FIR Steady State Kalman Filter for Motion Estimation and Prediction

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Abstract: - Motion models play an essential role in navigation and tracking applications. Constant velocity, constant acceleration, and periodic motion state space models are described. It is shown that the models are observable and controllable. The Kalman filter has been used in order to estimate or predict the position of moving objects. It is shown that a steady state always exists for these models, and steady state Kalman filters for estimation and prediction are derived. It is also shown that the Finite Impulse Response (FIR) form of the steady state Kalman filter always exists, and the FIR form of the steady state Kalman filters for estimation and prediction are derived. Estimation and prediction algorithms that combine the steady state Kalman filter and its FIR form are proposed. Simulation results show the satisfactory behavior of the proposed estimation and prediction algorithms.

Key-Words: - Kalman Filter, State Space, Estimation, Prediction, Steady State, Finite Impulse Response.

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1 Introduction

Motion models play a crucial role in navigation applications and in tracking applications in order to estimate or predict the possible location of moving objects. There are various motion models in the literature [1], such as constant velocity, constant acceleration, periodic, and constant turn. Constant velocity motion model is presented in [2], navigation model of constant acceleration vehicle movement is given in [3], models for vehicle location estimation presented in [4], a vehicle tracking problem is considered where a vehicle is constrained to move with constant velocity or constant acceleration [5], visual tracking models are referred in [6], [7], constant acceleration motion model using a continuous-time white noise assumption is given in [8]. Typical periodic motion is described in [9].

The problem of estimating the position and velocity can be faced using the Kalman filter [10], [11], which is the most well-known estimation algorithm that has been widely used in various applications: autonomous orbit determination [12], satellite orbit determination [13], power generation prediction [14], cases prediction of Covid-19 [15], multi-observation fusion applications related to timescale [16], control effectiveness estimation on airplanes [17], applications with time-correlated measurement errors [18], estimation with unlimited sensing measurements [19], multi-target localization [20], GPS position estimation [21], stock price prediction [22]. Kalman filter has been used for target tracking [23], and linear Kalman filters for object tracking are accessible in [24]. In fact, in the case of linear models, a time-invariant Kalman filter has been used. In this work, we are going to investigate the existence of a steady state for the time-invariant models and the applicability of the steady state Kalman filter for motion estimation and prediction. In addition, we are going to investigate the existence of the FIR form of the steady state Kalman filter.

In this work, the constant velocity, constant acceleration, and periodic motion models are considered. The novelty of this work concerns: (a) the confirmation of all the models' observability and controllability, (b) the guarantee of the existence of steady state (c) the derivation of steady state Kalman filter for estimation and prediction (d) the proof of the existence of the FIR form of steady state Kalman filter, (e) the derivation of the FIR form of steady state Kalman filter for estimation and prediction, (g) the derivation of estimation and prediction algorithms which combine the steady state Kalman filter and its FIR form.

The paper is organized as follows: Constant velocity, constant acceleration, and periodic motion state space models are described in section 2, and their observability and controllability are confirmed. Steady state Kalman Filters for estimation and prediction are derived in section 3. In section 4, the FIR form of the steady state Kalman filters for estimation and prediction are derived and estimation and prediction algorithms that combine the steady state Kalman filter and its FIR form are proposed. Simulation examples are presented in section 5. Section 6 summarizes the conclusions.

2 State Space Motion Models

In this section we present time-invariant state space models for constant velocity, constant acceleration and periodic motions. Consider the discrete time, $k \geq 0$, invariant linear system, which is traditionally formulated by the state space equations [11]:

$$x(k+1) = F \cdot x(k) + B \cdot u(k) + G \cdot w(k) \qquad (1)$$

$$z(k) = H \cdot x(k) + v(k) \qquad (2)$$

Here, $x(k)$ defines the state vector of dimension n, $z(k)$ defines the measurement or observation vector of dimension m, $u(k)$ defines the input vector of dimension ℓ, $w(k)$ defines the input noise vector of dimension r, $v(k)$ defines the output noise vector of dimension m. In addition, F is the transition matrix, B is the input matrix, G is the input noise matrix and H is the output or observation matrix. Moreover, $w(k)$ and $v(k)$ are Gaussian of zero mean and covariances Q and R. All the model parameters F, B, G, H, Q, R are constant. Finally, the initial state $x(0)$ is a Gaussian random vector variable with mean x_0 and variance P_0.

2.1 Constant Velocity Motion Models

Consider the general case where it is desired to estimate the position and the velocity of a moving object. For example, in target tracking problems, this can be accomplished using measurements provided by a range sensor and an angle sensor; a radar system can provide range and angle measurement and a combination of a camera and a rangefinder can do the same, [3].

2.1.1 Position Only Driven Velocity Motion Model

One dimension motion

The constant velocity motion equation is:

$$p(k) = p_0 + \upsilon \cdot k$$

where $p(k)$ is the position, $\upsilon(k) = \upsilon$ is the constant velocity and p_0 is the initial position.

Then we get

$$p(k+1) = p(k) + T \cdot \upsilon(k)$$

where T is the sampling period.

It is clear that $T > 0$; the value $T = 1$ is very common in many applications, [3], [22].

The state consists of one element: the position $p(k)$.

The constant velocity $\upsilon(k) = \upsilon$ can be regarded as a zero mean Gaussian noise process with variance ${\sigma_\upsilon}^2 > 0$.

Then

$$x(k+1) = x(k) + G \cdot w(k)$$

where

$$G = T$$

$w(k) = \upsilon(k)$ is a zero mean Gaussian process with known variance $Q = {\sigma_\upsilon}^2$

$G \cdot w(k) = T \cdot \upsilon(k)$ is a zero mean Gaussian process with known variance $G^2 \cdot Q = {\sigma_\upsilon}^2 \cdot T^2$

The measurement has one element, the position.

$$z(k) = p(k) + v(k)$$

$v(k)$ is a zero mean Gaussian noise process with variance $R = {\sigma_p}^2$, where position is supposed to be corrupted with noise with variance of ${\sigma_p}^2$.

Multiple dimensions motion

The state consists of the position in two dimensions:

$$x(k) = [p_x(k) \quad p_y(k)]^T$$

or three dimensions

$$x(k) = [p_x(k) \quad p_y(k) \quad p_z(k)]^T$$

(note that M^T denotes the transpose of M).

Remark. It is worth no mention that setting $T = 1$ and $F = H = G \cdot Q \cdot G^T = R = I$, where I is the identity matrix, we derive the random walk model.

2.1.2 General Velocity Motion Model

One Dimension Motion

The assumption of completely constant velocity is not realistic, particularly for real world problems. The introduction of piecewise constant white acceleration, which can be described by a zero mean

Gaussian white noise with variance ${\sigma_a}^2$, controls the level of relaxation of the constant velocity assumption [2].

The constant velocity motion equations are:
$p(k) = p_0 + \upsilon \cdot k$
$\upsilon(k) = \upsilon$
$a(k) = a = 0$
where $p(k)$ is the position, $\upsilon(k) = \upsilon$ is the constant velocity, p_0 is the initial position and $a(k) = a = 0$ is the zero acceleration.

Then we get
$p(k+1) = p(k) + \upsilon(k) \cdot T + \frac{1}{2} \cdot a(k) \cdot T^2$
$\upsilon(k+1) = \upsilon(k) + a(k) \cdot T$

The state consists of two elements: the position $p(k)$ and the velocity $\upsilon(k)$: $x(k) = \begin{bmatrix} p(k) \\ \upsilon(k) \end{bmatrix}$.

The zero acceleration $a(k) = a = 0$ can be considered as a zero mean white noise sequence process with variance ${\sigma_a}^2 > 0$.

