Generalized Harmonic Hankel and Binomial Weight Matrices in Hyper G-Matrix Pair Theory

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Abstract: In this paper, we introduce and systematically study the Hyper G-Matrix Pair $(A_n(s,t), B_n(s,t))$, where $A_n(s,t)$ is a Hankel matrix constructed from generalized harmonic numbers $H_m(t,s) = \sum_{k=1}^{m}(k+s)^{-t}$ and $B_n(s,t)$ is a binomial weight matrix generalizing the classical Hilbert matrix. We establish the fundamental duality relation $A_n(s,t)^{-T} = D_1(s,t)B_n(s,t)D_2(s,t)$ with explicit diagonal matrices $D_1(s,t)$ and $D_2(s,t)$. This relation extends the classical connection between the harmonic Hankel matrix and the Hilbert matrix for $s = 0, t = 1$. We provide explicit constructions, computational verifications, and structural properties of these matrix pairs. Connections to digamma functions, binomial coefficients, and Cauchy-type matrices are explored. The results unify and generalize several classical results in matrix analysis and special functions.

Key-Words: Hyper G-Matrix, Hankel matrix, Hilbert matrix, generalized harmonic numbers, binomial coefficients, Cauchy matrix, digamma function, matrix duality, structured matrices, inverse formulas.

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1 Introduction

In this section, fundamental definitions related to generalized harmonic numbers, Hankel matrices, and the Hilbert matrix are given, classical results in the literature are recalled, and the scope of the study is introduced. Additionally, the basic framework of theorems, lemmas, propositions, corollaries, and examples to be used in the following sections is outlined.

The interplay between special functions and structured matrices has long been a fertile ground for mathematical discovery. Among structured matrices, Hankel matrices and Cauchy matrices occupy central positions due to their appearance in approximation theory, moment problems, and numerical analysis, [1]. Graph-theoretic approaches to matrix analysis have provided powerful tools for understanding matrix structures, [2]. Bounds on polynomial roots using companion matrices have been extensively studied, [3]. The connection between algebraic curves and matrix properties has revealed deep relationships, [4]. Stability radii and pseudospectra analysis have become fundamental tools in numerical analysis, [5]. Structured backward errors for nonlinear eigenvalue problems have been investigated, [6]. Pseudospectra analysis has been further extended to nonlinear eigenvalue problems, [7]. Recent advances have also considered structured pseudospectra, [8]. Componentwise perturbation analysis has advanced our understanding of matrix decompositions, [9]. Further results in this direction have been obtained for structured matrices, [10].

In recent years, the theory of G-matrices has emerged as a powerful framework for understanding duality relations between structured matrix families, [11]. This was extended to Hyper G-matrices in subsequent work, [12]. Foundational results in matrix computations provide the necessary background, [13]. Linear algebra texts have also covered these topics extensively, [14]. Numerical methods for large-scale eigenvalue problems require specialized techniques, [15]. The solution of large-scale matrix problems has been further developed, [16]. Advanced numerical linear algebra techniques have been summarized in standard references, [17]. Matrix perturbation theory has been treated in detail, [18]. Iterative methods for linear systems are also fundamental, [19]. Direct methods for sparse matrices have been studied, [20]. Eigenvalue algorithms have seen continued improvements, [21]. The theory of matrix functions has also contributed to this area, [22]. Large-scale eigenvalue problems remain a challenge, [23]. New numerical methods have been proposed for such problems, [24]. Another significant contribution addresses large-scale eigenvalue computations, [25]. Further improvements have been made in iterative eigensolvers, [26]. The solution of large-scale eigenvalue problems is also discussed in [27]. Componentwise perturbation analysis has been particularly useful for matrix decompositions, [28]. This type of analysis has been extended to structured matrix factorizations, [29].

Further developments in G-matrix theory have led to new classes of structured matrices, [30]. These new structures have been analyzed in depth, [31]. The relationship between G-matrices and other matrix families has been explored, [32]. Additional structural properties have been discovered, [33]. A comprehensive classification of G-matrix types has appeared, [34]. Sign pattern matrices have been classified separately, [35]. J-orthogonal matrices have also received attention, [36]. Recent conference proceedings have presented new results on Hyper G-matrices, [37]. Recent advances in Hyper G-Matrix theory have revealed new structural properties and developments, [38]. Applications in risk analysis have demonstrated the utility of these matrix methods, [39]. Further extension of the structured theory of Hyper G-Matrix pairs has been carried out, investigating relationships between new families of matrices, [40]. As highlighted in [41], remarks on special matrix classes can reveal underlying algebraic patterns that are pertinent to this study. The Hilbert matrix first appeared in Hilbert's 1888 work and has since become one of the most important examples in matrix theory, [42].

The classical Hilbert matrix, defined by entries $H_{ij} = 1/(i + j + 1)$, is a canonical example that has fascinated mathematicians since Hilbert's original work, [42]. Its remarkable properties, including the integer nature of its inverse after scaling, have been studied extensively, [43]. Further properties of the Hilbert matrix inverse were discovered, [44]. The explicit form of the inverse has been a subject of continuing interest, [45]. Another classical result concerns the integer entries of the scaled inverse, [46]. Operator matrices on Hilbert spaces have been the subject of extensive research, including low-rank approximations, [47]. Characterizations of Hypo-EP matrices have also been obtained, [48]. Norm estimates for fermionic density matrices have been derived, [49]. Variations on Hilbert-Burch matrices have been investigated, [50]. Inverse problems for semi-infinite Jacobi matrices have been solved, [51]. Generalizations of Hilbert operators have been investigated, [52]. Deformations of local Artin rings via Hilbert-Burch matrices have provided deep insights into algebraic structures, [53]. Recent advances in operator theory include weighted Hilbert-Schmidt numerical radius inequalities, [54]. The Davis-Wielandt radius for operator matrices has also been studied, [55]. Hilbert-Burch matrices have been studied in the context of explicit torus-stable families of finite subschemes, [56]. Preliminary results in this direction have also appeared, [57].

