Matrix Representations for Ordinary Restricted Place Transition Nets

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Abstract: - Matrix representation provides for a concise representation of restricted or simple Petri nets and place transition nets. This main property is often ignored. Matrices are useful for identifying basic fundamental properties that are mainly related to the static structure of a net as presented in this work. Matrices can be used for identifying some very important basic properties. This paper contains the following sections: i) introduction, ii) motivation and problem why matrices can be used to represent basic petri nets iii) incidence matrix representation types for Petri nets iv) basic Petri net properties in terms of matrices are defined, discussed and explained v) simple examples of the properties observed from the matrices are given. Finally vi) some useful observations and vii) conclusions are given.

Key-Words: Matrices, Matrix Representation of Petri Nets, Modeling, Network Modeling, Petri Net Theory

1 Introduction

Petri nets are visual and mathematical formalisms or semi formalisms that have found widespread use for representing various types of system behaviour in their past three decades of coverage [1]-[3].

Real time, parallel, concurrent, asynchronous, sequential and many other modes of behaviour can be represented using Petri nets or place transition nets. Petri nets share some common properties with other types of diagrams like network diagrams, UML activity models and digraphs. Petri nets can be classified as bipartite digraphs having static and dynamic representation. The static part represents the structure of the net and normally does not change even though the net is executed. The dynamic part of the net depends on the static part. I.e. the structure of the net determines the firing sequence. Petri nets can be represented using equations and the static structure can be represented by a special type of matrix that sums the difference between the output and the input flows between places and transitions and vice-versa. This matrix is known as the incidence matrix. The incidence matrix represents the inflows and outflows in the structure [4]-[7].

Normally much attention is given to the static and dynamic properties inherent in Petri net structures [1],[2]. Analysis of the Petri net is based on properties like transition and place invariants, marking graph, reachability, safeness, liveness, siphons and traps, etc. If a Petri net is restricted and simple in form the incidence matrix or its components which are the input flow matrix and output flow matrix can serve for proper comprehension or analysis of some fundamental properties.

A concise way to represent data and certain types of equations in mathematics is by using matrices and vectors. The incidence matrix of a Petri net represents a summary of a Petri net's input and output in a matrix form [1].

The incident matrix shows the physical structure of the net which normally remains unchanged throughout the operations of the net. If places, arcs or transitions are added or removed then the incidence matrix will have to be updated accordingly to reflect these changes.

2 Motivation and Problem Formulation

Unfortunately, the incidence matrix represents the difference between the output flows and the input flows. Thus it cannot really represent the complete Petri net structure. It is possible to end up with flows that cancel each other out. Therefore for the full structure to be visible the input flow matrix and the output flow matrix need to be separately available. The rows of the input/output matrices represent Petri net places, whilst the columns represent the transitions. Hence, the size of the matrices depend on the number of places and transitions that are found in the network structure. I.e. a place transition net having 5 places and 4 transitions will have a 5x4 matrix for input and

output. A net that has 20 places and 15 transitions will require an incidence matrix of order 20x15. This is an issue as the larger the net the larger the matrices to represent the net. For this special tools might need to be used [4],[5].
Sometimes the possible analysis that can be done on input matrix, output matrix and resultant incidence matrix is often overlooked. For restricted nets, transition firing can be easily represented in terms of matrices. It is definitely possible to discover new properties in matrices that contribute to the comprehension and analysis of Petri nets.

Basic Petri nets are bipartite graph representation types of a sy stem's states and processes at an elementary level.

The motivation for this work is that simple or elementary restricted Petri nets or Place Transition nets can be easily represented in terms of matrices [1]-[5],[8]. This fundamental property is not given sufficient importance. It is possible to model many of the structural elements in terms of matrices. The matrix structure that is called the incidence matrix does not change throughout the operations of the net. As long as the net is of a finite size, some basic properties of the net are examinable. I.e. some basic properties can be analysed from the matrix. The incidence matrix is the sum of the difference between the inflows and outflows of the net. The incidence matrix for the Petri net can be constructed even if the net does not have invariant properties like place and transition invariants. The incidence matrix components which are the input flow matrix and the output flow matrix can be used to represent a net structure in a compacted form more than the visual graph diagram especially if the net is large in size.