Then
$x(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot x(k) + G \cdot w(k)$
where
$G = \begin{bmatrix} \frac{1}{2} \cdot T^2 \\ T \end{bmatrix}$
$w(k) = a(k)$ is a zero mean Gaussian process with known variance $Q = {\sigma_a}^2$
$G \cdot w(k)$ is a zero mean Gaussian process with known covariance
$G \cdot Q \cdot G^T = {\sigma_a}^2 \cdot \begin{bmatrix} \frac{1}{4} \cdot T^4 & \frac{1}{2} T^3 \\ \frac{1}{2} T^3 & T^2 \end{bmatrix}$.

Remark. An alternative approach is given in [5] by deriving the process noise covariance using a continuous-time white noise assumption instead using of a piece-wise constant white noise model; then $G \cdot Q \cdot G^T = {\sigma_a}^2 \cdot \begin{bmatrix} \frac{1}{3} \cdot T^3 & \frac{1}{2} T^2 \\ \frac{1}{2} T^2 & T \end{bmatrix}$.

The measurement has two elements, the position and the velocity.
$z(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot x(k) + v(k)$
$v(k)$ is a zero mean Gaussian noise process with covariance $R = \begin{bmatrix} {\sigma_p}^2 & 0 \\ 0 & {\sigma_\upsilon}^2 \end{bmatrix}$
where position and velocity are supposed to be corrupted with noise with variances of ${\sigma_p}^2$ and ${\sigma_\upsilon}^2$ respectively.

Multiple dimensions motion
The state consists of the position and velocity in two dimensions:
$x(k) = [p_x(k) \quad p_y(k) \quad \upsilon_x(k) \quad \upsilon_y(k)]^T$
or in three directions:
$x(k) = [p_x(k) \quad p_y(k) \quad p_z(k) \quad \upsilon_x(k) \quad \upsilon_y(k) \quad \upsilon_z(k)]^T$

2.1.3 Special Velocity Motion Model

The measurement consists only of the position. Then $H = [I_{d\times d} \quad 0_{d\times d}]$ and $R = {\sigma_p}^2 \cdot I_{d\times d}$ where d=1,2,3 expresses the motion dimensions.
Note that $I_{d\times d}$ is the $d \times d$ identity matrix and $0_{d\times d}$ is the $d \times d$ zero matrix.

2.2 Constant Acceleration Motion Models

Consider position and velocity measurements provided by GPS and acceleration measurements provided by accelerometer. Consider the general case where it is desired to estimate the position and the velocity of a moving object. For example, in navigation problems this can be accomplished using measurements provided by sensor output from an inertial measurement unit (IMU) and a global navigation satellite system (GNSS) receiver, [3].

2.2.1 General Acceleration Motion Model

One dimension motion
The constant acceleration motion equations are:
$p(k) = p_0 + \upsilon_0 \cdot k + \frac{1}{2} \cdot a \cdot k^2$
$\upsilon(k) = \upsilon_0 + a \cdot k$
$a(k) = a$
where $p(k)$ is the position, p_0 is the initial position, $\upsilon(k) = \upsilon$ is the velocity, υ_0 is the initial velocity, $a(k) = a$ is the constant acceleration.

Then we get
$p(k+1) = p(k) + \upsilon(k) \cdot T + \frac{1}{2} \cdot a(k) \cdot T^2$
$\upsilon(k+1) = \upsilon(k) + a(k) \cdot T$

The state consists of two elements: the position $p(k)$ and the velocity $\upsilon(k)$: $x(k) = \begin{bmatrix} p(k) \\ \upsilon(k) \end{bmatrix}$.

The motion acceleration is the accelerometer output $u(k)$ corrupted with noise $w(k)$, which is assumed to be zero mean Gaussian with variance ${\sigma_a}^2$.

Then
$a(k) = u(k) + w(k)$
and

$$x(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \cdot x(k) + \begin{bmatrix} \frac{1}{2} \cdot T^2 \\ T \end{bmatrix} \cdot u(k) + \begin{bmatrix} \frac{1}{2} \cdot T^2 \\ T \end{bmatrix} \cdot w(k)$$

where

$$B = \begin{bmatrix} \frac{1}{2} \cdot T^2 \\ T \end{bmatrix}$$

$$G = \begin{bmatrix} \frac{1}{2} \cdot T^2 \\ T \end{bmatrix}$$

$u(k) = a(k)$ is the input (accelerometer output).
$w(k)$ is a zero mean Gaussian process with known variance $Q = \sigma_a{}^2$
$G \cdot w(k)$ is a zero mean Gaussian process with known covariance

$$G \cdot Q \cdot G^T = \sigma_a{}^2 \cdot \begin{bmatrix} \frac{1}{4} \cdot T^4 & \frac{1}{2} \cdot T^3 \\ \frac{1}{2} \cdot T^3 & T^2 \end{bmatrix}.$$

Remark. An alternative approach is given in [5], [8] by deriving the process noise covariance using a continuous-time white noise assumption instead using of a piece-wise constant white noise model. In fact the state consists of three elements: the position $p(k)$, the velocity $\upsilon(k)$ and the acceleration $a(k)$: $x(k) = [p(k) \quad \upsilon(k) \quad a(k)]^T$.

Then

$$F = \begin{bmatrix} 1 & T & \frac{1}{2} \cdot T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G \cdot Q \cdot G^T = \sigma_a{}^2 \cdot \begin{bmatrix} \frac{1}{20} \cdot T^5 & \frac{1}{8} \cdot T^4 & \frac{1}{6} \cdot T^3 \\ \frac{1}{8} \cdot T^4 & \frac{1}{3} \cdot T^3 & \frac{1}{2} \cdot T^2 \\ \frac{1}{6} \cdot T^3 & \frac{1}{2} \cdot T^2 & T \end{bmatrix}$$

The measurement has two elements, the position and the velocity.

$$z(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot x(k) + v(k)$$

$v(k)$ is a zero mean Gaussian noise process with covariance $R = \begin{bmatrix} \sigma_p{}^2 & 0 \\ 0 & \sigma_\upsilon{}^2 \end{bmatrix}$
where position and velocity are supposed to be corrupted with noise with variances of $\sigma_p{}^2$ and $\sigma_\upsilon{}^2$ respectively.

Multiple dimensions motion
The state consists of the position and velocity in two dimensions:
$x(k) = [p_x(k) \quad p_y(k) \quad \upsilon_x(k) \quad \upsilon_y(k)]^T$
or in three directions:
$x(k) = [p_x(k) \quad p_y(k) \quad p_z(k) \quad \upsilon_x(k) \quad \upsilon_y(k) \quad \upsilon_z(k)]^T$

2.2.2 Special Acceleration Motion Model

The measurement consists only of the position. Then $H = [I_{d\times d} \quad 0_{d\times d}]$ and $R = \sigma_p{}^2 \cdot I_{d\times d}$ where d=1,2,3 expresses the motion dimensions.

2.3 Periodic Motion Models

2.3.1 General Periodic Motion Model

One dimension motion.
Assume a typical periodic motion [9] which is described by the equations:
$p(k) = \cos(k)$
$\upsilon(k) = \sin(k)$
where $p(k)$ is the position and $\upsilon(k)$ is the velocity.

Then we get
$p(k+1) = p(k) + \upsilon(k) \cdot T$
$\upsilon(k+1) = -T \cdot \upsilon(k) + \upsilon(k)$

The state consists of two elements: the position $p(k)$ and the velocity $\upsilon(k)$: $x(k) = \begin{bmatrix} p(k) \\ \upsilon(k) \end{bmatrix}$.

Then

$$x(k+1) = \begin{bmatrix} 1 & T \\ -T & 1 \end{bmatrix} \cdot x(k) + w(k)$$

where
$w(k)$ is a zero mean Gaussian noise process with covariance $Q = \begin{bmatrix} \sigma_p{}^2 & 0 \\ 0 & \sigma_\upsilon{}^2 \end{bmatrix}$

The measurement has two elements, the position and the velocity.

$$z(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot x(k) + v(k)$$

$v(k)$ is a zero mean Gaussian noise process with covariance $R = \begin{bmatrix} \sigma_p{}^2 & 0 \\ 0 & \sigma_\upsilon{}^2 \end{bmatrix}$
where position and velocity are supposed to be corrupted with noise with variances of $\sigma_p{}^2$ and $\sigma_\upsilon{}^2$ respectively.

