# Determinant Symmetry and Scaled Hyper G-Pair Relations of $T_c^{(n)}$ Matrices

HASAN KELEŞ*[1]
[1]Department of Mathematics
Karadeniz Technical University
Campus of Kanuni, Ortahisar, 61080 Trabzon
TÜRKİYE

*Abstract:* We investigate the family of structured matrices $T_c^{(n)}$ defined by a geometric sum $S_n = \sum_{k=0}^{n} c^k$ in the first entry and a descending power pattern in subsequent rows. A closed formula for $\det(T_c^{(n)})$ is derived, showing that for integer $c$ the determinant is nonzero exactly when $c \notin \{1, -1\}$ and equals 1 only for $c = 0$ (if $n \geq 3$). The matrices exhibit a reciprocal symmetry expressed through a *scaled Hyper Gpair* relation: there exist diagonal matrices $D_1, D_2$ and a third diagonal matrix $\Gamma(c)$ such that

$$(T_c^{(n)})^{-T} = \Gamma(c)\, D_1\, T_{1/c}^{(n)}\, D_2.$$

generalizes the classical notion of Hyper Gmatrices, where $\Gamma(c)$ would be a scalar, and reveals a richer algebraic structure linking $T_c^{(n)}$ with its reciprocal parameter counterpart. The results unify combinatorial determinant evaluation, modularlike matrix symmetries, and the theory of scaled Gpairs.

*Key-Words:* Structured matrices, Determinant formulas, Hyper G-matrices, Scaled reciprocal relations, Matrix symmetries, Integer parameter matrices, Block matrices, Diagonal congruences.

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## 1 Introduction

The investigation of sequences like $H_n = 3^n - 1$ connects with numerous areas of mathematics. Fibonacci numbers and their generalizations appear in various contexts, from combinatorial identities to matrix theory and number theory, [1]. This topic also relates to broader structural properties, [2]. The modular group and its subgroups provide a rich algebraic framework for studying matrix actions, continued fractions, and their connections to hyperbolic geometry, [3]. Further aspects of these group actions are examined in the literature, [4]. Additional studies explore related algebraic structures, [5]. The geometric interpretations of these actions are also significant, [6]. Connections to continued fractions have been established, [7]. Hyperbolic geometry provides a natural setting for these investigations, [8]. Further developments in this direction have been presented, [9]. Various matrix constructions involving Fibonacci and related sequences have been explored extensively, revealing deep structural properties and computational applications, [10]. These constructions often lead to new computational methods, [11]. The structural properties have been analyzed in detail, [12]. Applications in different areas of mathematics have also emerged, [13]. Meanwhile, combinatorial identities uncover intricate relationships between these special numbers and other mathematical objects, [14]. Such identities often reveal unexpected connections, [15]. The interplay between combinatorial structures and number theory is particularly fruitful, [16]. The relations among $O(n)$-invariants of several matrices have been systematically characterized, providing a comprehensive algebraic framework for understanding invariant theory with applications to matrix representations, [17]. These graphical methods offer new insights into group actions, [18]. A comprehensive study of matrices involving bidimensional Fibonacci numbers has been conducted, demonstrating that these matrices exhibit rich algebraic structures and satisfy novel recurrence relations that extend the well-known properties of traditional Fibonacci matrices, [19]. Suborbital graphs have been studied in various contexts, [20]. Their relevance to number theory continues to grow, [21]. Recent research on polynomial generalizations of Fibonacci numbers has expanded the classical theory to include hyperbolic, $k$-step, and other extended Fibonacci-type sequences, revealing new algebraic and analytical properties, [22]. These generalizations have led to new algebraic frameworks, [23]. Analytical properties of these sequences have been investigated, [24]. Hyperbolic Fibonacci-type sequences exhibit distinctive behavior, [25]. The $k$-step extensions have been characterized in detail, [26]. Further algebraic properties have been derived, [27]. New analytical methods have been applied to these sequences, [28]. Extensions of the classical theory continue to appear, [29]. Connections to other special sequences have been identified, [30]. The polynomial approach has proven particularly versatile, [31]. Additional results on hyperbolic cases have been obtained, [32]. The $k$-step case has been generalized further, [33]. These sequences appear in various mathematical disciplines, [34]. Their properties have been studied from multiple perspectives, [35]. Recent work has unified several generalizations, [36]. The analytical framework has been strengthened, [37]. Algebraic characterizations have been refined, [38]. Further extensions continue to be explored, [39]. Simultaneously, developments in multi-valued logic systems have established connections between logical algebras and algebraic structures related to Fibonacci sequences, opening new interdisciplinary research directions, [40]. These connections have led to novel logical systems, [41].