The incidence matrix can be used to reverse or invert the structure of the net along with other interesting applications.

3 Incidence Matrix Representation

If the *number of transitions* = *number of places* in the net then by default the incidence matrix will be square. Normally this is not the case because the number of transitions is different from the number of places. Square matrices have several interesting properties.

The most important part of the square matrix depends on whether it is a square matrix or not. To have a sq uare matrix the property where the no. of transitions = no. of places has to be enforced.

4 Basic Petri Net Definition and Properties in Terms of Matrices

4.1 Basic Petri Net Definition

A Petri net simply represented as a f our tuple set, PN = (P,T,F,W). P is a finite non e mpty set of places P = {p1,p2,p3,…,pn} and T is a finite non empty net of transitions T= {t1,t2,t3,…,tn}. F is a finite non e mpty set of flows from a place to a transition and vice-versa, given as $F \subseteq \{(PxT) \cup (TxP)$. Normally (PxT) represents the input arcs also denoted as I and (TxP) represent the output arcs denoted as O. W is a weight function or marking value for the tokens at a place p, given as $W : P \rightarrow \{1,2,3,..., n\}$. Places and transitions are disjoint i.e. $P \cup T = \phi$ and $P \cap T = \phi$. Nodes are not isolated. The initial marking of the net is denoted as Mo.

4.2 Ordinary vs Non-Ordinary Nets

A Petri net is ordinary **iff,** $\forall p \in P$, $t \in T$, $I(p,t) \leq 1$ and $O(t,p) \leq 1$. I.e all arcs have a multiplicity of 1. This property is directly observable from the input flow matrix and the output flow matrix. If there are no entries that are > 1 in both matrices then the Petri net is ordinary. Otherwise if there are entries in one or more of the matrices that are >1 then the net is non-ordinary. This property defining the net as non-ordinary is given as f ollows $\exists_{(p \in P)}; I(p,t) > 1$ or $\exists_{(p \in P)}; O(t,p) > 1$.

4.3 Nodes

A node in a Petri net refers to either a place or a transition, y is a node, **iff** $y \in P \cup T$. The input set or pre-set of a transition t implies the set of all input places to t. This can be given as $\bullet t = \{p : p \in P \cap I(p,t) \neq 0$. The output set or post-set of t is the set of all output places from t. $t \bullet = \{p : p \in P \cap O(t,p) \neq 0$. An *elementary path* in the Petri net is identified as a sequence of nodes: $a_1, a_2, ..., a_n$ where n>2 and $\exists$ arc (a_i, a_{i+1}) for $i \in N_{n-1}$ if n>1 and $a_i = a_j$ implying that i=j where $N_n = \{1,2,...., n\}$ possibly representing a self-loop.

4.4 Input Matrix

The input flow matrix denoted as I_{ij} or C_{ij}^{-} represents all the input arcs in the Petri net. I.e. the arcs connecting a p lace to a transition are the input arcs. In this matrix the rows represent places and the columns represent transitions. A directed connection from a place to a transition is represented by a value in the matching row and column. If there are no connections then a zero value is placed in the matrix.

4.5 Output Matrix

The output flow matrix denoted as O_{ij} or C_{ij}^{+} represents all the output arcs in the Petri net. I.e. the arcs connecting a transition to a place are the output arcs. In this matrix the rows represent places and the columns represent transitions. A directed connection from a transition to a place is represented by a value in the matching row and column. If there are no connections then a zero value is placed in the matrix.