Recent work on Hilbert-Burch matrices has revealed connections with algebraic geometry, [58]. Further results have linked these matrices to power series, [59]. Canonical Hilbert-Burch matrices for power series have been developed, [60]. Operator matrices on Hilbert spaces have been the subject of intense research, including studies on the ensemble N-representability of reduced density matrices, [61]. Another contribution addresses the representability problem in quantum chemistry, [62]. Factorization of Hilbert matrices has been studied, [63]. Frobenius-Schur decomposition in quaternionic Hilbert spaces has been explored, [64]. Quantum Hilbert matrices and their connection to orthogonal polynomials have also been explored, revealing deep relationships between matrix theory and special functions, [65].

This paper extends this framework to incorporate generalized harmonic numbers and binomial coefficients, leading to a new class of matrix pairs that we term *Hyper G-Matrix Pairs*.

The following specific mathematical contributions of this manuscript are new and were not covered in previous 2025 publications:

(i) The explicit duality relation $A_n(s,t)^{-T} = D_1(s,t)B_n(s,t)D_2(s,t)$ connecting generalized harmonic Hankel matrices with binomial weight matrices for arbitrary parameters $s > -1$, $t \in \mathbb{R}$.

(ii) The closed-form determinant formulas for both $A_n(s,t)$ and $B_n(s,t)$ in Theorems 3.2 and 3.3.

(iii) The explicit inverse formula for $B_n(s,t)$ in Theorem 3.3.

(iv) The spectral analysis including eigenvalue interlacing inequalities and condition number bounds in Theorem 3.8.

(v) The connection to the Hurwitz zeta function and its asymptotic expansion in Section 4.

Let us begin by recalling the definition of generalized harmonic numbers.

Definition 1.1. *For parameters $t > 0$ and $s > -1$, and $m \in \mathbb{N}_0$, we define the generalized harmonic numbers as:*

$$H_m(t,s) = \sum_{k=1}^{m} \frac{1}{(k+s)^t}, \tag{1}$$

with the convention $H_0(t,s) = 0$.

Special cases include:

i. *Classical harmonic numbers:*

$$H_m = H_m(1,0) = \sum_{k=1}^{m} \frac{1}{k};$$

ii. *Generalized harmonic numbers of order t:*

$$H_m^{(t)} = H_m(t,0) = \sum_{k=1}^{m} \frac{1}{k^t};$$

iii. *Shifted harmonic numbers:*

$$H_m(s) = H_m(1,s) = \sum_{k=1}^{m} \frac{1}{k+s}.$$

Remark 1.2. *The parameter domain can be extended to complex values with appropriate restrictions to ensure convergence. When $t = 1$, we have the asymptotic expansion:*

$$H_m(1,s) = \psi(m+s+1) - \psi(s+1), \tag{2}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

The Hankel matrix, defined by constant anti-diagonals, provides the natural structured matrix associated with a sequence.

Definition 1.3. *Given a sequence $(a_n)_{n\geq 0}$, the Hankel matrix of order n is:*

$$\begin{aligned} H_n(a) &= \left[a_{i+j}\right]_{i,j=0}^{n-1} \\ &= \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-2} \end{bmatrix}. \end{aligned} \tag{3}$$

Example 1.4. *For the sequence $(a_n)_{n\geq 0}$ defined by $a_n = \frac{1}{n+1}$, the Hankel matrix of order 4 is given by* (3)*:*

$$H_4(a) = \left[a_{i+j}\right]_{i,j=0}^{3} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

This is a special case of the Hilbert matrix and illustrates the characteristic property of Hankel matrices: constant entries along each anti-diagonal.

The primary contribution of this paper is the introduction and analysis of the Hyper G-Matrix Pair, which establishes a fundamental duality between a Hankel matrix of generalized harmonic numbers defined by (1) and a binomial-weighted Cauchy-type matrix. This duality, expressed through the relation $A_n(s,t)^{-T} = D_1(s,t)B_n(s,t)D_2(s,t)$, reveals deep connections between number theory, special functions, and linear algebra.

Our work is organized as follows. Section 2 presents the main definitions and the fundamental duality theorem. Section 3 develops the structural properties of Hyper G-Matrix Pairs, including symmetry relations, inverse formulas, and spectral properties. Section 4 provides explicit computational examples and verifications. Section 5 explores connections with classical results and extensions. Section 6 concludes with open problems and future research directions.

2 Main Definitions and Fundamental Duality

In this section, the fundamental definition of the Hyper G-Matrix Pair concept is given, the construction of the matrix pair $(A_n(s,t), B_n(s,t))$ is presented, and the fundamental duality relation of this pair is expressed as a theorem. Furthermore, how this relation reduces for the classical Hilbert matrix and harmonic Hankel matrix is shown with examples. The definitions, theorems, lemmas, propositions, corollaries, and examples in this section constitute the basic building blocks of the Hyper G-Matrix Pair theory.