Also, The study of density matrices and their relations has profound implications in quantum information theory, particularly for evenN qubit states, [42]. Stability results for A relations in $C^*$-algebras with tracial rank at most one provide further structural insights into operator algebras, [43]. Inequalities for the Euclidean operator radius of nn-tuple operators and operator matrices in Hilbert $C$-modules have been established, extending classical norm inequalities to modular settings, [44]. Early foundational work on first and secondorder density matrices of symmetryprojected singledeterminant wavefunctions remains highly relevant for understanding electronic structure calculations, [45]. The theory of (M,N)-coherent pairs of linear functionals and their associated Jacobi matrices connects classical orthogonal polynomials with matrix analysis, [46]. The relationship between the determinant and the permanent on symmetric matrices reveals subtle combinatorial differences between these two important matrix functions, [47]. Investigations into undergraduate students' performances in linear algebra, particularly regarding the factorization of determinants of nŒnnŒn dimensional matrices, highlight educational challenges in this area, [48]. Asymptotic expansions of Toeplitz determinants of an indicator function with discrete rotational symmetry, as well as powers of random unitary matrices, have been derived, contributing to random matrix theory, [49]. A homogeneous recurrence relation for the determinants of general pentadiagonal Toeplitz matrices has been established, providing efficient computational methods for this important class of structured matrices, [50]. Smalldeterminant directional dilation matrices for anisotropic multiresolution analysis have been developed, offering new tools for signal processing and wavelet theory, [51]. The relation between determinants of tridiagonal and certain pentadiagonal matrices has been explored, providing efficient computational formulas for structured matrices, [52]. Cofactor expansion approaches for generalized determinants of nonsquare matrices have been proposed, extending classical determinant theory to rectangular arrays, [53].

These diverse investigations provide a comprehensive background for understanding the intricate relationships among $H_n$, $T_c$, and Fibonacci sequences explored in this paper. The present work builds upon this rich mathematical landscape by investigating a specific family of structured matrices $T_c^{(n)}$, analyzing their determinant properties, reciprocal symmetries, and potential cryptographic applications. The interplay between matrix theory, number sequences, and algebraic structures forms the foundation for the exploration of scaled Hyper $G$-pair relations and diagonal congruence symmetries in these matrices.

## 2 Basic Preliminary Results

This paper investigates the determinant properties of a family of structured matrices denoted by $T_c^{(n)}$. These matrices arise in various combinatorial and algebraic contexts and exhibit interesting behavior depending on the parameter $c$.

For $n \geq 2$ and $c \in \mathbb{C}$, the matrix is defined as in (1):

$$T_c^{(n)} = \begin{bmatrix} S_n & -c^{n-1} & -c^{n-2} & \cdots & -c \\ c^n & 1-c^{n-1} & 0 & \cdots & 0 \\ c^{n-1} & 0 & 1-c^{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c^2 & 0 & 0 & \cdots & 1-c \end{bmatrix}, \quad (1)$$

where $S_n = \sum_{k=0}^{n} c^k$.

The matrix $T_c^{(n)}$ has a distinctive block structure: the first row contains the geometric sum $S_n$ followed by negative powers of $c$, while the remaining rows have a diagonal-like pattern with nonzero entries only in the first column and on the diagonal.

The primary objective is to characterize the determinant $\det(T_c^{(n)})$ when $c$ is an integer. Specifically, the aim is to determine:

(i) when the determinant is nonzero, and in particular when it equals 1;

(ii) when the determinant vanishes.

The results reveal a surprisingly simple classification that depends only on the values $c = 0, \pm 1$ and the dimension $n$.

The determinant of $T_c^{(n)}$ can be obtained by viewing the matrix in block form. To this end, the following decomposition is recorded. A proposition describing the block decomposition of the matrix is now presented.

**Proposition 2.1.** *For $n \geq 2$ and $c \in \mathbb{C}$, the matrix $T_c^{(n)}$ admits the block decomposition given in* (2)*:*

$$T_c^{(n)} = \begin{bmatrix} S_n & -\mathbf{r}^\top \\ \mathbf{v} & D \end{bmatrix}, \tag{2}$$

*where*

$$S_n = \sum_{k=0}^{n} c^k, \tag{3}$$

$$\mathbf{r} = \big(c^{n-1}, c^{n-2}, \ldots, c\big)^\top, \tag{4}$$

$$\mathbf{v} = \big(c^{n}, c^{n-1}, \ldots, c^2\big)^\top, \tag{5}$$

*and*

$$D = \operatorname{diag}\big(1 - c^{n-1}, 1 - c^{n-2}, \ldots, 1 - c\big). \tag{6}$$

*Proof.* By (1), the topleft entry is $S_n$ as defined in (3). The first row after $S_n$ equals $-\mathbf{r}^\top$ with $\mathbf{r}$ defined in (4). The first column below $S_n$ equals $\mathbf{v}$ defined in (5). The lowerright $(n-1) \times (n-1)$ block is diagonal with entries $1 - c^{n-1}, 1 - c^{n-2}, \ldots, 1 - c$, i.e., $D$ as in (6).

Thus $T_c^{(n)}$ coincides with the stated block matrix. $\square$

This is illustrated with a concrete example.

**Example 2.2.** For $n = 4$ and an arbitrary $c \in \mathbb{C}$, the matrix $T_c^{(4)}$ is given by (7):

$$T_c^{(4)} = \begin{bmatrix} S_4 & -c^3 & -c^2 & -c \\ c^4 & 1 - c^3 & 0 & 0 \\ c^3 & 0 & 1 - c^2 & 0 \\ c^2 & 0 & 0 & 1 - c \end{bmatrix}, \tag{7}$$

with $S_4$ defined in (8):

$$S_4 = 1 + c + c^2 + c^3 + c^4. \tag{8}$$

Its block decomposition according to Proposition 2.1 is given by (9):

$$T_c^{(4)} = \begin{bmatrix} S_4 & -\mathbf{r}^\top \\ \mathbf{v} & D \end{bmatrix}, \tag{9}$$

where $\mathbf{r}$ and $\mathbf{v}$ are as in (10):

$$\mathbf{r} = \begin{pmatrix} c^3 & c^2 & c \end{pmatrix}^\top, \qquad \mathbf{v} = \begin{pmatrix} c^4 & c^3 & c^2 \end{pmatrix}^\top, \tag{10}$$

and $D$ is as in (11):

$$D = \operatorname{diag}\big(1 - c^3,\ 1 - c^2,\ 1 - c\big). \tag{11}$$

A closed-form expression for $\det(T_c^{(n)})$ is now derived, which will be the foundation for all subsequent analysis. The main determinant formula is stated and proved.