4.6 Incidence Matrix

The incidence matrix for a Petri net given as Cij is the difference between the output flow matrix Oij and the input flow matrix Iij. The incidence matrix representation can also be written as $C_{ij} = C_{ij}^{+} - C_{ij}^{-}$ where $C_{ij}^{+} = W(i, j)$ if $t_j \in {}^{\bullet}P_i$ or else it is zero and $C_{ij}^{-} = W(i, j)$ if $t_j \in P_i^{\bullet}$ or else it is zero and *W(i,j)*= weight of arc from $i \rightarrow j$ or $j \rightarrow i$. The O_{ij} matrix is the complete set of output flows from transitions to places. I.e. ${}^{\bullet}p = \{t : (t, p) \in F)$. The I_{ij} matrix is the complete set of input flows from place to transitions. I.e. $p^{\bullet} = \{t : (p,t) \in F)$. If flows cancel each other out, the incidence matrix cannot be used for reverse engineering, i.e. it would not be possible to construct the proper Petri net. Hence, the individual matrices for the input and output flows could provide us with better detail.

4.7 Compaction

The incidence matrix, input flow and output flow matrix remain unchanged during the action of the Petri net or whatever state the Petri net is in. This is because the state of the net is determined by the tokens in the places and not by the static structure. Hence the matrices serve as an alternative way for representing the structure.

4.8 Firing the Net via Matrices and Transitions

The operation of firing the Petri net can be very simply represented using vectors and matrices. The basic equation for this operation is given as follows: $M_1 = M_o + Cf$. M_0 is the initial state of the net. C is the incidence matrix and f is a firing vector. M_0 is the resultant new marking of the net. A sequence of markings can be derived. Total marking sequence $= \{M_0, M_1, M_2, ..., M_n\}$. The values of the firing vector can be only non-negative integers or zeros. The results of the firing operations can be placed in matrices composed of the firing vectors. Actually transition firing is a vector based operation. It is possible to reduce transition firing to vector mathematics.

5 Interesting Properties of Matrix Representation

5.1 Elements of the Input, Output and Incidence Matrices

The addresses of the element in the matrices are determined by the values i,j, where i represents the row number, i.e. the place row and j represents the column, i.e. the transition column. i represents the place number and j represents the transition [9],[10]. E.g. if there is no relationship between a place and a transition in the input matrix then the corresponding i,j values are left empty. If one arc links a transition to a place then for the matching i,j values in the output matrix a one is placed. If more than one arc links a transition to a place then the value in the corresponding matrix has to represent this. If the total no. rows = total no. of columns then the matrix is a square one. This implies that the number of places and transitions are equal. An important fact is that the dimensions of the input, output flow and incidence matrices have to be identical. If there is a self-loop in the Petri net i.e. the input and output of a transition are identical this is cancelled out in the incidence matrix. This property is useful for reducing the net.

5.2 Dimensions of the Matrix

If we have 30 x 20 matrices for the Petri net then this implies that we have 30 places and 20 transitions. I.e. I >J. This could indicate that the structure is not properly balanced as there are more places. The larger the matrix obviously the more complex the net will be in terms of structure.

5.3 Square Incidence Matrices

If as already stated the no of places = no. of transitions the representational matrices will be square ones. This property could indicate if the net is balanced or unbalanced. By the term balanced it is implied that the number of transitions and places are equal.

5.4 Analysis of the Matrices

Analyzing or examining the matrices can provide useful information about the structure of the net.

E.g. if the sum of a particular row in the incidence matrix is zero then it implies that there are an equal amount of inputs and outputs or that there are none for a particular place, in this case the place is represented by row 1 . E.g. $a_{1,1}+ a_{1,2}+ a_{1,3}\ldots.. a_{1,n}=0$.

Similarly if the sum or a particular column in the incidence matrix is zero this can imply that the transition has an equal amount of inputs and outputs or that the transition is not connected. E.g. for a transition denoted by column 2 this is given as $a_{1,2}+ a_{2,2}+ a_{3,2}\ldots.. a_{n,2}=0$.