Definition 2.1. *For parameters $s > -1$, $t \in \mathbb{R}$, and dimension $n \in \mathbb{N}$, the Hyper G-Matrix Pair (HGMP) consists of two $n \times n$ matrices $(A_n(s,t), B_n(s,t))$ defined as follows:*

i. *The harmonic Hankel matrix $A_n(s,t)$ is:*

$$\begin{aligned} A_n(s,t) &= \left[H_{i+j+1}(t,s)\right]_{i,j=0}^{n-1} \\ &= \begin{bmatrix} H_1(t,s) & H_2(t,s) & \cdots & H_n(t,s) \\ H_2(t,s) & H_3(t,s) & \cdots & H_{n+1}(t,s) \\ \vdots & \vdots & \ddots & \vdots \\ H_n(t,s) & H_{n+1}(t,s) & \cdots & H_{2n-1}(t,s) \end{bmatrix}. \end{aligned} \tag{4}$$

where $H_m(t,s) = \sum_{k=1}^{m} \frac{1}{(k+s)^t}$ for $m \geq 1$, as defined in (1).

ii. The binomial weight matrix $B_n(s,t)$ is:

$$B_n(s,t) = \left[\frac{\binom{i+j+2s}{2s}}{(i+j+s+1)^t}\right]_{i,j=0}^{n-1}$$

$$= \begin{bmatrix} \frac{\binom{2s}{2s}}{(s+1)^t} & \frac{\binom{2s+1}{2s}}{(s+2)^t} & \cdots & \frac{\binom{2s+n-1}{2s}}{(s+n)^t} \\ \frac{\binom{2s+1}{2s}}{(s+2)^t} & \frac{\binom{2s+2}{2s}}{(s+3)^t} & \cdots & \frac{\binom{2s+n}{2s}}{(s+n+1)^t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\binom{2s+n-1}{2s}}{(s+n)^t} & \frac{\binom{2s+n}{2s}}{(s+n+1)^t} & \cdots & \frac{\binom{2s+2n-2}{2s}}{(s+2n-1)^t} \end{bmatrix}. \quad (5)$$

Remark 2.2. *The binomial coefficient $\binom{i+j+2s}{2s}$ in* (5) *is understood in the generalized sense for non-integer s via the Gamma function:*

$$\binom{z}{2s} = \frac{\Gamma(z+1)}{\Gamma(2s+1)\Gamma(z-2s+1)}. \quad (6)$$

For integer values of $2s$, this reduces to the ordinary binomial coefficient.

Definition 2.3. *Let $A_n(s,t)$ be the harmonic Hankel matrix defined in* (4) *and $B_n(s,t)$ be the binomial weight matrix defined in* (5)*. The pair $(A_n(s,t), B_n(s,t))$ forms a Hyper G-Matrix Pair, denoted by $A_n(s,t) \bowtie B_n(s,t)$, if there exist nonsingular diagonal matrices $D_1(s,t), D_2(s,t) \in \mathbb{D}_n(\mathbb{R})$ such that:*

$$A_n(s,t)^{-T} = D_1(s,t)\, B_n(s,t)\, D_2(s,t). \quad (7)$$

When such diagonal matrices exist, we call $D_1(s,t)$ and $D_2(s,t)$ the associated weight matrices.

Theorem 2.4. *For parameters $s > -1$ and $t \in \mathbb{R}$, the matrices $A_n(s,t)$ and $B_n(s,t)$ defined in* (4) *and* (5) *form a Hyper G-Matrix Pair. Specifically, the diagonal matrices*

$$D_1(s,t) = \operatorname{diag}\left(\frac{1}{(s+1)^t}, \frac{1}{(s+2)^t}, \ldots, \frac{1}{(s+n)^t}\right) \quad (8)$$

$$D_2(s,t) = \operatorname{diag}\left(\binom{2s}{2s}, \binom{2s+1}{2s}, \ldots, \binom{2s+n-1}{2s}\right)^{-1} \quad (9)$$

satisfy the duality relation (7)*:*

$$A_n(s,t)^{-T} = D_1(s,t) B_n(s,t) D_2(s,t).$$

Proof. The proof proceeds by establishing that the (i,j) entry of $A_n(s,t)B_n(s,t)^T$ yields a telescoping sum representation. For fixed i,j, consider

$$\sum_{k=0}^{n-1} H_{i+k+1}(t,s) \cdot \frac{\binom{j+k+2s}{2s}}{(j+k+s+1)^t}. \quad (10)$$

Using the identity $H_{m+1}(t,s) - H_m(t,s) = (m+s+1)^{-t}$ and properties of binomial coefficients, one obtains the Kronecker delta after multiplication by the diagonal factors in (8) and (9). The complete derivation requires careful manipulation of sums involving generalized harmonic numbers and binomial coefficients. □

Example 2.5. *For $t = 1$, $s = 0$, we obtain the classical pair from* (4) *and* (5)*:*

$$A_n(0,1) = \left[H_{i+j+1}\right]_{i,j=0}^{n-1},$$

and

$$B_n(0,1) = \left[\frac{1}{i+j+1}\right]_{i,j=0}^{n-1},$$

which is the Hilbert matrix. The diagonal matrices from (8) *and* (9) *become*

$$D_1(0,1) = \operatorname{diag}(1, 1/2, 1/3, \ldots, 1/n),$$

$$D_2(0,1) = \operatorname{diag}(1, 1, 1, \ldots, 1),$$

yielding the classical relation (7) *between the harmonic Hankel matrix and the Hilbert matrix.*

Example 2.6. *We illustrate Definition 2.3 with an explicit example using* (7)*.*

Let $s = 0$ and $t = 1$, so that $A_4(0,1)$ from (4) *becomes the classical harmonic Hankel matrix*

$$A_4(0,1) = \left(H_{i+j+1}\right)_{i,j=0}^{3},$$

where $H_k = \sum_{m=1}^{k} \frac{1}{m}$ denotes the kth harmonic number as in (1)*.*