**Proposition 2.3.** *Let $n \geq 2$ and $c \in \mathbb{C}$ satisfy $c^k \neq 1$ for $k = 1, \ldots, n-1$. For the matrix $T_c^{(n)}$, the determinant is given by* (12)*:*

$$\det(T_c^{(n)}) = \left[ \textstyle\sum_{k=0}^{n} c^k + \sum_{m=1}^{n-1} \frac{c^{2m+1}}{1 - c^m} \right] \prod_{k=1}^{n-1} (1 - c^k) \tag{12}$$

*Proof.* Write $T_c^{(n)}$ in block form as in (13):

$$T_c^{(n)} = \begin{bmatrix} S_n & -\mathbf{r}^T \\ \mathbf{v} & D \end{bmatrix}, \tag{13}$$

where $\mathbf{r}^T$ and $\mathbf{v}$ are defined in (14):

$$\begin{aligned} \mathbf{r}^T &= (c^{n-1}, c^{n-2}, \ldots, c), \\ \mathbf{v} &= (c^{n}, c^{n-1}, \ldots, c^2)^T, \end{aligned} \tag{14}$$

and $D$ is defined in (15):

$$D = \operatorname{diag}(1 - c^{n-1}, 1 - c^{n-2}, \ldots, 1 - c). \tag{15}$$

Since $c^k \neq 1$ for $k = 1, \ldots, n-1$, the diagonal matrix $D$ is invertible. Hence, by the Schur complement formula,

$$\det(T_c^{(n)}) = \det(D)\, \det\big(S_n - \mathbf{r}^T D^{-1} \mathbf{v}\big). \tag{16}$$

First, compute $\det(D)$ as in (17):

$$\det(D) = \prod_{k=1}^{n-1} (1 - c^k). \tag{17}$$

Next, note that $D^{-1}$ is given by (18):

$$D^{-1} = \operatorname{diag}\left( \frac{1}{1 - c^{n-1}}, \frac{1}{1 - c^{n-2}}, \ldots, \frac{1}{1 - c} \right). \tag{18}$$

A direct calculation yields (19):

$$\mathbf{r}^T D^{-1}\mathbf{v} = \sum_{m=1}^{n-1} \frac{c^{n-m}\, c^{n-m+1}}{1-c^{n-m}} = \sum_{m=1}^{n-1} \frac{c^{2m+1}}{1-c^m}, \quad (19)$$

where a change of index in the summation is used.

Therefore, the Schur complement reduces to (20):

$$S_n - \mathbf{r}^T D^{-1}\mathbf{v} = \sum_{k=0}^{n} c^k + \sum_{m=1}^{n-1} \frac{c^{2m+1}}{1-c^m}. \quad (20)$$

Combining (16), (17) and (20), it follows that

$$\det(T_c^{(n)}) = \left[ \sum_{k=0}^{n} c^k + \sum_{m=1}^{n-1} \frac{c^{2m+1}}{1-c^m} \right] \prod_{k=1}^{n-1}(1-c^k) \quad (21)$$

which proves the claim. □

The formula (12) is valid whenever all denominators $1-c^k$ are nonzero. For integer $c$, the only exceptions occur at $c = 1$ and $c = -1$, which are handled separately. The product term $\prod_{k=1}^{n-1}(1-c^k)$ plays a crucial role: it vanishes precisely when $c$ is a $k$-th root of unity for some $k \le n-1$.

Using (12), the invertibility of $T_c^{(n)}$ for integer $c$ can be completely characterized. The following lemma characterizes when the determinant is nonzero.

**Lemma 2.4.** *For integer $c \in \mathbb{Z}$,*

*i. If $n = 2$, then $\det(T_c^{(2)}) = 1 \neq 0$ for every $c$.*

*ii. If $n \ge 3$, then the condition (22) holds:*

$$\det(T_c^{(n)}) \neq 0 \iff c \notin \{1, -1\}. \quad (22)$$

*Proof.* For $n = 2$, direct computation yields $\det(T_c^{(2)}) = (1+c+c^2)(1-c)+c^3 = 1$.

For $n \ge 3$, if $\det(T_c^{(n)}) \neq 0$, then from (12) it must hold that $\prod_{k=1}^{n-1}(1-c^k) \neq 0$. For integer $c$, this forces $c \neq \pm 1$ because $1-(-1)^2 = 0$ and $1-1^k = 0$. Conversely, if $c \notin \{1,-1\}$, then either $c = 0$ or $|c| \ge 2$. In both cases $c^k \neq 1$ for all $k$, so the product is nonzero. Moreover, the bracketed factor in (12) cannot vanish for integer $c$ (a parity argument handles $|c| \ge 2$, while $c = 0$ gives the value 1). Hence $\det(T_c^{(n)}) \neq 0$. □

The singular cases $c = \pm 1$ are easily understood: for $c = 1$, rows 2 through $n$ become identical; for $c = -1$, the factor $1-(-1)^2 = 0$ makes the product vanish.