The values for a n x m incidence matrix for a given Petri net can be interpreted as follows. An element a_{ij} from the matrix can have the following values: i) + value i.e. place j is an output place of transition i. ii) – value place j is an input of transition i, iii) 0 value indicating that either place i is both input and output place of a transition i. In this case there is a self-loop which cancels out the value or else 0 because there are no connections.

If the values of summing a column are negative, this might indicate that there are more inputs than outputs for that particular transition. T he values could be positive thus indicating that there are more outputs than inputs for the transition.

If the values of summing a row are negative this can indicate various conditions. E.g. either that the place is being used for more inputs than outputs or that the place is used only for input, etc.

However, for more detailed analysis and precision, each matrix would have to be examined for every situation considering the structure of the net and both the input flow and output flow matrices would have to be examined.

The important point for consideration is that the matrix will tell us about the structure of the net. Many details and extra information are possible from individual analysis of the matrices.

5.5 Transposition of the Matrices

The transpose of a matrix is given as turning the rows of a given matrix into columns and vice-versa. C_{ij} is written as C_{ji}. I.e. RxC is transposed into CxR. This can be expressed as f ollows $C_{ij} = C_{ji} \forall_{i,j} (i \in I), (j \in J)$ where C_{ij} = 0. This property can be used to check if there are errors in the matrices or for creating structural inversion of the net. I.e. places are transformed into transitions and vice-versa. However, the edges of the net would remain untouched. I.e. the edges of the net are not changed.

5.6 Summing Two Matrices of two Nets

If two incidence matrices are of the same order. I.e. the matrices represent two different nets but if both nets have the same number of places and transitions it is possible to combine them by adding the incidence matrices together. However, this can also be done by using the input flow matrix and the output flow matrix for two nets that have identical matrices.

Some care has to be taken because if the incidence matrices are used it is possible for entries to cancel each other out.

5.7 Subtracting the Matrices of two Nets

Similarly if two matrices from two different nets are of the same order it is possible to subtract them from each other. The difference of the nets is in the resultant matrix.

5.8 Some Fundamental Properties

If two Petri nets are structurally identical then the incidence matrices for the nets are equal. I.e. they are Equal Matrices. This property is independent of the state of the nets.

5.9 Siphons and Trap Generation

A siphon is defined as a subset of non -empty places S. If ${}^{\bullet}S \subseteq S^{\bullet}$ i.e. every transition having an output place in S has an input place in S. If a siphon is token free under some marking, then it is token free for each successive marking. A trap can be defined as a non-empty set of places R, if $R^{\bullet} \subseteq {}^{\bullet}R$ i.e. every transition having an input place in R has an output place in R. A trap remains marked with at least one token under every successive marking starting off from an initial marking having at least one token. Different authors classify these into other sub-categories.

5.10 Construction of a Relationship Graph

The incidence matrix can be used to generate what is called a relationship graph in this paper. The relationship graph is a simplified graph that shows the structure of the Petri net, i.e. the relationship between places and transitions. The negative value on the edges implies that the place is an input one and a p ositive value implies that the place is an output place.

6 Elementary Examples

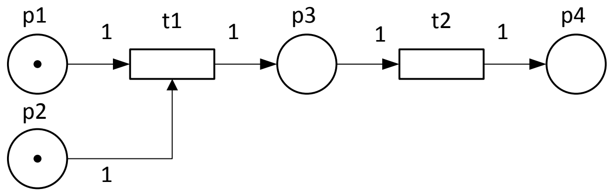

Fig. 1 A Basic Petri Net

6.1 A Basic Net and its Matrix Representation

Fig. 1 shows a basic Petri net. The Petri net places and transitions are properly labelled for clarity's sake. The matrices are given as follows.

$$C_{ij}^{-}=\begin{bmatrix}-1 & 0\\ -1 & 0\\ 0 & -1\\ 0 & 0\end{bmatrix}\quad C_{ij}^{+}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\quad C_{ij}=\begin{bmatrix}-1 & 0\\ -1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}$$

The first matrix from the l.h.s. is the input flow matrix followed by the output flow and incidence matrices. These are 4 x 2 non-square matrices. The incidence matrix in this case can be used to reverse engineer the Petri net as no flows cancel each other out.