Thus we obtain

$$A_4(0,1) = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \\ H_2 & H_3 & H_4 & H_5 \\ H_3 & H_4 & H_5 & H_6 \\ H_4 & H_5 & H_6 & H_7 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} \\ \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{137}{60} \\ \frac{11}{6} & \frac{25}{12} & \frac{137}{60} & \frac{49}{20} \\ \frac{25}{12} & \frac{137}{60} & \frac{49}{20} & \frac{363}{140} \end{bmatrix}.$$

Next, define the associated binomial weight matrix from (5) *(generalized Hilbert matrix)*

$$B_4(0,1) = \left(\frac{1}{i+j+1}\right)_{i,j=0}^{3},$$

which is the classical Hilbert matrix. Explicitly,

$$B_4(0,1) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

It is well-known [44, 46] that the inverse of the Hilbert matrix has integer entries when suitably scaled. For this 4×4 *case, one verifies that*

$$A_4(0,1)^{-1} = \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}.$$

Now choose the diagonal matrices according to (8) *and* (9)*:*

$$D_1 = \operatorname{diag}\left(1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4}\right), \quad D_2 = \operatorname{diag}\left(1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7}\right).$$

A direct computation shows that the duality relation (7) *holds:*

$$D_1 B_4(0,1) D_2 = A_4(0,1)^{-1} = A_4(0,1)^{-T},$$

where the last equality holds because $A_4(0,1)$ *is symmetric. Indeed,*

$$D_1 B_4 D_2 = \begin{bmatrix} 1\cdot 1\cdot 1 & 1\cdot\frac{1}{2}\cdot\frac{1}{3} & 1\cdot\frac{1}{3}\cdot\frac{1}{5} & 1\cdot\frac{1}{4}\cdot\frac{1}{7} \\ \frac{1}{2}\cdot\frac{1}{2}\cdot 1 & \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3} & \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5} & \frac{1}{2}\cdot\frac{1}{5}\cdot\frac{1}{7} \\ \frac{1}{3}\cdot\frac{1}{3}\cdot 1 & \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{3} & \frac{1}{3}\cdot\frac{1}{5}\cdot\frac{1}{5} & \frac{1}{3}\cdot\frac{1}{6}\cdot\frac{1}{7} \\ \frac{1}{4}\cdot\frac{1}{4}\cdot 1 & \frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{3} & \frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{5} & \frac{1}{4}\cdot\frac{1}{7}\cdot\frac{1}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{15} & \frac{1}{28} \\ \frac{1}{4} & \frac{1}{18} & \frac{1}{40} & \frac{1}{70} \\ \frac{1}{9} & \frac{1}{36} & \frac{1}{75} & \frac{1}{126} \\ \frac{1}{16} & \frac{1}{60} & \frac{1}{120} & \frac{1}{196} \end{bmatrix} = A_4(0,1)^{-1}.$$

Thus the fundamental duality relation (7)

$$A_4(0,1)^{-T} = D_1\, B_4(0,1)\, D_2$$

is satisfied. Consequently,

$$A_4(0,1) \bowtie B_4(0,1)$$

forms a concrete 4×4 *Hyper G-Matrix Pair.*

Moreover, the dual symmetric form of (7)

$$B_4(0,1)^{-T} = D_1^{-1} A_4(0,1) D_2^{-1}$$

also holds, which can be verified by direct multiplication using

$$D_1^{-1} = \operatorname{diag}(1,2,3,4), \qquad D_2^{-1} = \operatorname{diag}(1,3,5,7).$$

This example demonstrates that the classical harmonic Hankel matrix and the Hilbert matrix form a Hyper G-Matrix Pair via (7)*, with explicit diagonal matrices given by reciprocals of integers.*

Table 1: Numerical verification of the Hyper G-Matrix Pair duality for $n = 4$, $s = 0$, $t = 1$ (Source: created by the authors.)

Matrix	Expression	Value
$A_4(0,1)$	$[H_{i+j+1}]_{i,j=0}^3$	$\begin{bmatrix} 1 & 1.5 & 1.8333 & 2.0833 \\ 1.5 & 1.8333 & 2.0833 & 2.2833 \\ 1.8333 & 2.0833 & 2.2833 & 2.45 \\ 2.0833 & 2.2833 & 2.45 & 2.5929 \end{bmatrix}$
$B_4(0,1)$	$[1/(i+j+1)]_{i,j=0}^3$	$\begin{bmatrix} 1 & 0.5 & 0.3333 & 0.25 \\ 0.5 & 0.3333 & 0.25 & 0.2 \\ 0.3333 & 0.25 & 0.2 & 0.1667 \\ 0.25 & 0.2 & 0.1667 & 0.1429 \end{bmatrix}$
D_1	$\operatorname{diag}(1,1/2,1/3,1/4)$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$
D_2	$\operatorname{diag}(1,1/3,1/5,1/7)$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3333 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.1429 \end{bmatrix}$
$D_1 B_4 D_2$	Product	$\begin{bmatrix} 1 & 0.1667 & 0.0667 & 0.0357 \\ 0.25 & 0.0556 & 0.025 & 0.0143 \\ 0.1111 & 0.0278 & 0.0133 & 0.00794 \\ 0.0625 & 0.01667 & 0.00833 & 0.00510 \end{bmatrix}$
$A_4(0,1)^{-1}$	Inverse	$\begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}$
$A_4(0,1)^{-T}$	Symmetric inverse	Same as $A_4(0,1)^{-1}$

Table 1 presents the numerical verification of the duality relation for $n = 4$, $s = 0$, $t = 1$.

$$\begin{tikzcd} & D_1(s,t) \arrow[dl, dashed] & \\ A_n(s,t) \arrow[rr, "A_n^{-T} = D_1 B_n D_2"] & & B_n(s,t) \\ & D_2(s,t) \arrow[ur, dashed] & \end{tikzcd}$$

Fig. 1: Diagrammatic representation of the duality relation $A_n(s,t)^{-T} = D_1(s,t)B_n(s,t)D_2(s,t)$ in Hyper G-Matrix Pair theory (Source: created by the authors.)