A natural question is when the determinant takes the minimal nonzero value 1. The answer turns out to be quite restrictive. An immediate consequence is (23):

**Corollary 2.5.** *For integer $c \in \mathbb{Z}$,*

$$\det(T_c^{(n)}) = 1 \iff \begin{cases} \text{any } c \in \mathbb{Z}, & n = 2, \\ c = 0, & n \ge 3. \end{cases} \quad (23)$$

*Proof.* For $n = 2$ it holds that $\det(T_c^{(2)}) = 1$ universally. For $n \ge 3$, if $\det(T_c^{(n)}) = 1$, then by Lemma 2.4 it must be that $c \notin \{1,-1\}$. If $|c| \ge 2$, then $|1-c^k| \ge 2^k - 1$, so $|\prod_{k=1}^{n-1}(1-c^k)| > 1$, forcing $|\det(T_c^{(n)})| > 1$ — a contradiction. Thus $c = 0$. The converse is immediate from (12). □

Consider the following example illustrating the determinant being equal to 1.

**Example 2.6.** For $n = 4$ and $c = 0$, we have $S_4 = \sum_{k=0}^{4} 0^k = 1$ and $T_0^{(4)}$ as in (24):

$$T_0^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

This is clearly the identity matrix $I_4$, so $\det(T_0^{(4)}) = 1$, confirming Corollary 2.5 that for $n \ge 3$, the determinant equals 1 only when $c = 0$.

In contrast, for $c = 2$ (as shown in Example 2.7), $\det(T_2^{(4)}) = 125 \neq 1$, and for $c = 1$ (as shown in Example 2.8), $\det(T_1^{(4)}) = 0$. This illustrates the sharp classification provided by (23): among integer $c$, the unique case yielding determinant 1 for $n \ge 3$ is $c = 0$.

The following table, constructed by the author based on the preceding analysis, summarizes the complete behavior of $\det(T_c^{(n)})$ for integer $c$.

Table 1: Determinant behavior of $T_c^{(n)}$ for integer $c$ (Source: created by the authors).

| $n$ | Condition on $c$ | $\det(T_c^{(n)})$ | Remark |
|---|---|---|---|
| $n=2$ | any $c \in \mathbb{Z}$ | 1 | always nonsingular |
| $n \ge 3$ | $c = 0$ | 1 | unique case with determinant 1 |
| | $c = \pm 1$ | 0 | singular |
| | $c \neq 0, \pm 1$ | $\neq 0, \neq 1$ | nonsingular, determinant not 1 |

The determinant behavior is summarized in Table 1. In words:

- For $n = 2$ the determinant is identically 1, independent of $c$.
- For $n \ge 3$ the determinant is nonzero exactly when $c \notin \{1, -1\}$.
- For $n \ge 3$ the determinant equals 1 only when $c = 0$.

This classification is remarkably simple and demonstrates how the dimension $n$ drastically changes the determinant's dependence on $c$.

To make the abstract results concrete, two explicit examples are presented that highlight the different behaviors. An example with a specific matrix and its determinant is provided.

**Example 2.7.** For $n = 4$ and $c = 2$, we have $S_4 = 31$ and $T_2^{(4)}$ as in (25):

$$T_2^{(4)} = \begin{bmatrix} 31 & -8 & -4 & -2 \\ 16 & -7 & 0 & 0 \\ 8 & 0 & -3 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix}. \tag{25}$$

Applying (12) yields $\det(T_2^{(4)}) = 125$. Since $2 \notin \{1, -1\}$, Lemma 2.4 guarantees a nonzero determinant, and indeed $125 \neq 0$. Moreover, $125 \neq 1$, which is consistent with Corollary 2.5 stating that for $n \geq 3$ only $c = 0$ gives determinant 1.

Now an example of a singular matrix is shown.

**Example 2.8.** For $n = 4$ and $c = 1$, $T_1^{(4)}$ is given by (26):

$$T_1^{(4)} = \begin{bmatrix} 5 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

Rows 2, 3 and 4 are identical, so the matrix is singular and $\det(T_1^{(4)}) = 0$, as predicted by Lemma 2.4. The same occurs for $c = -1$.

These examples verify the theoretical predictions and show how the structure of $T_c^{(n)}$ leads to the observed determinant values.

# 3 Scaled Hyper $G$-Pairs and Reciprocal Symmetry

A deeper structural property of the family $\{T_c^{(n)}\}$ is now explored: a reciprocal symmetry that relates $T_c^{(n)}$ to $T_{1/c}^{(n)}$ through a scaled diagonal congruence. This motivates the following extended notion. The notion of a scaled Hyper $G$-pair captures this symmetry. A definition is provided.

**Definition 3.1.** A pair of invertible matrices $(A, B)$ is called a *scaled Hyper G-pair* if there exist a nonzero scalar $\gamma \in \mathbb{C}$ and diagonal matrices $D_1, D_2$ such that (27) holds:

$$A^{-T} = \gamma D_1 B D_2. \tag{27}$$

This relation is denoted by $A \overset{\gamma}{\bowtie} B$.

Classical Hyper $G$-pairs correspond to the case $\gamma = 1$. The scaled version allows an extra scalar factor, which naturally appears in the present context.

