Split Detour Monophonic Sets in Graph

M. MAHENDRAN[1], R. KAVITHA[2]
[1]Department of Humanities and Science,
Rajalakshmi Institute of Technology,
Kuthambakkam Post, Chennai, 600 124,
INDIA

[2]Department of Mathematics and Statistics,
Faculty of Science and Humanities,
SRM Institute of Science and Technology, Kattankulathur Campus,
Chengalpattu, TamilNadu,
INDIA

Abstract: - A subset $T \subseteq V$ is a detourmonophonic set of G if each node (vertex) x in G contained in an p-q detourmonophonic path where p, q ∈ T.. The number of points in a minimum detourmonophonic set of G is called as the detourmonophonic number of G, dm(G). A subset $T \subseteq V$ of a connected graph G is said to be a split detourmonophonic set of G if the set T of vertices is either T = V or T is detoumonophonic set and V – T induces a subgraph in which is disconnected. The minimum split detourmonophonic set is split detourmonophonic set with minimum cardinality and it is called a split detourmonophonic number, denoted by $dm_s(G)$. For certain standard graphs, defined new parameter was identified. Some of the realization results on defined new parameters were established.

Key-Words: - Distance, geodesic, monophonic path, detour, monophonic number, detourmonophonic path, detourmonophonic number, split detourmonophonic number.

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1 Introduction

In the whole paper, we assume finite, undirected and connected graphs represented by G = (V,E), where V denotes the set of vertices (nodes or points) and E the set of edges (lines) (without loops or multiple edges), with |V| = n and |E| = m respectively. For basic concepts, we refer to [1]. The distance, [2], d(p,q) in G is defined as the length of the shortest path between p and q in G. A p-q path of length d(p,q) is referred to as a p-q geodesic. A vertex x is extreme if the neighbors of x induce a complete graph. A set S ⊆ V (G) is termed a geodetic set of G if every vertex of G lies on a x-y geodesic, where x and y are in S. The cardinality of a minimum geodetic set is known as the geodetic number of G, denoted by g(G). The concept of geodetic numbers was introduced and studied in [3], [4], [5].

In a path $x_1, x_2, \ldots, x_n$, an edge x_ix_j with $j \geq i + 2$ is termed a chord. For nodes p and q in G, a p-q path is considered a monophonic path if this p-q path is the chordless path. A monophonic set T of G contains every vertex of G in the monophonic path of some pair of points in T. The number of points in the minimum monophonic set is called the monophonic number and is denoted by m(G). This concept was discussed in detail in [6], [7].

A split monophonic set T of the graph is T is either equal to V or T is a monophonic set and the subgraph [V – T] is disconnected. This concept was studied in [8]. A p – q chordless path with maximum length is known as a p – q detourmonophonic path. A set T ⊆ V is a detourmonophonic set of graphs if every nodes x of G contained in a p – q detourmonophonic path for any nodes p and q in T. The detourmonophonic number of the graph notated by dm(G), the number of vertices in a minimum detourmonophonic set. This concept was discussed in [9], [10].

A detour monophonic set T is connected if the subgraph induced by T is connected. The number of vertices in the minimum connected detourmonophonic set is the connected detourmonophonic number, denoted by dmc(G). This concept was discussed in detail in [11]. The concept of outer connected in detour was discussed in detail in [12].

Result 1.1. [9], Every detourmonophonic set contains all its extreme nodes of graph.

2 Split Detourmonophonic Number of a Graph

Definition 2.1. A set T of vertices in a connected graph G is a split detour monophonic set if either T is a detour monophonic set and the subgraph induced by V –T is disconnected or T = V . A split detourmonophonic set with minimum cardinality is a minimum splitdetour monophonic set and this number is the split detourmonophonic number $dm_s(G)$.

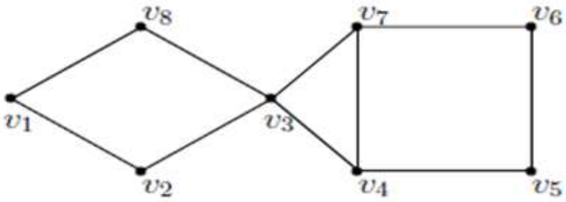

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Fig. 1: A graph with $dx_s(G) = 4$

Example 2.2. In a given G, Figure 1, S = {v_1, v_5, v_6} is a minimum detourmonophonic set and dm(G) = 3. It was noticed that V − S is not disconnected. Let S_1 = {v_2, v_8, v_5, v_6}. S_1 is minimum split detourmonophonic set and $dm_s(G) = 4$.