Multiple dimensions motion
The state consists of the position and velocity in two dimensions:
$x(k) = [p_x(k) \quad p_y(k) \quad \upsilon_x(k) \quad \upsilon_y(k)]^T$
or in three directions:
$x(k) = [p_x(k) \quad p_y(k) \quad p_z(k) \quad \upsilon_x(k) \quad \upsilon_y(k) \quad \upsilon_z(k)]^T$

2.3.2 Special Periodic Motion Model

The measurement consists only of the position. Then $H = [I_{d\times d} \quad 0_{d\times d}]$ and $R = {\sigma_p}^2 \cdot I_{d\times d}$ where d=1,2,3 expresses the motion dimensions.

Table 1 summarizes the constant velocity, constant acceleration and periodic motion models. It is worth to point out that the constant velocity and periodic motion models are no-input models, in contrast with the constant acceleration motion models.

Table 1. Constant velocity, constant acceleration and periodic motion models

model	**motion dim**	$\mathbf{x(k)}$ **n**	$\mathbf{u(k)}$ $\boldsymbol{\ell}$	$\mathbf{w(k)}$ **r**	$\mathbf{z(k)}$ **m**	$\mathbf{v(k)}$ **m**
position only driven velocity motion model	1	1	-	1	1	1
	2	2	-	2	2	2
	3	3	-	3	3	3
general velocity motion model	1	2	-	1	2	2
	2	4	-	2	4	4
	3	6	-	3	6	6
special velocity motion model	1	2	-	1	1	1
	2	4	-	2	2	2
	3	6	-	3	3	3
general acceleration motion model	1	2	1	1	2	2
	2	4	2	2	4	4
	3	6	3	3	6	6
special acceleration motion model	1	2	1	1	1	1
	2	4	2	2	2	2
	3	6	3	3	3	3
general periodic motion model	1	2	-	2	2	2
	2	4	-	4	4	4
	3	6	-	6	6	6
special periodic motion model	1	2	-	2	1	1
	2	4	-	4	2	2
	3	6	-	6	3	3

Source: created by the authors.

model	**F**	**B**	**H**	**G**	**Q**	**R**
	n × n	**n × ℓ**	**m × n**	**n × r**	**r × r**	**m × m**
position only driven velocity motion model	I	-	I	$T \cdot I$	${\sigma_\upsilon}^2 \cdot I$	${\sigma_p}^2 \cdot I$
general velocity motion model	$\begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$	-	$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_a}^2 \cdot I & 0 \\ 0 & {\sigma_a}^2 \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_p}^2 \cdot I & 0 \\ 0 & {\sigma_\upsilon}^2 \cdot I \end{bmatrix}$
special velocity motion model	$\begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$	-	$[I \quad 0]$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_a}^2 \cdot I & 0 \\ 0 & {\sigma_a}^2 \cdot I \end{bmatrix}$	${\sigma_p}^2 \cdot I$
general acceleration motion model	$\begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_a}^2 \cdot I & 0 \\ 0 & {\sigma_a}^2 \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_p}^2 \cdot I & 0 \\ 0 & {\sigma_\upsilon}^2 \cdot I \end{bmatrix}$
special acceleration motion model	$\begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$[I \quad 0]$	$\begin{bmatrix} \frac{1}{2} \cdot T^2 \cdot I \\ T \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_a}^2 \cdot I & 0 \\ 0 & {\sigma_a}^2 \cdot I \end{bmatrix}$	${\sigma_p}^2 \cdot I$
general periodic motion model	$\begin{bmatrix} I & I \\ -I \cdot T & \end{bmatrix}$	-	$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$	-	$\begin{bmatrix} {\sigma_p}^2 \cdot I & 0 \\ 0 & {\sigma_\upsilon}^2 \cdot I \end{bmatrix}$	$\begin{bmatrix} {\sigma_p}^2 \cdot I & 0 \\ 0 & {\sigma_\upsilon}^2 \cdot I \end{bmatrix}$
special periodic motion model	$\begin{bmatrix} I & I \\ -I \cdot T & \end{bmatrix}$	-	$[I \quad 0]$	-	$\begin{bmatrix} {\sigma_p}^2 \cdot I & 0 \\ 0 & {\sigma_\upsilon}^2 \cdot I \end{bmatrix}$	${\sigma_p}^2 \cdot I$

I is the identity matrix of dimension equal to motion dimensions
F, B, G, H, Q, R consist of block diagonal matrices
Source: created by the authors.

2.4 Observability

It is worth to note that all the models are observable, i.e. all the states can be uniquely determining from the observations. In fact **the pair $(\mathbf{F}, \mathbf{H})$ is observable**, since the $(m \cdot n) \times n$ dimensional observability matrix

$$OM = \begin{bmatrix} H \\ H \cdot F \\ \cdots \\ H \cdot F^{n-1} \end{bmatrix} \tag{3}$$

satisfies the observability criterion:

$$\mathrm{rank}(OM) = n \tag{4}$$

Proof

a) position only driven velocity motion model
In this case we have
$F = I, H = I$
$$OM = \begin{bmatrix} I \\ I \\ \cdots \\ I \end{bmatrix}$$

Then $\operatorname{rank}(OM) = n$
b) general velocity motion model and general acceleration motion model
In this case we have
$$F = \begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$$
$$H = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$OM = \begin{bmatrix} I \\ F \\ \cdots \\ F^{n-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ I & I \cdot T \\ 0 & I \\ & \cdots \\ I & I \cdot (n-1) \cdot T \\ 0 & I \end{bmatrix}$$

Then $\operatorname{rank}(OM) = n$
c) special velocity motion model and special acceleration motion model

In this case we have
$$F = \begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$$
$H = [I \quad 0]$
$$OM = \begin{bmatrix} H \\ H \cdot F \\ \cdots \\ H \cdot F^{n-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & I \cdot T \\ & \cdots \\ I & I \cdot (n-1) \cdot T \end{bmatrix}$$

Then $\operatorname{rank}(OM) = n$
d) general periodic model

In this case we have
$$F = \begin{bmatrix} I & I \cdot T \\ -I \cdot T & I \end{bmatrix}$$
$$H = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$OM = \begin{bmatrix} I \\ F \\ \cdots \\ F^{n-1} \end{bmatrix}$$

Then $\operatorname{rank}(OM) = n$
e) special periodic motion model
In this case we have
$$F = \begin{bmatrix} I & I \cdot T \\ -I \cdot T & I \end{bmatrix}$$
$H = [I \quad 0]$
$$OM = \begin{bmatrix} I \\ F \\ \cdots \\ F^{n-1} \end{bmatrix}$$

Then $\operatorname{rank}(OM) = n$

2.5 Controllability

It is worth to note that all the models are controllable, i.e. there is the ability to transfer the system from any initial state to any desired final state in a finite time. In fact **the pair $(\mathbf{F}, \mathbf{B})$ is controllable**, since the $n \times (n \cdot \ell)$ dimensional controllability matrix
$$CM = [B \quad F \cdot B \quad \cdots \quad F^{n-1} \cdot B] \tag{5}$$

satisfies the controllability criterion:
$$\operatorname{rank}(CM) = n \tag{6}$$

Proof
general acceleration motion model and special acceleration motion model
In this case we have
$$F = \begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$$
$$B = \begin{bmatrix} I \cdot \frac{1}{2} \cdot T^2 \\ I \cdot T \end{bmatrix}$$
$$CM = \begin{bmatrix} I \cdot \frac{1}{2} \cdot T^2 & I \cdot \frac{3}{2} \cdot T^2 & \cdots & I \cdot \frac{2 \cdot n-1}{2} \cdot T^2 \\ I \cdot T & I \cdot T & & I \cdot T \end{bmatrix}$$
Then $\operatorname{rank}(CM) = n$