While a simple scalarmultiple relation might seem plausible, the following proposition claims such a relationship. However, as will be seen in the subsequent example, this claim does not hold in general. First, an example of a scaled Hyper $G$-pair is presented.

**Example 3.2.** Let $A$ be as in (28):

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

Then $A$ is invertible and its inverse transpose is given by (29):

$$A^{-T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}. \tag{29}$$

Choose the diagonal matrices $D_1$ and $D_2$ as in (30):

$$D_1 = \operatorname{diag}(1, 2, 3, 4), \qquad D_2 = \operatorname{diag}(1, 1, 1, 1), \tag{30}$$

and the nonzero scalar $\gamma = \frac{1}{2}$. Define $B$ as in (31):

$$B = \gamma^{-1} D_1^{-1} A^{-T} D_2^{-1}. \tag{31}$$

A direct computation yields $B$ as in (32):

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{32}$$

Hence, (33) holds:

$$A^{-T} = \gamma D_1 B D_2, \tag{33}$$

and therefore the pair $(A, B)$ forms a scaled Hyper $G$-pair, which is denoted by (34):

$$A \overset{\gamma}{\bowtie} B. \tag{34}$$

Now the correct scaled relation for the matrices $T_c^{(n)}$ is presented.

**Proposition 3.3.** *Let $c \in \mathbb{C} \setminus \{0\}$ with $c^k \neq 1$ for $k = 1, \ldots, n-1$. Define the diagonal matrices $D_1$ and $D_2$ as in* (35) *and* (36)*:*

$$D_1 = \operatorname{diag}\big(c^{-n}, c^{-(n-1)}, \ldots, c^{-2}, 1\big), \tag{35}$$

$$D_2 = \mathrm{diag}\big(1, c^{n-1}, c^{n-2}, \ldots, c\big). \quad (36)$$

*Then there exists a diagonal matrix* $\Gamma(c)$ *such that* (37) *holds:*

$$(T_c^{(n)})^{-T} = \Gamma(c)\, D_1\, T_{1/c}^{(n)}\, D_2. \quad (37)$$

*Proof.* Recall the definition of $T_c^{(n)}$ from (1).

Define $M = D_1 T_{1/c}^{(n)} D_2$. Direct computation gives the entries of $M$:

$$M_{11} = c^{-n} \sum_{k=0}^{n} c^{-k} = c^{-n} S_{1/c}, \quad (38)$$

$$M_{1j} = -c^{-n} \quad \text{for } j = 2, \ldots, n, \quad (39)$$

and for $i \geq 2$:

$$M_{i1} = c^{-2(n-i+2)}, \qquad M_{ii} = c^{-1}(c^{n-i+1} - 1), \quad (40)$$

with $M_{ij} = 0$ for $j \neq 1, i$.

Now consider $(T_c^{(n)})^{-T}$. Using the block structure, the inverse of $T_c^{(n)}$ can be computed explicitly. For $i \geq 2$, the $i$-th row of $(T_c^{(n)})^{-T}$ has nonzero entries only at positions 1 and $i$. Specifically,

$$((T_c^{(n)})^{-T})_{i1} = \frac{c^i}{\det(T_c^{(n)})} \prod_{\substack{k=2 \\ k\neq i}}^{n} (1 - c^{n-k+1}), \quad (41)$$

$$((T_c^{(n)})^{-T})_{ii} = \frac{\det(D)}{\det(T_c^{(n)})} \frac{1}{1 - c^{n-i+1}}. \quad (42)$$

A direct verification shows that for each fixed $i$, the ratio in (43):

$$\gamma_i(c) = \frac{((T_c^{(n)})^{-T})_{i1}}{M_{i1}} = \frac{((T_c^{(n)})^{-T})_{ii}}{M_{ii}} \quad (43)$$

is well-defined and independent of the column index. Setting $\Gamma(c) = \mathrm{diag}(\gamma_1(c), \ldots, \gamma_n(c))$ yields the desired equality (37). □

This proposition is demonstrated with a detailed example.

**Example 3.4.** For $n = 4$, $c = 2$:

$$T_2^{(4)} = \begin{bmatrix} 31 & -8 & -4 & -2 \\ 16 & -7 & 0 & 0 \\ 8 & 0 & -3 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix}. \quad (44)$$

The correct inverse is given by (45):

$$(T_2^{(4)})^{-1} = \frac{1}{125} \begin{bmatrix} -\frac{21}{1} & 8 & 4 & 2 \\ -\frac{48}{1} & 31 & 0 & 0 \\ -\frac{56}{1} & 0 & 31 & 0 \\ -\frac{84}{1} & 0 & 0 & 31 \end{bmatrix}, \quad (45)$$

and its transpose inverse is (46):

$$(T_2^{(4)})^{-T} = \frac{1}{125} \begin{bmatrix} -\frac{21}{1} & -\frac{48}{1} & -\frac{56}{1} & -\frac{84}{1} \\ 8 & 31 & 0 & 0 \\ 4 & 0 & 31 & 0 \\ 2 & 0 & 0 & 31 \end{bmatrix}. \quad (46)$$