In fig. 2 a relationship graph has been drawn from the incidence matrix presented above on the r.h.s. The relationship graph is just a symbolic or simplified representation of the Petri net in fig. 1. The signs on the edges indicate whether a place loses or gains tokens.

By transposing the incidence matrix the following matrix is obtained. The structure of the net is not entirely different. It is just that transitions are transformed into places and places are transformed into transitions. The edges remain unchanged as shown in the matrix below.

$$C_{ij}=\begin{bmatrix}-1 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$$

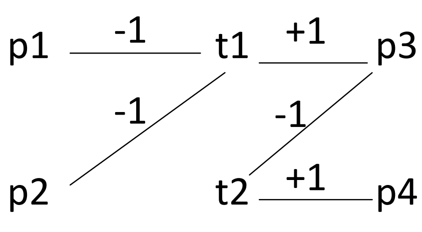

Fig. 2 A Relationship Graph for the Net in fig.1

6.2 A Petri Net with a Square Incidence Matrix

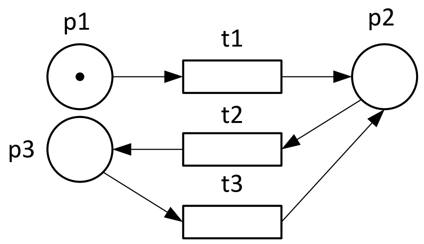

Fig. 3 A Petri Net Having a Square Incidence Matrix

The Petri net depicted in fig. 3 has a square incidence matrix that is given below.

$$C_{ij}=\begin{bmatrix}-1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}$$

If the edges and nodes in fig. 2 are considered as objects, then there are 11 objects or entries used to create fig. 2. On the other hand the matrix just has 9 entries that represent the entire structure, hence the matrix is more compact. By summing the first column the result would be zero indicating that the number of inputs and outputs to transition t1 are balanced i.e. equal. If the second column is summed the result is 1 implying that transition t2 has more outputs than inputs. The transpose of the incidence matrix for fig. 3 is identical to the original incidence matrix implying that the matrix is a symmetric matrix as $C^T = C$ or $C_{ij} = C_{ji}$. Hence transposing the incidence matrix will still give the same net structure. This Petri net could be called a symmetric Petri net.

6.3 Summing the Incidence Matrix for Two Nets

The Petri net in fig. 4 has the incidence matrix shown below. The incidence matrices for the Petri nets in fig. 4 and fig. 3 have the same dimensions and can be summed. Note that it is possible to sum even nets that have different incidence matrices, provided some adjustments are made.

$$C_{ij}=\begin{bmatrix}-1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}$$

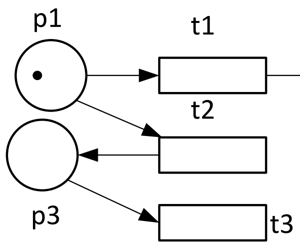

Fig. 4 Petri Net with same Dimensions as Fig. 3

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

The result of summing the incidence matrices of the two nets yields the Petri Net drawn in Fig. 5.