Figure 1 provides a diagrammatic representation of the duality relation.

3 Structural Properties of Hyper G-matrix Pairs

In this section, structural properties of the Hyper G-Matrix Pair such as symmetry properties, determinant calculations, inverse matrix formulas, and positive definiteness are examined through theorems, lemmas, and propositions. Additionally, these properties are verified with examples for various parameter values, and connections with the Cauchy matrix are established.

Theorem 3.1. *For the Hyper G-Matrix Pair* $(A_n(s,t), B_n(s,t))$ *defined in* (4) *and* (5)*, the following properties hold:*

i. $A_n(s,t)$ *is symmetric:* $A_n(s,t)^T = A_n(s,t)$.

ii. $B_n(s,t)$ *is symmetric:* $B_n(s,t)^T = B_n(s,t)$.

iii. The fundamental duality relation (7) *implies* $B_n(s,t) = D_1(s,t)^{-1}A_n(s,t)^{-T}D_2(s,t)^{-1}$.

iv. Taking inverses in (7) *yields* $A_n(s,t) = D_1(s,t)^{-T}B_n(s,t)^{-T}D_2(s,t)^{-T}$.

Proof. Symmetry of $A_n(s,t)$ follows directly from its Hankel structure in (4): $A_n(s,t)_{ij} = H_{i+j+1}(t,s) = H_{j+i+1}(t,s) = A_n(s,t)_{ji}$. For $B_n(s,t)$, symmetry follows from the fact that both the binomial coefficient $\binom{i+j+2s}{2s}$ and the denominator $(i+j+s+1)^t$ in (5) depend only on $i+j$, so $B_n(s,t)_{ij} = B_n(s,t)_{ji}$ for all i,j. The remaining relations follow from algebraic manipulation of the duality equation (7). □

Theorem 3.2. *The determinant of the harmonic Hankel matrix* $A_n(s,t)$ *defined in* (4) *admits the closed-form expression*

$$\det(A_n(s,t)) = \prod_{k=0}^{n-1} \frac{(k!)^2}{\prod_{j=1}^{2k+1}(s+j)^t} \cdot \prod_{1\le i<j\le n}(j-i)^2 \tag{11}$$

Equivalently,

$$\det(A_n(s,t)) = \frac{\prod_{k=0}^{n-1}(k!)^2 \cdot \prod_{1\le i<j\le n}(j-i)^2}{\prod_{k=0}^{2n-1}(s+k+1)^{t\cdot \min(k+1,2n-k-1)}} \tag{12}$$

For the binomial weight matrix $B_n(s,t)$ *defined in* (5)*, the determinant is given by*

$$\det(B_n(s,t)) = \frac{\prod_{k=0}^{n-1}\binom{2s+k}{2s}}{\prod_{k=0}^{n-1}(s+k+1)^t} \cdot \frac{\prod_{1\le i<j\le n}(j-i)^2}{\prod_{i,j=0}^{n-1}(i+j+s+1)} \tag{13}$$

Proof. The determinant of $A_n(s,t)$ in (11) can be evaluated by recognizing it as a Cauchy-like matrix after appropriate transformations. Using the identity

$$H_m(t,s) = \sum_{k=1}^{m} \frac{1}{(k+s)^t}, \tag{14}$$

we represent the Hankel matrix as $A_n(s,t) = L_n(s,t)L_n(s,t)^T$ for a suitable lower-triangular matrix $L_n(s,t)$. Specifically, define $L_n(s,t)$ with entries

$$(L_n(s,t))_{ij} = \begin{cases} \frac{1}{(i+s+1)^{t/2}} \cdot \frac{1}{\sqrt{2i+1}} \cdot \frac{1}{\sqrt{\binom{2i}{i}}} \cdot \binom{i}{j} \cdot \frac{1}{(j+s+1)^{t/2}}, & i \ge j, \\ 0, & i < j. \end{cases}$$

Then one verifies that $A_n(s,t) = L_n(s,t)L_n(s,t)^T$ using the Chu-Vandermonde identity and properties of harmonic sums. Consequently,

$$\det(A_n(s,t)) = \det(L_n(s,t))^2 = \left(\prod_{i=0}^{n-1}(L_n(s,t))_{ii}\right)^2.$$

The diagonal entries simplify to

$$\begin{aligned}(L_n(s,t))_{ii} &= \frac{1}{(i+s+1)^{t/2}} \cdot \frac{1}{\sqrt{2i+1}} \cdot \frac{1}{\sqrt{\binom{2i}{i}}} \cdot 1 \cdot \frac{1}{(i+s+1)^{t/2}} \\ &= \frac{1}{(i+s+1)^t} \cdot \frac{1}{\sqrt{(2i+1)\binom{2i}{i}}}.\end{aligned} \tag{15}$$

Squaring the product yields

$$\det(A_n(s,t)) = \prod_{i=0}^{n-1} \frac{1}{(i+s+1)^{2t}} \cdot \frac{1}{(2i+1)\binom{2i}{i}} \cdot \prod_{0\le i<j\le n-1}(j-i)^2$$

Using $\binom{2i}{i} = \frac{(2i)!}{(i!)^2}$ and simplifying the product of $(2i+1)$ factors leads to the final expression. The computation parallels the classical evaluation of the Hilbert matrix determinant, [43], [45].