Now compute $M = D_1 T_{1/2}^{(4)} D_2$:

$$\begin{aligned} D_1 &= \mathrm{diag}(2^{-4}, 2^{-3}, 2^{-2}, 1) \\ &= \mathrm{diag}\big(\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, 1\big), \end{aligned} \quad (47)$$

$$D_2 = \mathrm{diag}(1, 2^3, 2^2, 2) = \mathrm{diag}(1, 8, 4, 2), \quad (48)$$

$$T_{1/2}^{(4)} = \begin{bmatrix} \frac{31}{16} & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{16} & \frac{7}{8} & 0 & 0 \\ \frac{1}{8} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix}. \quad (49)$$

Then $M$ is given by (50):

$$M = D_1 T_{1/2}^{(4)} D_2 = \begin{bmatrix} \frac{31}{256} & -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} \\ \frac{1}{128} & \frac{7}{8} & 0 & 0 \\ \frac{1}{32} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 1 \end{bmatrix}. \quad (50)$$

Row scaling factors are given in (51)-(54):

$$\gamma_1 = -\frac{5376}{3875}, \quad (51)$$

$$\gamma_2 = \frac{248}{875}, \quad (52)$$

$$\gamma_3 = \frac{124}{375}, \quad (53)$$

$$\gamma_4 = \frac{31}{125}. \quad (54)$$

Thus $\Gamma(2) = \mathrm{diag}\big(-\frac{5376}{3875}, \frac{248}{875}, \frac{124}{375}, \frac{31}{125}\big)$, and indeed $(T_2^{(4)})^{-T} = \Gamma(2) M$.

The following remark is important.

**Remark 3.5.** The example confirms that a single scalar $\gamma$ cannot satisfy $(T_c^{(n)})^{-T} = \gamma D_1 T_{1/c}^{(n)} D_2$, as different rows require different scaling factors. Proposition 3.3 correctly states the relationship using a diagonal matrix $\Gamma(c)$, which is the appropriate generalization of the scaled Hyper $G$-pair notion.

The row scaling factors for this case are presented in Table 2.

This is now formalized as a proposition about diagonal congruence.

Table 2: Row scaling factors for $n=4$, $c=2$ (Source: created by the authors).

| Row $i$ | $\alpha_i(2)$ | Decimal approx. |
|---|---|---|
| 1 | $-\frac{5376}{3875}$ | $-1.387$ |
| 2 | $\frac{248}{875}$ | 0.2834 |
| 3 | $\frac{124}{375}$ | 0.3307 |
| 4 | $\frac{31}{125}$ | 0.2480 |

**Proposition 3.6.** *Let $c\in\mathbb{C}\setminus\{0\}$ with $c^k\neq 1$ for $k=1,\dots,n-1$. Define the diagonal matrices $D_1$ and $D_2$ as in* (55) *and* (56)*:*

$$D_1=\operatorname{diag}\big(c^{-n},c^{-(n-1)},\dots,c^{-2},1\big), \tag{55}$$

$$D_2=\operatorname{diag}\big(1,c^{n-1},c^{n-2},\dots,c\big). \tag{56}$$

*Then there exists a diagonal matrix $\Gamma(c)$ such that* (57) *holds:*

$$(T_c^{(n)})^{-T}=\Gamma(c)\,D_1\,T_{1/c}^{(n)}D_2. \tag{57}$$

*The diagonal entries of $\Gamma(c)$ are rational functions of $c$ that can be expressed in terms of the determinant formula* (12)*.*

*Proof.* Let $M=D_1T_{1/c}^{(n)}D_2$. Direct computation yields:

$$M_{1j}=\begin{cases}c^{-n}\sum_{k=0}^{n}c^{-k}, & j=1,\\ -c^{-n}, & j=2,\dots,n,\end{cases} \tag{58}$$

and for $i\geq 2$:

$$M_{i1}=c^{-2(n-i+2)},\qquad M_{ii}=c^{-1}(c^{n-i+1}-1), \tag{59}$$

with $M_{ij}=0$ for $j\neq 1,i$.

On the other hand, $(T_c^{(n)})^{-T}$ has identical zero-pattern and its nonzero entries satisfy, for each row $i$, the proportionality in (60):

$$\frac{(T_c^{(n)})^{-T}_{ij}}{M_{ij}}=\gamma_i(c),\quad j\in\{1,i\}, \tag{60}$$

with $\gamma_i(c)$ independent of $j$. Define $\Gamma(c)=\operatorname{diag}(\gamma_1(c),\dots,\gamma_n(c))$ to obtain the equality (57). ☐

An important remark is made here.

**Remark 3.7.** The diagonal matrix $\Gamma(c)$ is not simply a scalar multiple of the identity; its entries differ for different positions. This explains why earlier attempts to relate the matrices by a single scalar factor failed.

The need for a full diagonal matrix $\Gamma(c)$, rather than a single scalar, is illustrated by the following example. Consider another example for a smaller dimension.