From Example 2.2, it is observed that detour monophonic number differs from the split detour monophonic number.

Result 2.3. Split detour monophonic set of any graph G may or may not contain the cutvertex of the given graph.

Proof. We prove this result by inspecting Example 2.2, it is observed that the sets S_1 = {v_2, v_5, v_6, v_8}, and S_2 = {v_1, v_3, v_5, v_6} are the split detour monophonic sets. Also, v_3 is cutvertex of G and $v_3 \in S_2$ but $v_3 \notin S_1$. Hence, the cutvertex need not be a member of the split detour monophonic set.

Let us consider another example to express that dm, dm_s, dm_c are different

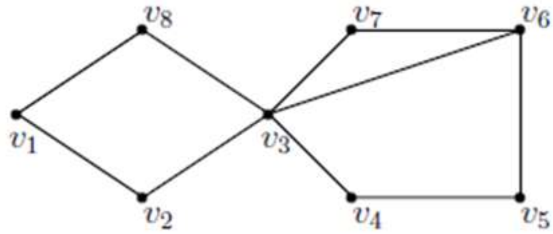

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Fig. 2: A graph with $dm(G) \neq dm_s(G) \neq dm_c(G)$

Example 2.4 In above graph Figure 2, T = {v_1, v_5, v_6, v_7} - minimum detourmonophonic set and dm(G) = 4. Also, the set T_1 = {v_1, v_3, v_5, v_6, v_7} is a minimum split detourmonophonic set, $dm_s(G) = 5$. Clearly, T_1 = {v_1, v_2, v_3, v_4, v_5, v_6, v_7} is the minimum connected detourmonophonic set, $dm_c(G) = 7$. Hence, the $dm(G) \neq dm_s(G) \neq dm_c(G)$.

We know that the detourmonophonic set is contained in split detourmonophonic set and this implies $dm(G) \leq dm_s(G)$. Also, a split detourmonophonic set is contained in a connected detourmonophonic set. Hence, $dm_s(G) \leq dm_c(G)$.
Combining the above result, we can state the theorem as

Result 2.5 For graph with order n, $2 \leq dm(G) \leq dms(G) \leq dmc(G) \leq n$, $dms(G) \neq n - 1$.

Result 2.6 Every extreme node of the graph in split detourmonophonic set.

Proof. By Result 1.1, Every extreme vertex contained in the detourmonophonic set and also, every detourmonophonic set is a subset of the split detourmonophonic set. Hence every extreme vertex belongs to a split detourmonophonic set.

Corollary 2.7 In complete graph $K_n(n \geq 2)$, $dm_s(Kn) = n$.

Remark 2.8 Converse of the above fact need not be true. For the graphs P_3 and P_4, it is clear that $dm_s(P_3) = 3$ and $dm_s(P_4) = 4$.

Result 2.9 For any cycle $G = C_n(n \geq 4)$, $dm_s(G) = \begin{cases} 2 \; if \; n \; even \\ 3 \; if \; n \; odd \end{cases}$

Proof. Let the cycle $G = C_n(n \geq 4)$ be Cn : $v_1, v_2, \ldots, v_n, v_1$ of order n. For n even. the set S = {v_1, $v_{n/2+1}$} is a minimum split detourmonophonic set. In general, we can generate $\frac{n}{2}$ minimum split detourmonophonic sets which can be represented as S_i = {v_i, $v_{n/2+i}$} where i = 1, 2, . . . , n/2 and $dm_s(G) = 2$.
For n odd. On verifying the set S = {v_1, $v_{\frac{n+1}{2}}$,$v_{\frac{n+3}{2}}$}is a smallest split detourmonophonic set. In general,

we can generate n minimum split detourmonophonic which can be represented as Si = $\{v_i, v_j, v_{j+1}\}$ such that dm(v_i, v_j) = $\frac{n-1}{2}$ where $1 \le i \le n$ and $1 \le j \le n$. Hence $dm_s(G) = 3$.

Result 2.10 For any path G = $P_n (n \ge 5)$, dms(G) = 3.
Proof. Consider the Path $P_n = v_1v_2v_3 \ldots v_{n-1}v_n$. On the view, the set S = $\{v_1, v_n\}$ forms minimum detourmonophonic set, dm(G) = 2. Notified that Si = $\{v_1, v_i, v_n\}$ where $d(v_1, v_i) \ge 2$ and $d(v_i, v_n) \ge 2$ is a smallest split detourmonophonic set. Hence $dm_s(G) = 3$.