3 Steady State Kalman Filter

3.1 Time-Invariant Kalman Filter

Kalman filter [10], [11] computes the state estimation $x(k|k)$ and the estimation error covariance $P(k|k)$ as well as the state prediction $x(k+1|k)$ and the prediction error covariance $P(k+1|k)$, using the Kalman filter gain $K(k)$. The equations of the Time-Invariant Kalman Filter are as follows:

Time Invariant-Kalman Filter (TIKF)
$K(k) = P(k|k-1) \cdot H^T \cdot [H \cdot P(k|k-1) \cdot H^T + R]^{-1}$
$x(k|k) = [I - K(k) \cdot H] \cdot x(k|k-1) + K(k) \cdot z(k)$
$P(k|k) = [I - K(k) \cdot H] \cdot P(k|k-1)$
$x(k+1|k) = F \cdot x(k|k) + B \cdot u(k)$
$P(k+1|k) = G \cdot Q \cdot G^T + F \cdot P(k|k) \cdot F^T$
for $k = 0,1, \ldots,$
with initial conditions
$x(0|-1) = x_0, P(0|-1) = P_0$

Note that the existence of the inverse of the matrices in the Kalman filter gain equation is ensured assuming that the covariance matrix R is positive definite; this has the significance that no measurement is exact.

It is worth to highlight that, for the constant velocity, constant acceleration and periodic motion models, the model parameters F, B, G, H, Q, R and the initial condition P_0 consist of block diagonal matrices. Then $P(k|k), P(k+1|k), K(k)$ also consist of block diagonal matrices.

Furthermore it is worth to note that computation of the prediction error covariance, the estimation error covariance and the Kalman filter gain do not depend on the measurements. In addition, combining the above equations we derive the Riccati Equation:

$$\begin{aligned} P(k+1|k) &= G \cdot Q \cdot G^T + F \cdot P(k|k-1) \cdot F^T \\ &-F \cdot P(k|k-1) \cdot H^T \\ &\cdot [H \cdot P(k|k-1) \cdot H^T + R]^{-1} \\ &\cdot H \cdot P(k|k-1) \cdot F^T \end{aligned} \tag{7}$$

3.2 Steady State

Steady State means that the estimation error covariance, the prediction error covariance and the Kalman filter gain remain constant.

Steady State existence

It is known that the existence of the steady state is ensured

(a) if the signal process model is time-invariant and asymptotically stable, i.e. all the eigenvalues of F lie inside the Unit Circle, [11].

Note that in our signal process models are not asymptotically stable models; in fact for the constant velocity and constant acceleration models, the eigenvalues of F lie on the Unit Circle since $\lambda = 1$ and for the periodic motion models the eigenvalues of F lie outside the Unit Circle since $\lambda = 1 \pm j \cdot T$ and so $|\lambda| = \sqrt{1+T^2} > 1$

(b) if the pair $\left(F^T, H^T\right)$ is stabilizable and the pair $\left(F, G \cdot Q \cdot G^T\right)$ is controllable [25]

Note that in our signal process models the pair $\left(F^T, H^T\right)$ is stabilizable, since the pair (F, H) is observable.

Then it is necessary that $\left(F, G \cdot Q \cdot G^T\right)$ is controllable.

Proof

a) position only driven velocity motion model

In this case we have

$F = I$

$G \cdot Q \cdot G^T = {\sigma_\upsilon}^2 \cdot T^2 \cdot I$

$CM = [{\sigma_\upsilon}^2 \cdot T^2 \cdot I \quad {\sigma_\upsilon}^2 \cdot T^2 \cdot I \quad \cdots \quad {\sigma_\upsilon}^2 \cdot T^2 \cdot I]$

Then $\mathrm{rank}(OM) = n$

b) general velocity motion model and special velocity motion model

general acceleration motion model and special acceleration motion model

In these cases we have

$F = \begin{bmatrix} I & I \cdot T \\ 0 & I \end{bmatrix}$

$G \cdot Q \cdot G^T = {\sigma_a}^2 \cdot \begin{bmatrix} \frac{1}{4} \cdot T^4 & \frac{1}{2} \cdot T^3 \\ \frac{1}{2} \cdot T^3 & T^2 \end{bmatrix}$

$$CM = {\sigma_a}^2 \cdot \begin{bmatrix} \frac{1}{4} \cdot T^4 & \frac{1}{2} \cdot T^3 & \cdots & \left(\frac{1}{4} + \frac{n-1}{2}\right) \cdot T^4 & \left(\frac{1}{2} + n - 1\right) \cdot T^3 \\ \frac{1}{2} \cdot T^3 & T^2 & & \frac{1}{2} \cdot T^3 & T^2 \end{bmatrix}$$

Then $\mathrm{rank}(CM) = n$

In addition, the existence of the steady state is related to the symplectic matrix [25]

$$S = \begin{bmatrix} F^T + H^T \cdot R^{-1} \cdot H \cdot F^{-1} \cdot G \cdot Q \cdot G^T & -H^T \cdot R^{-1} \cdot H \cdot F^{-1} \\ -F^{-1} \cdot G \cdot Q \cdot G^T & F^{-1} \end{bmatrix} \tag{8}$$

where no eigenvalue of S lies on the Unit Circle.

We consider the time-invariant model and the steady state case where the estimation error covariance, the prediction error covariance and the Kalman filter gain remain constant. In fact, the steady state prediction error covariance P_p satisfies the algebraic Riccati Equation:

$$\begin{aligned} P_p &= G \cdot Q \cdot G^T + F \cdot P_p \cdot F^T \\ &-F \cdot P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \cdot H \cdot P_p \cdot F^T \end{aligned} \tag{9}$$

Then the steady state Kalman filter gain K is:

$$K = P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \tag{10}$$

and the steady state estimation error covariance P_e is:

$$P_e = [I - K \cdot H] \cdot P_p \tag{11}$$

It is worth to highlight that, for the constant velocity, constant acceleration and periodic motion models, the matrices P_p, K, P_e consist of block diagonal matrices.

The Riccati equation is very important in stability theory and so there are a variety of iterative or algebraic solution algorithms, [11].

Remark. In the case of the random walk model, where $F = H = G \cdot Q \cdot G^T = R = I$, the solution of the corresponding Riccati equation becomes:

$P_p = \varphi \cdot I$

where

$\varphi = \frac{\sqrt{5}+1}{2} \approx 1{,}618$ is the golden ratio

Then
$P_e = K = \alpha \cdot I$
where $\alpha = \frac{\sqrt{5}-1}{2} \approx 0{,}618$ is the golden section

3.3 Steady State Kalman Filter for Estimation

In the steady state case, from the time-invariant Kalman filter equations, the Steady State Kalman Filter for estimation is derived:

Steady State Kalman Filter for Estimation (SSKFE)
$x(k+1\|k+1) = D_x \cdot x(k\|k) + D_z \cdot z(k+1) + D_u \cdot u(k)$
for $k = 0,1,\ldots,$
with initial condition $x(0\|0)$

where

$$D_x = [I - K \cdot H] \cdot F \quad (12)$$

$$D_z = K \quad (13)$$

$$D_u = [I - K \cdot H] \cdot B \quad (14)$$

It is worth to note that the coefficients D_x, D_z, D_u are a-priori computed, i.e. before the filter implementation, by first off-line solving the corresponding algebraic Riccati equation.

Note that the initial condition is calculated by:
$K(0) = P(0|-1) \cdot H^T \cdot [H \cdot P(0|-1) \cdot H^T + R]^{-1}$
$x(0|0) = [I - K(0) \cdot H] \cdot x(0|-1) + K(0) \cdot z(0)$
with $x(0|-1) = x_0, P(0|-1) = P_0$

Another alternative initial condition is:
$x(0|0) = [I - K \cdot H] \cdot x(0|-1) + K \cdot z(0)$
with $x(0|-1) = x_0$

It is worth to highlight that, for the constant velocity, constant acceleration and periodic motion models, the matrices D_x, D_z, D_u consist of block diagonal matrices.