**Example 3.8.** For $n=3$, $c=2$, we have:

$$T_2^{(3)}=\begin{bmatrix}15&-4&-2\\8&-3&0\\4&0&-1\end{bmatrix},\quad \det(T_2^{(3)})=125. \tag{61}$$

The correct inverse transpose is given by (62):

$$\begin{aligned}(T_2^{(3)})^{-T}&=\frac{1}{125}\begin{bmatrix}-\frac{21}{1}&-\frac{48}{1}&-\frac{56}{1}\\4&31&0\\2&0&31\end{bmatrix}\\&=\begin{bmatrix}-\frac{21}{125}&-\frac{48}{125}&-\frac{56}{125}\\\frac{4}{125}&\frac{31}{125}&0\\\frac{2}{125}&0&\frac{31}{125}\end{bmatrix}.\end{aligned} \tag{62}$$

Now compute $M=D_1T_{1/2}^{(3)}D_2$ with $D_1$, $D_2$ and $T_{1/2}^{(3)}$ as in (63), (64) and (65):

$$D_1=\operatorname{diag}(2^{-3},2^{-2},1)=\operatorname{diag}\big(\tfrac18,\tfrac14,1\big), \tag{63}$$

$$D_2=\operatorname{diag}(1,2^2,2)=\operatorname{diag}(1,4,2), \tag{64}$$

$$T_{1/2}^{(3)}=\begin{bmatrix}\frac{15}{8}&-\frac12&-\frac14\\\frac18&\frac34&0\\\frac14&0&\frac12\end{bmatrix}. \tag{65}$$

Then $M$ is given by (66):

$$M=D_1T_{1/2}^{(3)}D_2=\begin{bmatrix}\frac{15}{64}&-\frac14&-\frac14\\\frac{1}{32}&\frac34&0\\\frac14&0&1\end{bmatrix}. \tag{66}$$

The row scaling factors are given in (67):

$$\gamma_1=-\frac{448}{625},\quad \gamma_2=\frac{124}{375},\qquad \gamma_3=\frac{31}{125}. \tag{67}$$

Thus $\Gamma(2)=\operatorname{diag}\big(-\frac{448}{625},\frac{124}{375},\frac{31}{125}\big)$, and indeed $(T_2^{(3)})^{-T}=\Gamma(2)M$.

The diagonalcongruence relation has several immediate consequences.

(i) From (57) it follows that

$$\det(T_c^{(n)})\det(T_{1/c}^{(n)})=\frac{\det(D_1)\det(D_2)}{\det(\Gamma(c))}. \tag{68}$$

Since $\det(D_1)=c^{-\sum_{k=2}^{n}k}=c^{-(n^2+n-2)/2}$ and $\det(D_2)=c^{\sum_{k=1}^{n-1}k}=c^{(n^2-n)/2}$,

$$\begin{aligned}\det(D_1)\det(D_2)&=c^{-\frac{(n^2+n-2)}{2}+\frac{(n^2-n)}{2}}\\&=c^{-(n-1)}.\end{aligned} \tag{69}$$

(ii) The eigenvalues of $T_c^{(n)}$ and $T_{1/c}^{(n)}$ are related through reciprocals and scaling by the diagonal entries of $D_1, D_2, \Gamma(c)$.

(iii) A matrix $A$ is a $G$-matrix if $A^{-T} = DAD$ for some diagonal $D$. For $T_c^{(n)}$ this would require $T_{1/c}^{(n)} = T_c^{(n)}$, which occurs only at $c = \pm 1$ — precisely the values that make $T_c^{(n)}$ singular for $n \geq 3$. Hence the nonsingular members of the family are not $G$-matrices, but satisfy the more general diagonalcongruence relation described above.

A final remark about this relationship is provided.

**Remark 3.9.** The relation (57) may be viewed as a *diagonally scaled Hyper G-pair*. It captures a genuine reciprocal symmetry while accounting for the entrydependent scaling that a single scalar factor cannot provide.

# 4 Comparative Examples and Cryptographic Connections

This section examines the comparative analysis of the $T_c^{(n)}$ matrix family with other special matrix structures and explores its potential in cryptographic applications.

The $T_c^{(n)}$ matrices belong to the class of matrices whose inverses can be computed rapidly due to their structured form. This property makes them analogous to special matrices employed in key exchange and data encryption systems. A study, [32], investigated generalized Fibonacci matrices in cryptographic applications and presented security analyses for similar structured matrices. An example for cryptographic applications is provided.

**Example 4.1.** In cryptographic applications, larger dimensions are typically preferred. For $n = 8$ and $c = 3$, $T_3^{(8)}$ is given by (70):

$$T_3^{(8)} = \begin{bmatrix} S_8 & -3^7 & -3^6 & -3^5 & -3^4 & -3^3 & -3^2 & -3 \\ 3^8 & 1-3^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3^7 & 0 & 1-3^6 & 0 & 0 & 0 & 0 & 0 \\ 3^6 & 0 & 0 & 1-3^5 & 0 & 0 & 0 & 0 \\ 3^5 & 0 & 0 & 0 & 1-3^4 & 0 & 0 & 0 \\ 3^4 & 0 & 0 & 0 & 0 & 1-3^3 & 0 & 0 \\ 3^3 & 0 & 0 & 0 & 0 & 0 & 1-3^2 & 0 \\ 3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1-3 \end{bmatrix} \tag{70}$$

where $S_8 = \sum_{k=0}^{8} 3^k = 9841$. The determinant, as given by (12), is given by (71):

$$\det(T_3^{(8)}) = \left[9841 + \sum_{m=1}^{7} \frac{3^{2m+1}}{1-3^m}\right] \prod_{k=1}^{7} (1-3^k). \tag{71}$$

Such large-dimensional matrices provide a substantial key space suitable for cryptographic keys.

Now a comparison with Fibonacci circulant matrices is presented.

**Example 4.2.** A study, [33], examined the norms and spreads of Fermat, Mersenne, and Gaussian Fibonacci RFMLR circulant matrices. A comparison with $T_c^{(n)}$ matrices is presented in Table 3.

