned} & c(F_L) + c(P_L) \\ &\leqslant (2\kappa + 2)\,\mathrm{LP}(\mathcal{I}_L) + (\kappa + \eta)(\mathrm{LP}(\mathcal{I}_L) - D_L) \\ &= (3\kappa + \eta + 2)\,\mathrm{LP}(\mathcal{I}_L) - (\kappa + \eta) D_L, \end{aligned}$$

where $\mathrm{LP}(\mathcal{I}_L) = \sum_{U \in \mathcal{L} :\, U \subset L} 2y_U$.
Further, by [21, Th. 4.1],

$$\begin{aligned} c(E(B)) &\leqslant 2\,\mathrm{LP}(\mathcal{I}') \\ &= 2 \cdot \Big(\sum_{L \in \mathcal{L} \cap \mathcal{L}'} 2y_L + \sum_{L \in \mathcal{L}_{\bar{S}}} (2y_L + D_L)\Big). \end{aligned}$$

Finally, taking into account that $D_L \leqslant \mathrm{LP}(\mathcal{I})$ and

$$\sum_{L \in \mathcal{L}_{\bar{S}}} 2y_L + \sum_{v \notin V(B) :\, \{v\} \in \mathcal{L} \cap \mathcal{L}'} 2y_{\{v\}} + \sum_{L \in \mathcal{L}_{\bar{S}}} \mathrm{LP}(\mathcal{I}_L) \leqslant \mathrm{LP}(\mathcal{I}),$$

we obtain

$$\begin{aligned} & c(E(B)) + c(F') + \sum_{L \in \mathcal{L}_{\bar{S}}} \big(c(F_L) + c(P_L)\big) \\ &\leqslant (\kappa + 2) \sum_{L \in \mathcal{L} \cap \mathcal{L}'} 2y_L + (\kappa + \eta + 2) \sum_{L \in \mathcal{L}_{\bar{S}}} (2y_L + D_L) \\ &+ \eta \sum_{v \notin V(B) :\, \{v\} \in \mathcal{L} \cap \mathcal{L}'} 2y_{\{v\}} + \sum_{L \in \mathcal{L}_{\bar{S}}} \big((3\kappa + \eta + 2)\,\mathrm{LP}(\mathcal{I}_L) \\ &- (\kappa + \eta) D_L\big) \leqslant (3\kappa + \eta + 2)\,\mathrm{LP}(\mathcal{I}) + 2 \sum_{L \in \mathcal{L}_{\bar{S}}} D_L \\ &\leqslant (3\kappa + \eta + 4)\,\mathrm{LP}(\mathcal{I}). \end{aligned}$$

Lemma is proved. □

Now, we are ready to formulate our main results, which easily follow from the proved lemmas.

Theorem 1. *For an arbitrary $k \geqslant 1$ and $\varepsilon > 0$, there exists a polynomial time algorithm that assigns to an arbitrary instance (G, c, S) of the Min-k-SCCP$_S$ an approximate solution (V, F_S) of cost $c(F_S) \leqslant (24 + \varepsilon) \cdot \mathcal{P}^*$, where $\mathcal{P}$ is an LP-relaxation of the MILP-model* (2)–(5).

Theorem 2. *For an arbitrary $k \geqslant 1$ and $\varepsilon > 0$, the Min-k-SCCP can be approximated within a ratio $24 + \varepsilon$ by a polynomial time algorithm.*

6 Conclusion

In this paper, by extending the breakthrough results of O. Svensson, J. Tarnawski, L. Végh, V. Traub, and J. Vygen, we proposed the first fixed-ratio approximation algorithm for the problem of computing a minimum cost cover of a digraph by at most k cycles (Min-k-SCCP). We showed that, for any $k > 1$ and $\varepsilon > 0$, the Min-k-SCCP can be approximated in polynomial time within the ratio $24 + \varepsilon$. The believe that the our approach can be extended to more wide class of asymmetric routing problems.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Michael Khachay and Katherine Neznakina proposed general ideas of the proposed approximation framework and formulations of main theorems.
Ksenia Rizhenko designed Algorithm A.
Daniil Khachai and Ksenia Rizhenko proved all the lemmas.

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Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.