It is obvious that the steady state Kalman filter for estimation is faster than the time-invariant Kalman filter.

3.4 Steady State Kalman Filter for Prediction

In the steady state case, from the time-invariant Kalman filter equations, the Steady State Kalman Filter for prediction is derived:

Steady State Kalman Filter for Prediction (SSKFP)
$x(k+1\|k) = C_x \cdot x(k\|k-1) + C_z \cdot z(k) + C_u \cdot u(k)$
for $k = 0,1,\ldots,$
with initial condition $x(0\|-1) = x_0$

where

$$C_x = F \cdot [I - K \cdot H] \quad (15)$$

$$C_z = F \cdot K \quad (16)$$

$$C_u = B \quad (17)$$

It is worth to note that the coefficients C_x, C_z, C_u are a-priori computed, i.e. before the filter implementation, by first off-line solving the corresponding algebraic Riccati equation.

It is worth to highlight that, for the constant velocity, constant acceleration and periodic motion models, the matrices C_x, C_z, C_u consist of block diagonal matrices.

It is clear that the steady state Kalman filter for prediction has the same computational requirements as the steady state Kalman filter for estimation, so the steady state Kalman filter for prediction is faster than the time-invariant Kalman filter.

4 FIR Form of the Steady State Kalman Filter

4.1 FIR Form of the Steady State Kalman Filter for Estimation

From the estimation equation of the Prediction Steady State Kalman Filter we get:
$x(1|1) = D_x \cdot x(0|0) + D_z \cdot z(1) + D_u \cdot u(0)$

$x(2|2) = D_x \cdot x(1|1) + D_z \cdot z(2) + D_u \cdot u(1)$
$= D_x^2 \cdot x(0|0)$
$+ D_x \cdot D_z \cdot z(1) + D_z \cdot z(2)$
$+ D_x \cdot D_u \cdot u(0) + D_u \cdot u(1)$

$x(3|3) = D_x \cdot x(2|2) + D_z \cdot z(3) + D_u \cdot u(2)$
$= D_x^3 \cdot x(0|0)$
$+ D_x^2 \cdot D_z \cdot z(1) + D_x \cdot D_z \cdot z(2) + D_z \cdot z(3)$
$+ D_x^2 \cdot D_u \cdot u(0) + D_x \cdot D_u \cdot u(1) + D_u \cdot u(2)$

…

$x(k+1|k+1) = D_x^{k+1} \cdot x(0|0)$
$+ D_x^k \cdot D_z \cdot z(1) + \cdots + D_z \cdot z(k+1)$
$+ D_x^k \cdot D_u \cdot u(0) + \cdots + D_u \cdot u(k)$

It is known [26], [27] that if the spectral radius $\rho(M)$ of a matrix M is less than 1, then the powers of the matrix can be expected to converge to zero, i.e. there exists a positive integer L, such that $\|M^L\| < \varepsilon$, where ε is the convergence criterion and $\|M\|$ denotes the norm-2 of M. Hence, due to

computational accuracy, there exists some positive integer L, such that:
$M^L \neq 0, M^{L+1} = 0$

If the coefficient $D_x = [I - K \cdot H] \cdot F$ has this property, then
$x(L+1|L+1)$
$= D_x^L \cdot D_z \cdot z(1) + \cdots + D_z \cdot z(L+1)$
$+ D_x^L \cdot D_u \cdot u(0) + \cdots + D_u \cdot u(L)$

$x(L+2|L+2)$
$= D_x \cdot x(L+1|L+1)$
$+ D_z \cdot z(L+2) + D_u \cdot u(L+1)$
$= D_x^{L+1} \cdot D_z \cdot z(1) + D_x^L \cdot D_z \cdot z(2)$
$+ \cdots + D_x \cdot D_z \cdot z(L+1) + D_z \cdot z(L+2)$
$+ D_x^{L+1} \cdot D_u \cdot u(0) + D_x^L \cdot D_u \cdot u(1)$
$+ \cdots + D_x \cdot D_u \cdot u(L) + D_u \cdot u(L+1)$
$= D_x^L \cdot D_z \cdot z(2) + \cdots + D_z \cdot z(L+2)$
$+ D_x^L \cdot D_u \cdot u(1) + \cdots + D_u \cdot u(L+1)$

Then
$x(k+1|k+1)$
$= D_x^L \cdot D_z \cdot z(k+1-L) + \cdots + D_z \cdot z(k+1)$
$+ D_x^L \cdot D_u \cdot u(k-L) + \cdots + D_u \cdot u(k)$

Assuming that $z(k) = 0, k < 0$ and $u(k) = 0, k < 0$, the Finite Impulse Response form (FIR) form of the estimation steady state Kalman filter is derived:

FIR Steady State Kalman Filter for Estimation (FIRSSKFE)

$$x(k+1|k+1) = \sum_{i=0}^{L} D_x^i \cdot D_z \cdot z(k+1-i) + \sum_{i=0}^{L} D_x^i \cdot D_u \cdot u(k-i)$$

for $k = 0,1, \ldots$

where
$D_x = [I - K \cdot H] \cdot F$
$D_z = K$
$D_u = [I - K \cdot H] \cdot B$
and

$$\|D_x^L\| \geq \varepsilon, \|D_x^{L+1}\| < \varepsilon \quad (18)$$

The coefficients D_x, D_z, D_u are a-priori computed, i.e. before the filter implementation, by first off-line solving the corresponding algebraic Riccati equation. It is worth to note that the computation of the estimation is not done iteratively, since the knowledge of the previous estimate is not required; in fact the computation of the estimation requires the knowledge of the last $L+1$ observations and inputs. It is obvious that the FIR order L depends on the desired convergence criterion ε.

4.2 Steady State Kalman Filter and its FIR Form Combination for Estimation

In order to avoid the lack of measurements and inputs for negative time, we proposed the following approach, which combines SSKFE and FIRSSKFE:

SSKFE-FIRSSKFE combination
use of SSKFE for $k = 0,1, \ldots, L$
use of FIRSSKFE for $k = L+1, L+2, \ldots$

4.3 FIR form of Steady State Kalman Filter for Prediction

From the prediction equation of the Prediction Steady State Kalman Filter we get:

$x(1|0) = C_x \cdot x(0|-1) + C_z \cdot z(0) + C_u \cdot u(0)$
$= C_x \cdot x_0 + C_z \cdot z(0) + C_u \cdot u(0)$

$x(2|1) = C_x \cdot x(1|0) + C_z \cdot z(1) + C_u \cdot u(1)$
$= C_x^2 \cdot x_0$
$+ C_x \cdot C_z \cdot z(0) + C_z \cdot z(1)$
$+ C_x \cdot C_u \cdot u(0) + C_u \cdot u(1)$

$x(3|2) = C_x \cdot x(2|1) + C_z \cdot z(2) + C_u \cdot u(2)$
$= C_x^3 \cdot x_0$
$+ C_x^2 \cdot C_z \cdot z(0) + C_x \cdot C_z \cdot z(1) + C_z \cdot z(2)$
$+ C_x^2 \cdot C_u \cdot u(0) + C_x \cdot C_u \cdot u(1) + C_u \cdot u(2)$

...