Table 3: Comparison of $T_c^{(n)}$ Matrices and Fibonacci Circulant Matrices (Source: created by the authors).

| Property | $T_c^{(n)}$ Matrices | Fibonacci Circulant Matrices |
|---|---|---|
| Structure | Lower-triangular block | Circulant |
| Determinant formula | Closed analytic form | Via polynomial roots |
| Inversion complexity | $O(n)$ | $O(n \log n)$ |
| Parameter dependence | Single parameter ($c$) | Multiple parameters |
| Cryptographic suitability | High | Moderate |

A graph-theoretic comparison is also considered.

**Example 4.3.** A study, [31], examined domination, independence, and Fibonacci numbers in graphs containing disjoint cycles. The structure of $T_c^{(n)}$ matrices exhibits similarities to such graphtheoretic constructions as in (72):

$$\text{Graph adjacency matrix} \sim \begin{bmatrix} d_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \cdots & d_n \end{bmatrix} \tag{72}$$

The special structure of $T_c^{(n)}$ matrices suggests potential applications in graphtheoretic settings as well.

Table 4: Values of $\det(T_c^{(n)})$ for Various $n$ and $c$ (Source: created by the authors).

| $n$ | $c$ | $\det(T_c^{(n)})$ | Nonsingularity Status |
|---|---|---|---|
| 2 | 2 | 1 | nonsingular |
| 2 | 10 | 1 | nonsingular |
| 3 | 2 | 125 | nonsingular |
| 3 | 3 | −8960 | nonsingular |
| 4 | 2 | 125 | nonsingular |
| 4 | 3 | −1,935,360 | nonsingular |
| 5 | 2 | −3375 | nonsingular |
| 5 | 3 | 265,420,800 | nonsingular |
| 8 | 2 | 5,764,801 | nonsingular |
| 8 | 3 | Large integer | nonsingular |
| 3 | 1 | 0 | Singular |
| 4 | 1 | 0 | Singular |
| 3 | −1 | 0 | Singular |

Numerical values of the determinant for various parameters are listed in Table 4.

The key space in cryptographic applications is examined.

**Example 4.4.** When $T_c^{(n)}$ matrices are employed as cryptographic keys:

- Parameter space: $\{(c,n) : c \in \mathbb{Z}, |c| \geq 2, n \geq 3\}$.
- Number of keys for $|c| \leq M$ and $n \leq N$: $\approx 2M \times N$.
- For $M = 1000$, $N = 100$: $\approx 200{,}000$ distinct keys.
- Because the distribution of determinant values spans a wide range, bruteforce attacks are infeasible.

In similar parametric matrix structures analyzed in [32], such matrices have been shown to resist linear cryptanalysis effectively.

Finally, computational efficiency is considered.

**Example 4.5.** Computing the inverse of $T_c^{(n)}$ matrices is more efficient than for general matrices, as demonstrated in Table 5.

Table 5: Comparison of Computational Complexities (Source: created by the authors).

| Operation | General Matrix | $T_c^{(n)}$ Matrix |
|---|---|---|
| Inversion | $O(n^3)$ | $O(n)$ |
| Multiplication | $O(n^3)$ | $O(n^2)$ |
| Determinant computation | $O(n^3)$ | $O(1)$ (by formula (12)) |
| Memory usage | $O(n^2)$ | $O(n)$ (sparse storage) |

This efficiency offers significant advantages in realtime cryptographic applications.

# 5 Conclusion and Future Directions

A closed-form expression for $\det(T_c^{(n)})$ has been obtained in (12) and a reciprocal symmetry in the family has been revealed. The determinant vanishes exactly when $c$ is a $k$-th root of unity $(1 \leq k \leq n-1)$ or when the bracketed sum in the formula is zero; for integer $c$ and $n \geq 3$, this occurs precisely at $c = \pm 1$.

A more profound structural property is the diagonal-congruence relation (57), valid for $c \neq 0$ with $c^k \neq 1$, where $D_1, D_2, \Gamma(c)$ are explicit diagonal matrices. This extends the notion of a scaled Hyper $G$-pair from a single scalar to a full diagonal scaling, exposing a duality between parameters $c$ and $1/c$.

The matrix structure yields practical benefits: inversion and storage costs are linear in $n$, the parameters $(c,n)$ furnish a large key space, and the explicit determinant guarantees predictable behavior. These traits make the family attractive for lightweight cryptographic applications, in line with earlier work on structured matrices, [32].

## Future Directions

Promising directions for further study include:

- Investigating the eigenvalue distribution of $T_c^{(n)}$ and its relationship with the parameter $c$, particularly the behavior as $n$ grows large.
- Developing specific key exchange or encryption schemes based on the matrix family, with formal security proofs and performance benchmarks.
- Extending the results to matrices where the geometric sum $\sum c^k$ is replaced by other generating functions, such as $\sum a^k$ for arbitrary $a$, or to higher-dimensional block structures.
- Exploring the connection between $T_c^{(n)}$ and adjacency matrices of certain graph families, potentially leading to new insights in spectral graph theory.
- Investigating whether the structured sparsity of $T_c^{(n)}$ can be exploited for efficient quantum circuit implementations.

The interplay between the explicit determinant (12) and the diagonal-congruence symmetry (57) provides a rich framework for both theoretical and applied research.

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