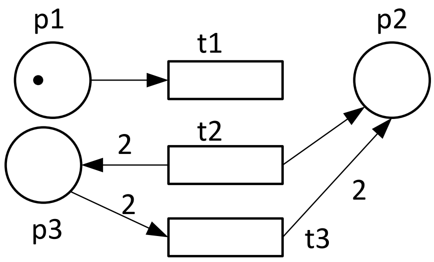

Fig. 5 New Petri Net Result of Summing

6.4 Subtracting the Incidence Matrix for Two Nets

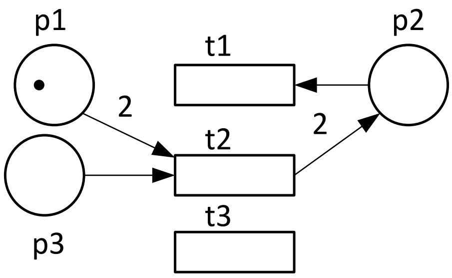

Fig. 6 Resultant Net that shows the Differences between fig. 3 less fig. 4 Incidence Matrices

Subtracting the incidence matrices from one another should yield the differences in the two nets. I.e. if A-B is used the result is the difference. A s an example the incidence matrix of the net in fig. 3 is subtracted from the incidence matrix for the net in fig. 4. What is left is the difference between the two nets. This difference is shown in fig. 6. Fig. 6 is not a well formed Petri net due to the fact that there are unconnected nodes. Normally these nodes can be removed because they do not have any value unless they form part of the net structure.

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

7 Findings and Observations

Some results are observed from this work and summarized as follows. i) Elementary or simple nets can easily be represented using matrices. ii) Most basic Petri net operations like transition firing can be represented using matrices and vectors. iii) The representational matrices can be used to create relationship graphs which indicate the structure of the Petri net. iv) Matrices are more compact than the Petri net itself. v) The matrices of similar order derived from different nets can be summed or subtracted from each other. vi) Analysis of the net and its structure can be obtained directly from the incidence or input/output flow matrices.

Some observations are that i) the resultant net of the addition or subtraction of the incidence matrices might not be live or have unconnected nodes. ii) Some matrices can have interesting properties like symmetry but this property depends highly on t he structure of the net and will not always be present. iii) Siphons and traps can be identified from the matrices. iv) The matrices indicate how the net is structured, if it is well formed or balanced.

For simple matrices, even visual inspection can reveal some interesting properties. The techniques or ideas presented in this paper can be extended to other net structures. The biggest problem is related to the size of the matrices. In this case the analysis is not so straightforward.

It has to be noted that in the case of incidence matrices, $A - B$ does not imply $B - A$.

The matrices can be reduced to vectors either for places or transitions. It is stated that these normally do not change even though the Petri net is executed. By modifying the incidence matrix the static structure of the Petri net can be modified.

The relationship graph gives a good indication of the structure of the net in a simpler form. The matrices are a co mpact representation of a net. However the complete picture is not necessary visible because the tokens cannot be seen, i.e. the state of the net is missing.

In essence the findings give a good indication that elementary or basic Petri net structures can be adequately represented using matrices. These matrices have interesting properties that can be used for other forms of analysis and verification as has been mentioned in sections 5 and 7.

8 Conclusion

Petri nets are powerful formalisms or semi-formalisms that can be used for representing diverse system structures. Petri nets share some similarities with other network structures. This work has just discussed some of the significant uses of matrices for Petri Net representation. This is often overlooked by Petri Net researchers and users. Matrices provide a concise and interesting way of representing simple Petri Net structures. Matrices have important properties that can be used for detailed analysis. This work has not dealt with these properties.

The basic methodology that was used for representing the structures is related to the dual nature of Petri nets. Petri nets can be represented both i) using matrices and ii) diagrammatic notations. Obviously the matrices cannot represent all the details that are captured in the visual notation. However the structural details are simple to capture in matrix form. This idea is suitable for reduced net structures or elementary nets. Matrices have served to represent other types of network graphs and structures, so there is still a lot of work that can be done in this area as regards to Petri net representation.

One significant problem in this work is the size of the matrix. This depends on the size of the Petri Net. To solve this issue computational programs would have to be used.

It is advisable to use both the diagrammatic notation along with the matrix notation when representing Petri nets. The matrix structure cannot contain all the visual properties of the net.

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