$x(k+1|k) = C_x^{k+1} \cdot x_0$
$+ C_x^k \cdot C_z \cdot z(0) + \cdots + C_z \cdot z(k)$
$+ C_x^k \cdot C_z \cdot u(0) + \cdots + C_u \cdot u(k)$

It is known [26], [27] that if the spectral radius $\rho(M)$ of a matrix M is less than 1, then the powers of the matrix can be expected to converge to zero, i.e. there exists a positive integer L, such that $\|M^L\| < \varepsilon$, where ε is the convergence criterion and $\|M\|$ denotes the norm-2 of M. Hence, due to computational accuracy, there exists some positive integer L, such that:
$M^L \neq 0, M^{L+1} = 0$

If the coefficient $C_x = F \cdot [I - K \cdot H]$ has this property, then
$x(L+1|L)$
$= C_x^L \cdot C_z \cdot z(0) + \cdots + C_z \cdot z(L)$

$+C_x^L \cdot C_z \cdot u(0) + \cdots + C_u \cdot u(L)$

$x(L+2|L+1) = C_x \cdot x(L+1|L)$
$+C_z \cdot z(L+1) + C_u \cdot u(L+1)$
$= C_x^{L+1} \cdot C_z \cdot z(0) + C_x^L \cdot C_z \cdot z(1)$
$+ \cdots + C_x \cdot C_z \cdot z(L) + C_z \cdot z(L+1)$
$+C_x^{L+1} \cdot C_z \cdot u(0) + C_x^L \cdot C_z \cdot u(1)$
$+ \cdots + C_x \cdot C_u \cdot u(L) + C_u \cdot u(L+1)$
$= C_x^L \cdot C_z \cdot z(1) + \cdots + C_z \cdot z(L+1)$
$+C_x^L \cdot C_z \cdot u(1) + \cdots + C_u \cdot u(L+1)$
Then
$x(k+1|k)$
$= C_x^L \cdot C_z \cdot z(k-L) + \cdots + C_z \cdot z(k)$
$+C_x^L \cdot C_u \cdot u(k-L) + \cdots + C_u \cdot u(k)$

Assuming that $z(k) = 0, k < 0$ and $u(k) = 0, k < 0$, the Finite Impulse Response form (FIR) form of the prediction steady state Kalman filter is derived:

FIR Steady State Kalman Filter for Prediction (FIRSSKFP)

$$x(k+1|k) = \sum_{i=0}^{L} C_x^i \cdot C_z \cdot z(k-i) + \sum_{i=0}^{L} C_x^i \cdot C_u \cdot u(k-i)$$

for $k = 0,1,\ldots$

where
$C_x = F \cdot [I - K \cdot H]$
$C_z = F \cdot K$
$C_u = B$
and

$$\|C_x^L\| \geq \varepsilon, \|C_x^{L+1}\| < \varepsilon \quad (19)$$

The coefficients C_x, C_z, C_u are a-priori computed, i.e. before the filter implementation, by first off-line solving the corresponding algebraic Riccati equation. It is worth to note that the computation of the prediction is not done iteratively, since the knowledge of the previous estimate is not required; in fact the computation of the prediction requires the knowledge of the last $L+1$ observations and inputs. It is obvious that the FIR order L depends on the desired convergence criterion ε.

4.4 Steady State Kalman Filter and its FIR Form Combination for Prediction

In order to avoid the lack of measurements and inputs for negative time, we proposed the following approach, which combines SSKFP and FIRSSKFP:

SSKFP-FIRSSKFP combination
use of SSKFP for $k = 0,1,\ldots,L-1$
use of FIRSSKFP for $k = L, L+1,\ldots$

4.5 Existence of FIR Form of Steady State Kalman Filter

It is obvious that the existence of the FIR form of Steady State Kalman Filter depends on the eigenvalues of the matrices $D_x = [I - K \cdot H] \cdot F$ and $C_x = F \cdot [I - K \cdot H]$.

It is known [11] that the existence of the steady state implies that all the eigenvalues of $C_x = F \cdot [I - K \cdot H]$ lie inside the Unit Circle.

It is worth to note that $C_x = F \cdot [I - K \cdot H]$ and $D_x = [I - K \cdot H] \cdot F$ have the same eigenvalues since F and $[I - K \cdot H]$ are square matrices of the same dimensions.

Then it is clear that all the eigenvalues of $C_x = F \cdot [I - K \cdot H]$ and $D_x = [I - K \cdot H] \cdot F$ lie inside the Unit Circle.

It is worth to note that the eigenvalues of $C_x = F \cdot [I - K \cdot H]$ and $D_x = [I - K \cdot H] \cdot F$ are also the eigenvalues of the symplectic matrix S in (8) that lie inside the unit circle. The proof is in the Appendix.

5 Simulation Examples

Constant velocity, constant acceleration and periodic motions are simulated. The examples are summarized in Table 2.

Table 2. Examples

example	real motion	motion model
1	constant velocity $\upsilon_x = 5, \upsilon_y = 6, \upsilon_z = 2$	special velocity
2	constant velocity $\upsilon_x = 5, \upsilon_y = 6, \upsilon_z = 2$	general velocity
3	constant velocity $\upsilon_x = 5, \upsilon_y = 6, \upsilon_z = 2$	general acceleration
4	constant acceleration $a_x = 2, a_y = 3, a_z = 1$	general acceleration
5	periodic $p_x = 5 \cdot \cos(k), \upsilon_x = -5 \cdot \sin(k)$ $p_y = 8 \cdot \cos(k), \upsilon_y = -8 \cdot \sin(k)$	general periodic

Source: created by the authors.

Simulation parameters

Constant velocity and constant acceleration motions. The sampling period is $T = 1$.

The position, velocity and acceleration are supposed to be corrupted with noise with variances $\sigma_p{}^2 = 3^2$, $\sigma_\upsilon{}^2 = 1^2$, $\sigma_a{}^2 = 0.3^2$ respectively.

The initial condition of SSKF for prediction is:
$x(0|-1) = [2 \quad 2 \quad 2 \quad 5 \quad 5 \quad 0]^T$

Periodic motion.

The position and velocity are supposed to be corrupted with noise with variances $\sigma_p{}^2 = 0.3^2$, $\sigma_\upsilon{}^2 = 0.2^2$ respectively.

The initial condition of SSKF for prediction is:
$x(0|-1) = [5 \quad 8 \quad -5 \quad -8]^T$

Algorithms
SSKF and SSKF-FIRSSKF for estimation and prediction were implemented.

Table 3 records spectral radius and FIR order for estimation and prediction for the examples.

Table 3. Examples

example	**spectral radius estimation**	**FIR order estimation (ε=0.01)**	**spectral radius prediction**	**FIR order prediction (ε=0.01)**
1	0.8000	25	0.8000	24
2	0.7647	17	0.7647	17
3	0.7647	17	0.7647	17
4	0.7647	17	0.7647	17
5	0.3985	5	0.3985	5

Source: created by the authors.

Figure 1, Figure 2, Figure 3, Figure 4, Figure 5 depict the true position and velocity as well as the estimated and predicted position and velocity for the examples, when SSKF-FIRSSKF for estimation and prediction were implemented.

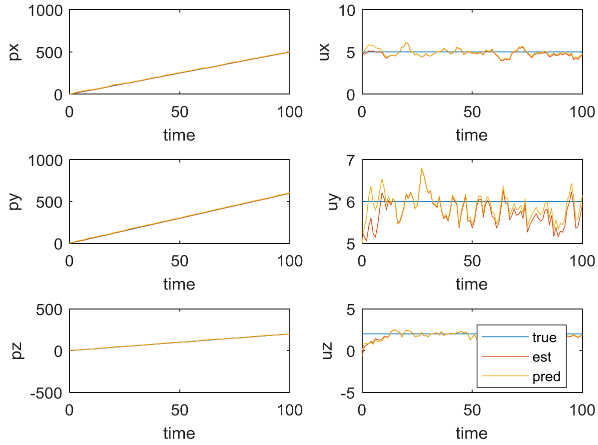

Fig. 1: Example 1. True, estimated and predicted position and velocity
Source: created by the authors.