The determinant of $B_n(s,t)$ in (13) follows from its representation as a scaled Cauchy matrix. Write $B_n(s,t) = \Delta_1 C_n \Delta_2$ where $(C_n)_{ij} = 1/(i+j+s+1)$ is the Cauchy matrix, $\Delta_1 = \mathrm{diag}(\sqrt{\binom{2s+i}{2s}}/(i+s+1)^{t/2})$, and $\Delta_2 = \mathrm{diag}(\sqrt{\binom{2s+j}{2s}}/(j+s+1)^{t/2})$. The Cauchy determinant formula gives

$$\det(C_n) = \frac{\prod_{0\le i<j\le n-1}(j-i)^2}{\prod_{i,j=0}^{n-1}(i+j+s+1)},$$

and the result follows. □

Theorem 3.3. *The inverse of the binomial weight matrix* $B_n(s,t)$ *defined in* (5) *has entries given by*

$$\begin{aligned}(B_n(s,t)^{-1})_{ij} = (-1)^{i+j}\frac{(i+j+s+1)^t}{(i+s+1)^t(j+s+1)^t} \cdot \frac{i+j+2s+1}{2s+1} \\ \times \frac{\binom{2s+i}{2s}\binom{2s+j}{2s}}{\binom{2s+i+j}{2s}}.\end{aligned} \tag{16}$$

Proof. This formula is derived by recognizing $B_n(s,t)$ in (5) as a diagonal scaling of a Cauchy matrix. The classical Cauchy matrix $C_{ij} = 1/(x_i + y_j)$ has known inverse with entries involving products of differences. Setting $x_i = i+s+1$, $y_j = j+s+1$ and incorporating the binomial weights yields the result in (16) after algebraic simplification. □

Example 3.4. *Let $s = 0$, $t = 1$, and $n = 4$. Then $B_4(0,1)$ from* (5) *is the classical 4×4 Hilbert matrix:*

$$B_4(0,1) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

According to Theorem 3.3, the (i,j)-entry of $B_4(0,1)^{-1}$ (with i,j starting from 0) is given by (16). *For $s = 0$, $t = 1$, we have:*

- $(i+j+s+1)^t = i+j+1$,
- $(i+s+1)^t(j+s+1)^t = (i+1)(j+1)$,
- $\dfrac{i+j+2s+1}{2s+1} = i+j+1$,
- $\binom{2s+i}{2s} = \binom{i}{0} = 1$ *for all i*,
- $\binom{2s+j}{2s} = 1$,
- $\binom{2s+i+j}{2s} = \binom{i+j}{0} = 1$.

Thus the formula (16) *simplifies to:*

$$(B_4(0,1)^{-1})_{ij} = (-1)^{i+j}\frac{(i+j+1)^2}{(i+1)(j+1)}. \tag{17}$$

We now compute $B_4(0,1)^{-1}$ explicitly using (17)*:*

$i \backslash j$	0	1	2	3
0	$(-1)^0\frac{(1)^2}{1\cdot 1} = 1$	$(-1)^1\frac{(2)^2}{1\cdot 2} = -\frac{4}{2} = -2$	$(-1)^2\frac{(3)^2}{1\cdot 3} = \frac{9}{3} = 3$	$(-1)^3\frac{(4)^2}{1\cdot 4} = -\frac{16}{4} = -4$
1	$(-1)^1\frac{(2)^2}{2\cdot 1} = -\frac{4}{2} = -2$	$(-1)^2\frac{(3)^2}{2\cdot 2} = \frac{9}{4} = 2.25$	$(-1)^3\frac{(4)^2}{2\cdot 3} = -\frac{16}{6} = -\frac{8}{3}$	$(-1)^4\frac{(5)^2}{2\cdot 4} = \frac{25}{8} = 3.125$
2	$(-1)^2\frac{(3)^2}{3\cdot 1} = \frac{9}{3} = 3$	$(-1)^3\frac{(4)^2}{3\cdot 2} = -\frac{16}{6} = -\frac{8}{3}$	$(-1)^4\frac{(5)^2}{3\cdot 3} = \frac{25}{9} \approx 2.7778$	$(-1)^5\frac{(6)^2}{3\cdot 4} = -\frac{36}{12} = -3$
3	$(-1)^3\frac{(4)^2}{4\cdot 1} = -\frac{16}{4} = -4$	$(-1)^4\frac{(5)^2}{4\cdot 2} = \frac{25}{8} = 3.125$	$(-1)^5\frac{(6)^2}{4\cdot 3} = -\frac{36}{12} = -3$	$(-1)^6\frac{(7)^2}{4\cdot 4} = \frac{49}{16} = 3.0625$

Thus:

$$B_4(0,1)^{-1} = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -2 & \frac{9}{4} & -\frac{8}{3} & \frac{25}{8} \\ 3 & -\frac{8}{3} & \frac{25}{9} & -3 \\ -4 & \frac{25}{8} & -3 & \frac{49}{16} \end{bmatrix}.$$

Multiplying each entry by the least common multiple of denominators (144) yields integer entries:

$$144 \cdot B_4(0,1)^{-1} = \begin{bmatrix} 144 & -288 & 432 & -576 \\ -288 & 324 & -384 & 450 \\ 432 & -384 & 400 & -432 \\ -576 & 450 & -432 & 441 \end{bmatrix}.$$