Result 2.11 For G = $K_{p,q}(2 \le p \le q)$, $dm_s(G) = p$.
Proof. Let X = $\{x_1, x_2, \ldots, x_p\}$ and Y = $\{y_1, y_2, \ldots, y_q\}$ be the partite sets of G. It is seen that the set S = X forms a split detourmonophonic set with smallest cardinality and so $dm_s(G) = |U| = p$.
Result 2.12 For $W_n = K_1 + C_{n-1} (n \ge 5)$, $dm_s(W_n) = \begin{cases} 3 & if\ n\ even \\ 4 & if\ n\ odd \end{cases}$
Proof. Let $W_n = K_1 + C_{n-1} (n \ge 5)$, $K_1 = \{x\}$, C_{n-1} : $v_1v_2v_3 \ldots, v_{n-1}v_nv_1$ be the wheel of order n. For n even. It is noticed that the set S = $\{v_1, v_{n/2+1}, x\}$ is a minimum split detourmonophonic set. In general, we can generate $n/2$ minimum split detourmonophonic sets which can be represented as $S_i = \{v_i, v_{n/2+i}, x\}$ where i = 1, 2,... ,n/2 and $dm_s(Wn) = 3$. For n odd. Clearly the set S = $\{v_1, v_{(n+1)/2}, v_{(n+3)/2}, x\}$ is a minimum split detourmonophonic set. In general, we can generate n minimum split detourmonophonic which can be represented as $S_i = \{v_i, v_j, v_{j+1}, x\}$ such that $dm(v_i, v_j) = (n-1)/2$ where $1 \le i \le n$ and $1 \le j \le n$. Hence $dm_s(Wn) = 4$.

Open Problem 2.13 Can you find an interconnected undirected circuit G for which $dm_s(G) = n$.

The concept of a split detourmonophonic set can be applied in one of the computational intelligence methods namely neural network as witnessed in [13] and also our new design could be facilitated in graph neural network which is studied in [14]. The application of our new design can be evolved in [15], [16].

The concept of a split detourmonophonic set may be applied in artificial intelligence and further can be studied in [17], [18]. [19].

3 Some Existence Results

Result 3.1 There exists interconnected undirected circuit G of order n, as arbitrary $dm_s(G) = k$ and $dm_c(G) = k + 1$ where n, k integers with $2 < k < n$,

Proof. Let P_3 : $u_1u_2u_3$ be the path of order 3. Now join the vertices $v_1, v_2, \ldots, v_{n-k}$ to u_1 as well as with u_3. Further, add $k - 3$ vertices such as $w_1, w_2, w_3, \ldots, w_{k-3}$ to the vertex v_{n-k}. Hence, the desirable graph G of order n as given in Figure 3 is obtained.

It is noticed that S = $\{w_1, w_2, \ldots, w_{k-2}\}$ does not form a split detourmonophonic set which is set of all extreme vertices. Let $S_1 = S \cup \{u_1, u_2\}$. Notified that S_1 is the smallest split detourmonophonic set and so $dm_s(G) = k$.

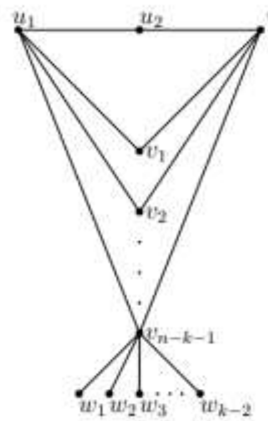

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Fig. 3: A graph G of order n, as arbitrary $dm_s(G) = k$ and $dm_c(G) = k + 1$

But S_1 is a disconnected detourmonophonic set. Let $S_2 = S \cup \{v_{n-k}\}$. Now, the S_2 becomes connected. Hence $dm_c(G) = k + 1$.

Result 3.2 There exits an interconnected undirected circuit G of order n as an arbitrary dm(G) = a, $dm_s(G) = a + 1$ and $dm_c(G) = a + 2$ where a, n integers with $5 \le a < n$.

Proof. Let $K_{n-a-1,3}$ be a complete bipartite graph. Let U = $\{u_1, u_2, u_3, \ldots, u_{n-a-1}\}$ and W = $\{w_1, w_2, w_3\}$ be the partite sets of $K_{n-a-1,3}$. Now, joining a vertex x to each vertex of U. Also, add $a - 3$ vertices say $v_1, v_2, \ldots, v_{a-3}$ with the vertex x. As a result, we obtained the desired graph as given in Figure 4 of order n.