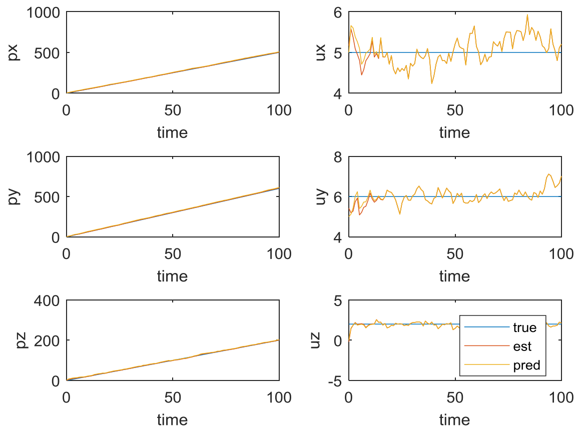

Fig. 2: Example 2. True, estimated and predicted position and velocity
Source: created by the authors.

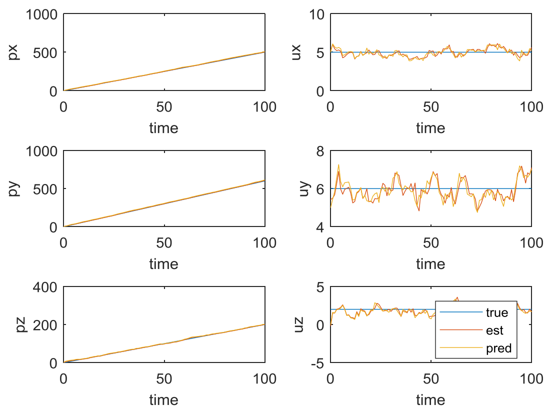

Fig. 3: Example 3. True, estimated and predicted position and velocity
Source: created by the authors.

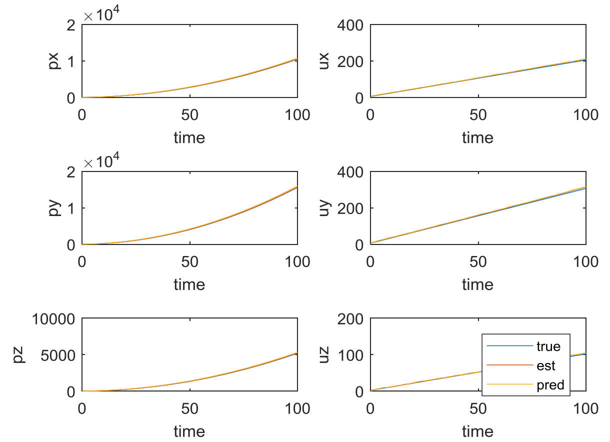

Fig. 4: Example 4. True, estimated and predicted position and velocity
Source: created by the authors.

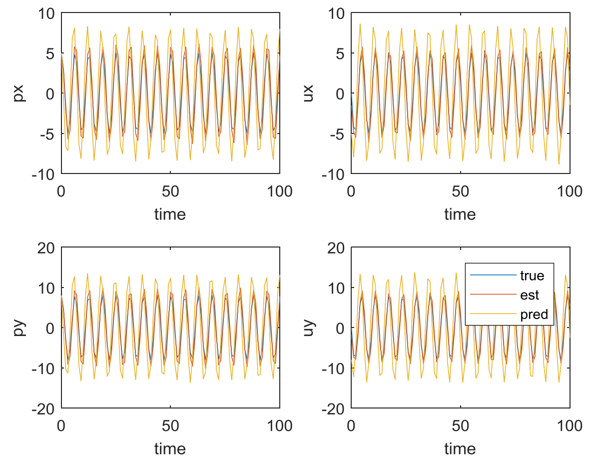

Fig. 5: Example 5. True, estimated and predicted position and velocity
Source: created by the authors.

Statistics
The following error metrics were calculated for the position and velocity estimation and prediction using SSKF:

Estimation Mean Absolute Error (MAE)

$$= \frac{1}{N}\sum_{k=1}^{N}|x(k|k) - x(k)|$$

Estimation Root Mean Squared Error (RMSE)

$$= \sqrt{\frac{1}{N}\sum_{k=1}^{N}[x(k|k) - x(k)]^2}$$

Prediction Mean Absolute Error (MAE)

$$= \frac{1}{N}\sum_{k=1}^{N}|x(k+1|k) - x(k)|$$

Prediction Root Mean Squared Error (RMSE)

$$= \sqrt{\frac{1}{N}\sum_{k=1}^{N}[x(k+1|k) - x(k)]^2}$$

where N is the number of observations

Table 4 and Table 5 record MAE and RMSE for estimation and prediction for the examples, when SSKF and SSKF-FIRSSKF for estimation and prediction were implemented.

Table 4. MAE and RMSE for estimation and prediction – SSKF

example	state	estimation MAE	prediction MAE	estimation RMSE	prediction RMSE
1	px	1.5124	5.1793	1.8341	5.6186
	py	1.1648	6.3997	1.4589	6.6135
	pz	1.2949	2.3284	1.6373	2.7179
	υx	0.2768	0.3079	0.3714	0.3996
	υy	0.2560	0.2329	0.3303	0.2960
	υz	0.3639	0.3626	0.5177	0.4876
2	px	1.3833	4.8084	1.6273	5.1214
	py	0.9457	5.8832	1.2115	6.0749
	pz	1.0155	1.9889	1.3442	2.4002
	υx	0.2551	0.2535	0.3129	0.3148
	υy	0.2731	0.2616	0.3554	0.3451
	υz	0.2284	0.2281	0.3448	0.3317
3	px	1.5949	4.5515	1.8495	4.9450
	py	1.1943	5.6379	1.4630	5.8663
	pz	1.2884	1.8126	1.6573	2.3188
	υx	0.4151	0.4330	0.5163	0.5395
	υy	0.4482	0.4813	0.5319	0.5669
	υz	0.4860	0.5146	0.6215	0.6519
4	px	5.1760	105.4145	5.4506	120.8870
	py	8.0614	156.9324	8.2076	179.8427
	pz	2.2404	51.8745	2.6242	59.2382
	υx	0.7945	1.7702	0.9134	1.8425
	υy	1.2237	2.7196	1.3281	2.7864
	υz	0.4707	0.7547	0.6165	0.9145
5	px	0.8822	3.1671	0.9898	3.5308
	py	1.3757	5.1340	1.5365	5.7027
	υx	0.6194	3.5011	0.7072	3.9049
	υy	0.9322	5.6714	1.0415	6.3217

Source: created by the authors.

Table 5. MAE and RMSE for estimation and prediction – SSKF-FIRSSKF

example	state	estimation MAE	prediction MAE	estimation RMSE	prediction RMSE
1	px	1.5606	4.6260	1.9747	5.1128
	py	1.3691	5.6930	1.7158	5.9927
	pz	1.3217	2.1706	1.6831	2.5643
	υx	0.3344	0.3265	0.4317	0.4207
	υy	0.3341	0.2603	0.4134	0.3282
	υz	0.3778	0.3687	0.5251	0.4905
2	px	2.0721	6.2061	2.5426	6.6754
	py	1.9848	7.5633	2.4253	7.8798
	pz	0.9545	2.4395	1.2704	2.7745
	υx	0.2715	0.2699	0.3341	0.3358
	υy	0.2835	0.2719	0.3769	0.3670
	υz	0.2188	0.2185	0.3353	0.3218
3	px	2.0728	5.9479	2.5338	6.4725
	py	1.8072	7.3173	2.2335	7.6098
	pz	1.1091	2.1567	1.5043	2.6340
	υx	0.4143	0.4284	0.5002	0.5262
	υy	0.4221	0.4464	0.5243	0.5370
	υz	0.4739	0.4973	0.6157	0.6372
4	px	20.7928	122.3249	27.2303	144.2073
	py	1.1014	181.8828	39.8135	214.2454
	pz	9.8641	60.1911	12.7382	70.6982
	υx	2.1071	3.1066	2.5375	3.4366
	υy	3.1896	4.6956	3.7279	5.1059
	υz	0.9806	1.3839	1.2107	1.6025
5	px	0.8665	3.1757	0.9726	3.5398
	py	1.3528	5.1461	1.5096	5.7167
	υx	0.6093	3.4994	0.6962	3.9038
	υy	0.9157	5.6701	1.0233	6.3203

Source: created by the authors.