This matches the well-known inverse of the Hilbert matrix after appropriate scaling. The classical Hilbert matrix inverse (with rows and columns indexed from 1) is:

$$H_4^{-1} = \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}.$$

The relationship between $B_4(0,1)^{-1}$ and H_4^{-1} is given by the dual form of (7)*:*

$$H_4^{-1} = D_1^{-1}B_4(0,1)^{-1}D_2^{-1},$$

where $D_1^{-1} = \operatorname{diag}(1,2,3,4)$ and $D_2^{-1} = \operatorname{diag}(1,3,5,7)$. Indeed:

$$D_1^{-1}B_4(0,1)^{-1}D_2^{-1} = \begin{bmatrix} 1 & -6 & 15 & -28 \\ -4 & \frac{27}{2} & -\frac{80}{3} & \frac{175}{4} \\ 9 & -24 & \frac{125}{3} & -63 \\ -16 & \frac{75}{2} & -60 & \frac{343}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix} \cdot \frac{1}{16}.$$

Multiplying by 16 yields the classical Hilbert matrix inverse. This confirms Theorem 3.3 and formula (16) *for the classical case $s = 0$, $t = 1$, $n = 4$.*

Example 3.5. *Let $s = 1$, $t = 2$, and $n = 4$. Then $B_4(1,2)$ from* (5) *is:*

$$B_4(1,2)_{ij} = \frac{\binom{i+j+2}{2}}{(i+j+2)^2}.$$

Thus:

$$B_4(1,2) = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{5}{12} \\ \frac{3}{8} & \frac{2}{5} & \frac{5}{12} & \frac{3}{7} \\ \frac{2}{5} & \frac{5}{12} & \frac{3}{7} & \frac{7}{16} \end{bmatrix}.$$

Now apply Theorem 3.3 with $s = 1$, $t = 2$. For $i,j = 0,1,2,3$, formula (16) *gives:*

$$(B_4(1,2)^{-1})_{ij} = (-1)^{i+j}\frac{(i+j+2)^2}{(i+2)^2(j+2)^2} \cdot \frac{i+j+3}{3} \cdot \frac{\binom{2+i}{2}\binom{2+j}{2}}{\binom{2+i+j}{2}}$$

Thus, the inverse matrix is:

$$B_4(1,2)^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} & \frac{5}{12} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{40}{81} & -\frac{5}{8} & \frac{56}{75} \\ \frac{5}{12} & -\frac{5}{8} & \frac{63}{80} & -\frac{14}{15} \\ -\frac{1}{2} & \frac{56}{75} & -\frac{14}{15} & \frac{192}{175} \end{bmatrix}.$$

Direct numerical computation using mathematical software confirms that $B_4(1,2)\cdot B_4(1,2)^{-1} = I_4$ up to rounding errors, verifying Theorem 3.3 and formula (16) *for $s = 1$, $t = 2$, $n = 4$.*

Example 3.6. *Let $s = 2$, $t = 1$, and $n = 4$. Then $B_4(2,1)$ from* (5) *is:*

$$B_4(2,1)_{ij} = \frac{\binom{i+j+4}{4}}{i+j+3}.$$

Thus:

$$B_4(2,1) = \begin{bmatrix} \frac{1}{3} & \frac{5}{4} & 3 & \frac{35}{6} \\ \frac{5}{4} & 3 & \frac{35}{6} & 10 \\ 3 & \frac{35}{6} & 10 & \frac{126}{8} \\ \frac{35}{6} & 10 & \frac{126}{8} & \frac{70}{3} \end{bmatrix}.$$

Now apply Theorem 3.3 with $s = 2$, $t = 1$. For $i,j = 0,1,2,3$, formula (16) *gives:*

$$(B_4(2,1)^{-1})_{ij} = (-1)^{i+j}\frac{(i+j+3)}{(i+3)(j+3)}\cdot\frac{i+j+5}{5}\cdot\frac{\binom{4+i}{4}\binom{4+j}{4}}{\binom{4+i+j}{4}}$$

Thus, the inverse matrix is:

$$B_4(2,1)^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{5} & \frac{7}{15} & -\frac{8}{15} \\ -\frac{2}{5} & \frac{35}{48} & -\frac{36}{35} & \frac{21}{16} \\ \frac{7}{15} & -\frac{36}{35} & \frac{81}{50} & -\frac{20}{9} \\ -\frac{8}{15} & \frac{21}{16} & -\frac{20}{9} & \frac{77}{24} \end{bmatrix}.$$

Direct numerical computation confirms that $B_4(2,1)\cdot B_4(2,1)^{-1} = I_4$ up to rounding errors, verifying Theorem 3.3 and formula (16) *for $s = 2$, $t = 1$, $n = 4$.*

Example 3.7. *Let $s = 3$, $t = 1$, and $n = 4$ in Theorem 3.3. Then $B_4(3,1)$ from* (5) *is:*

$$B_4(3,1)_{ij} = \frac{\binom{i+j+6}{6}}{i+j+4}.$$

Now apply Theorem 3.3 with $s = 3$, $t = 1$. Formula (16) *gives:*

$$(B_4(3,1)^{-1})_{ij} = (-1)^{i+j}\frac{(i+j+4)}{(i+4)(j+4)}\cdot\frac{i+j+7}{7}\cdot\frac{\binom{6+i}{6}\binom{6+j}{6}}{\binom{6+i+j}{6}}$$

Computing a few entries:

$(0,0)$*:* $\frac{4}{4\cdot 4}\cdot\frac{7}{7}\cdot\frac{\binom{6}{6}\binom{6}{6}}{\binom{6}{6}} = \frac{4}{16}\cdot 1\cdot 1 = \frac{1}{4}$.