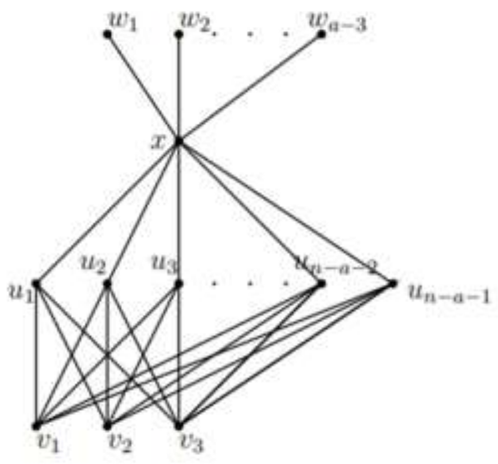

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Fig. 4: A graph G of order n as an arbitrary dm(G) = a, $dm_s(G) = a + 1$ and $dm_c(G) = a + 2$

The set W = $\{v_1,v_2,v_3,w_1,w_2,... ,w_{a-3}\}$ forms a smallest detourmonophonic set, dm(G) = a. But the subgraph [V – W] is not disconnected. Let $W_1 = W \cup \{x\}$. Now, the set W_1 becomes a minimum split detourmonophonic set. Hence $dm_s(G) = a + 1$. Moreover, the subgraph induced by $W_2 = W_1 \cup \{u_1\}$ is connected. Therefore, W_2 becomes smallest connected detourmonophonic set, $dm_c(G) = a + 2$.

Result 3.3 There is an interconnected undirected circuit G of order n with $dm_s(G) = a$ and $dm_c(G) = b$ where n, a, b are integers with $3 \leq a \leq b \leq n$,

Proof. Let P_{b-a+3}: $v_1v_2 ... v_{b-a+3}$ be the path of order b−a+3. Let $w_1,w_2,w_3,... ,w_{n-b}$ be a set of n − b vertices. Now, join each vertex $w_i(1 \leq i \leq n - b)$ with u_1 and u_3 and add the set of new vertices $y_1,y_2,... ,y_{a-3}$ with w_1. Therefore, the desirable circuit G with n vertices as shown in Figure 5.

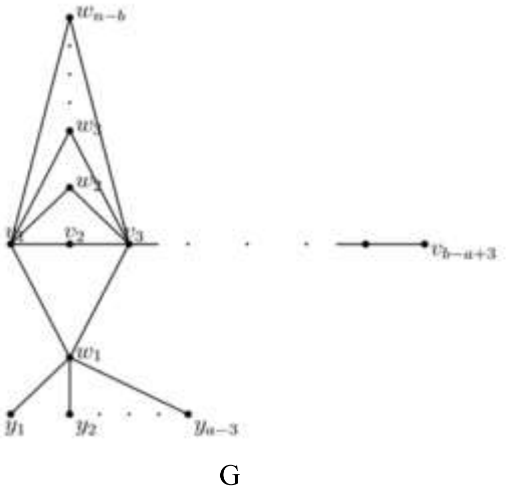

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Fig. 5: A graph G of order n with $dm_s(G) = a$ and $dm_c(G) = b$

The extreme vertices set S = $\{y_1,y_2,... ,y_{a-3},v_{b-a+3}\}$ is not detourmonophonic set. Let $S_1 = S \cup \{v_1, v_3\}$. It observed that S_1 is the unique smallest split detourmonophonic set and $dm_s(G) = a$. Also, the set S_1 is not connected. Now, let $S_2 = S_1 \cup \{w_1, v_4, v_5, ... , v_{b-a+2}\}$. Observed S_2 is the smallest connected detourmonophonic set, $dm_c(G) = b$.

4 Conclusion

The work contains a new parameter 'split detourmonophonic number'. It helps in a circuit or network to find removal nodes in the longest connecting path between two nodes so that it split the circuit or network. Also, we studied the relationship and realization between connected and split of the longest path between nodes. This design of the circuit may facilitate thedevelopment algorithms in Locating Capacitors, [20], to split the network and to one of step detection, [21], of fault in the longest monophonic path by splitting the network

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References:

[1] Harary F., *Graph Theory*, Addison-Wesley, 1969.
[2] Buckley F. and Harary F., Distance in graphs, *Addison-Wesley, Redwood city, CA*, 1990.
[3] Chartrand G., Palmer E. M. and Zhang P., The geodetic number of a graph: A survey, *Congr. Numer.,* 156 (2002), 37-58.
[4] Chartrand G., Harary F., Swart H. C., and Zhang P., Geodomination in graphs, *Bulletin of the ICA*, 31(2001), 51-59.
[5] Muntean R. and Zhang P., On geodomination in graphs, *Congr. Numer.,* 143(2000), 161-174.
[6] Santhakumaran A. P., Titus P. and Ganesamoorthy K., On the monophonic number of a graph, *J. Appl. Math. Informatics*, Vol. 32 (2014), No. 1 - 2, pp. 255 - 266.
[7] Titus P. and Ganesamoorthy K., The connected monophonic number of a graph,

Graph and Combinatorics, Vol. 30 (1) (2014), pp. 237-245.

[8] Mahendran M., Nithyaraj. R, Balaganesan P. and Somasundari M., The Split Monophonic Number of a Graph, *Journal of Adv Research in Dynamical Control Systems*, Vol. 11, 01-Special Issue, 2019.

[9] Titus P. and Ganesamoorthy K., On the Detourmonophonic Number of a Graph, *Ars Combinatoria*, 129, pp. 33-42, (2016).

[10] Titus P., Ganesamoorthy K. and Balakrishnan P., The Detour Monophonic Number of a Graph, *J. Combin. Math. Combin. Comput*., 84, pp. 179-188, (2013).

[11] Titus P., Santhakumaran A.P. and Ganesamoorthy K., The Connected Detour Monophonic Number of a Graph, *TWMS Journal of Applied and Engineering Mathematics*, Vol. 6, No.1, pp. 75-86, (2016).

[12] Johnwin Beaula N.E., Joseph Robin S., The Outer Connected Detourmonophonic Number of a Graph, *RATIO MATHEMATICA*, Vol. 44, 2022.

[13] Ilin, Vladimir & Simic, Dragan, A review of computational intelligence methods for traffic management systems. *Journal of Road and Traffic Engineering*. 67. 25-30. 10.31075/67.04.05, 2021.

[14] Jiawei Xue, Nan Jiang, Senwei Liang, Qiyuan Pang, Takahiro Yabe, Satish V. Ukkusuri & Jianzhu Ma, Quantifying the spatial homogeneity of urban road networks via graph neural networks., *Nature Machine Intelligence* volume 4, pages246–257 (2022)

[15] Xiangyu Li; Bodong Shang; Caiguo Li; Zhuhang Li., Coverage in Cooperative LEO Satellite Networks, *Journal of Communication and Information Networks*, Vol. 8, Issue 4, pp.329-340 December 2023.

[16] Zalán Heszberger, AndrásGulyás, József Bíró, Gábor Rétvári 1, Márton Novák, Attila Kőrösi, Mariann Slíz The role of detours in individual human navigation patterns of complex networks, *Scientific Reports*, Vol. 10, 1098 (2020).

[17] Rawat, Khushi & Kapoor, Chirag & Goyal, Himanshu & Sharma, Sachin.. Artificial Intelligence Based Optimized Traffic Diversion System in Smart Cities. *Advanced Communication and Intelligent Systems* (CCIS), Vol. 1921 pp. 97-108 (2023).

[18] T. Parsons and J. Seo, "FS-ACO: An Algorithm for Unsafe U-Turn Detours in Service Vehicle Route Optimization Applications," *23rd International Conference on Control, Automation and Systems (ICCAS), Yeosu, Korea, Republic of, 2023*, pp. 937-940, 2023. doi: 10.23919/ICCAS59377.2023.10316850.

[19] Maja Sareveska, Overview of RBF NN and Antenna Systems, *WSEAS Transactions On Electronics*, Vol. 13 pp. 84-88, 2022, https://doi.org/10.37394/232017.2022.13.11.

[20] E.S. Ali, S.M. Abd Elasim, Optimal Sizing and Locations of Capacitors using Slime Model Algorithm, *WSEAS Transactions on Power Systems*, Vol. 17, 38 pp.382-390, (2022, https://doi.org/10.37394/232016.2022.17.38.

[21] Zaid S. AI-Shamaain, Hussein. D. AI-Majali, Bilal. H. AI-Majali, Out-of-Step Detection based on Phasor Measurement Unit, *WSEAS Transactions on Power Systems*, Vol. 18, 36 pp.354-363, 2023, https://doi.org/10.37394/232016.2023.18.36.

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The authors equally contributed in the present research, at all stages from the formulation of the new parameter to the final results and their proofs.

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Conflict of Interest

The authors have no conflicts of interest to declare.