From Table 4 and Table 5 it is clear that the proposed estimation and prediction algorithms which combine the steady state Kalman filter and its FIR form, possess a very good behavior, since they produce MAE and RMSE close to the corresponding of steady state Kalman filter.

6 Conclusions

Constant velocity, constant acceleration and periodic motion state space models were considered. The models are time-invariant and it was shown that they are observable and controllable. The popular linear time-invariant Kalman filter can be implemented in order to estimate or predict the position and the velocity of moving objects. Steady state appears when the estimation and prediction error covariances remain constant in time. It was shown that steady state always exists for these models and steady state Kalman filter for estimation and prediction are derived. It is worth to point out that steady state Kalman filter is faster than time-invariant Kalman filter. In addition, it was also shown that always exists the FIR form of the steady state Kalman filter and FIR form of the steady state Kalman filter for estimation and prediction are derived. It is worth to highlight that FIR form of the steady state filter is not iterative and requires the knowledge of a set of observations and inputs. Finally, estimation and prediction algorithms which combine the steady state Kalman filter and its FIR form were proposed. The adequate behavior of the proposed estimation and prediction algorithms was verified through simulation results.

A subject of future work may be the applicability of steady state Kalman filter and its FIR form to visual tracking applications such as intelligent surveillance, autonomous driving, robot navigation, human computer/robot interaction, augmented reality, medical applications, motion-based recognition, video indexing [2]. Another subject of future work may be the investigation of the applicability of steady state Kalman filter and its FIR form to circular motion.

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APPENDIX

Relation of the eigenvalues of C_x, D_x and S

The algebraic Riccati equation (9) can be written in the form:

$$P = G \cdot Q \cdot G^T + F \cdot [P^{-1} + H^T \cdot R^{-1} \cdot H]^{-1} \cdot F^T$$

or

$$P = c + a^T \cdot [P^{-1} + b]^{-1} \cdot a \quad (A.1)$$

by setting

$a = F^T$

$b = H^T \cdot R^{-1} \cdot H$

$c = G \cdot Q \cdot G^T$

The symplectic matrix S in (8) is related to the algebraic Riccati equation solution. In fact, taking $\Phi = S^{-1}$ we get

$$\Phi = \begin{bmatrix} a^{-1} & a^{-1} \cdot b \\ c \cdot a^{-1} & a^T + c \cdot a^{-1} \cdot b \end{bmatrix} \quad (A.2)$$

We take the eigenvalue decomposition of Φ:

$$\Phi = W \cdot L \cdot W^{-1}$$

where the columns of W are the eigenvectors of Φ and L is a diagonal matrix containing the eigenvalues of Φ.

The eigenvalues of the symplectic Φ occur in reciprocal pairs; then we are able to write

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

and

$$L = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}$$

where Λ is a diagonal matrix containing the eigenvalues that lie outside the unit circle.

Then, the algebraic Riccati equation solution is

$$P = W_{21} \cdot {W_{11}}^{-1} \quad (A.3)$$

From $\Phi \cdot W = W \cdot L$ we get

$$a^{-1} \cdot W_{11} + a^{-1} \cdot b \cdot W_{21} = W_{11} \cdot \Lambda$$

$$c \cdot a^{-1} \cdot W_{11} + c \cdot a^{-1} \cdot b \cdot W_{21} + a^T \cdot W_{21} = W_{21} \cdot \Lambda$$

Then substituting the first equation in the second, we get

$$c \cdot W_{11} \cdot \Lambda + a^T \cdot W_{21} = W_{21} \cdot \Lambda$$

$$c \cdot W_{11} + a^T \cdot W_{21} \cdot \Lambda^{-1} = W_{21}$$

$$c + a^T \cdot W_{21} \cdot \Lambda^{-1} \cdot {W_{11}}^{-1} = W_{21} \cdot {W_{11}}^{-1}$$

$$c + a^T \cdot W_{21} \cdot {W_{11}}^{-1} \cdot W_{11} \cdot \Lambda^{-1} \cdot {W_{11}}^{-1} = W_{21} \cdot {W_{11}}^{-1}$$

and using (A.3) we derive

$$P = c + a^T \cdot P \cdot W_{11} \cdot \Lambda^{-1} \cdot {W_{11}}^{-1} \quad (A.4)$$

The steady state gain is

$$K = P \cdot H^T \cdot [H \cdot P \cdot H^T + R]^{-1}$$

Then

$$\begin{aligned}[I - K \cdot H] \cdot P &= P - K \cdot H \cdot P \\ &= P - P \cdot H^T \cdot [H \cdot P \cdot H^T + R]^{-1} \cdot H \cdot P = [P^{-1} + H^T \cdot R^{-1} \cdot H]^{-1} \\ &= [P^{-1} + b]^{-1}\end{aligned}$$

and

$$[I - K \cdot H] = [P^{-1} + b]^{-1} \cdot P^{-1} \quad (A.5)$$

The steady state Kalman filter parameter C_x is

$$C_x = F \cdot [I - K \cdot H] \quad (A.6)$$

Then using (A.5) we get

$$C_x = F \cdot [P^{-1} + b]^{-1} \cdot P^{-1}$$

from where we get

$$F \cdot [P^{-1} + b]^{-1} = C_x \cdot P \quad (A.7)$$

$$[P^{-1} + b] = P^{-1} \cdot {C_x}^{-1} \cdot F \quad (A.8)$$

In the following, we write (A.1) as

$$P = c + a^T \cdot [P^{-1} + b]^{-1} \cdot a$$

$$P = c + a^T \cdot [P^{-1} + b]^{-1} \cdot [P^{-1} + b] \cdot [P^{-1} + b]^{-1} \cdot a$$

$$P = c + F \cdot [P^{-1} + b]^{-1} \cdot [P^{-1} + b] \cdot [P^{-1} + b]^{-1} \cdot F^T$$

$$P = c + F \cdot [P^{-1} + b]^{-1} \cdot [P^{-1} + b] \cdot (F \cdot [P^{-1} + b]^{-1})^T$$

and using (A.7) and (A.8) we get

$$P = c + [C_x \cdot P] \cdot \left[P^{-1} \cdot {C_x}^{-1} \cdot F\right] \cdot [C_x \cdot P]^T$$

$$P = c + C_x \cdot P \cdot P^{-1} \cdot {C_x}^{-1} \cdot F \cdot P \cdot {C_x}^T$$

$$P = c + F \cdot P \cdot {C_x}^T$$

So

$$P = c + a^T \cdot P \cdot {C_x}^T \quad (A.9)$$

Thus from (A.4) and (A.9) we get

$${C_x}^T = W_{11} \cdot \Lambda^{-1} \cdot {W_{11}}^{-1}$$

or

$$C_x = {W_{11}}^{-T} \cdot \Lambda^{-1} \cdot {W_{11}}^T \quad (A.10)$$

Hence C_x is similar to Λ^{-1}.

This means that the eigenvalues of C_x are the diagonal elements of Λ^{-1} that lie inside the unit circle. Recall that these are eigenvalues of Φ and consequently of S.

Thus, the eigenvalues of $C_x = F \cdot [I - K \cdot H]$ are also the eigenvalues of the symplectic matrix S in (8).

It is worth to note that $C_x = F \cdot [I - K \cdot H]$ and $D_x = [I - K \cdot H] \cdot F$ have the same eigenvalues since F and $[I - K \cdot H]$ are square matrices of the same dimensions.

Thus, the eigenvalues of $C_x = F \cdot [I - K \cdot H]$ and $D_x = [I - K \cdot H] \cdot F$ are also the eigenvalues of the symplectic matrix S in (8) that lie inside the unit circle.