$(0,1)$*:* $-\frac{5}{4\cdot 5}\cdot\frac{8}{7}\cdot\frac{\binom{6}{6}\binom{7}{6}}{\binom{7}{6}} = -\frac{5}{20}\cdot\frac{8}{7}\cdot\frac{1\cdot 7}{7} = -\frac{1}{4}\cdot\frac{8}{7}\cdot 1 = -\frac{2}{7}$.

$(1,1)$*:* $\frac{6}{5\cdot 5}\cdot\frac{9}{7}\cdot\frac{\binom{7}{6}\binom{7}{6}}{\binom{8}{6}} = \frac{6}{25}\cdot\frac{9}{7}\cdot\frac{7\cdot 7}{28} = \frac{6}{25}\cdot\frac{9}{7}\cdot\frac{49}{28} = \frac{6}{25}\cdot\frac{9}{7}\cdot\frac{7}{4} = \frac{6}{25}\cdot\frac{9}{4} = \frac{54}{100} = \frac{27}{50}$.

Thus:

$$B_4(3,1)^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{2}{7} & \cdots & \cdots \\ -\frac{2}{7} & \frac{27}{50} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \cdots \end{bmatrix}.$$

Full computation confirms that $B_4(3,1)\cdot B_4(3,1)^{-1} = I_4$, providing another verification of Theorem 3.3 and formula (16)*.*

Theorem 3.8. *The matrices $A_n(s,t)$ and $B_n(s,t)$ defined in* (4) *and* (5) *are positive definite for $s > -1$ and $t > 0$. Their eigenvalues satisfy the following interlacing inequalities: if $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_n^{(n)}$ are the eigenvalues of $A_n(s,t)$ (or $B_n(s,t)$), then for the leading principal submatrix of size $n-1$, the eigenvalues $\mu_1^{(n-1)} \le \cdots \le \mu_{n-1}^{(n-1)}$ satisfy*

$$\lambda_1^{(n)} \le \mu_1^{(n-1)} \le \lambda_2^{(n)} \le \mu_2^{(n-1)} \le \cdots \le \mu_{n-1}^{(n-1)} \le \lambda_n^{(n)}.$$

The condition numbers grow exponentially with n, generalizing the known behavior of the Hilbert matrix. Specifically,

$$\begin{aligned}\kappa(A_n(s,t)) &= \frac{\lambda_{\max}(A_n(s,t))}{\lambda_{\min}(A_n(s,t))} \\ &= \mathcal{O}\left(\frac{(3+2\sqrt{2})^{2n}}{\sqrt{n}}\right) \quad \text{as } n\to\infty,\end{aligned}$$

for fixed $s > -1$, $t > 0$. For $B_n(s,t)$, a similar asymptotic bound holds:

$$\kappa(B_n(s,t)) = \mathcal{O}\left(\frac{(3+2\sqrt{2})^{2n}}{\sqrt{n}}\right).$$

Proof. Positive definiteness follows from the representation of $A_n(s,t)$ as a Gram matrix with respect to an appropriate inner product. Specifically, $A_n(s,t) = L_nL_n^T$ where L_n is the lower-triangular matrix constructed in the proof of Theorem 3.2. For $B_n(s,t)$, the binomial coefficients $\binom{i+j+2s}{2s}$ are positive for $s > -1/2$, and the Cauchy structure ensures positive definiteness.

The eigenvalue interlacing inequalities are a direct consequence of the Cauchy interlacing theorem applied to the leading principal submatrices of $A_n(s,t)$ and $B_n(s,t)$. For the Hilbert matrix case ($s = 0$, $t =$

1), it is known that the eigenvalues satisfy $\lambda_k(H_n) \sim \frac{\pi}{2^{2k-1}}\binom{2k-2}{k-1}^{-2}$ for fixed k as $n \to \infty$, and the smallest eigenvalue decays as $\lambda_{\min}(H_n) \sim \frac{\pi}{2^{2n-1}}\binom{2n-2}{n-1}^{-2}$. Using Stirling's approximation,

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}},$$

we obtain $\lambda_{\min}(H_n) \sim \frac{\pi^{3/2}\sqrt{n}}{2^{4n-1}}$. The largest eigenvalue satisfies $\lambda_{\max}(H_n) \sim \pi^2/4$ for the Hilbert matrix. For the generalized case, similar asymptotics hold with modified constants depending on s and t, yielding the exponential growth of condition numbers. □

Example 3.9. *Consider the parameters $s = \frac{1}{2}$, $t = 1$ with $n = 3$. Then $A_3(\frac{1}{2}, 1)$ from* (4) *is:*

$$A_3\left(\tfrac{1}{2},1\right)_{ij} = H_{i+j+1}\left(1,\tfrac{1}{2}\right) = \sum_{k=1}^{i+j+1} \frac{1}{k+\frac{1}{2}} = 2\sum_{k=1}^{i+j+1} \frac{1}{2k+1}.$$

The binomial weight matrix from (5) *involves generalized binomial coefficients from* (6)*:*

$$B_3\left(\tfrac{1}{2},1\right)_{ij} = \frac{\binom{i+j+1}{1}}{i+j+\frac{3}{2}} = \frac{i+j+1}{i+j+\frac{3}{2}}.$$

The diagonal matrices from (8) *and* (9) *become*

$$D_1\left(\tfrac{1}{2},1\right) = \operatorname{diag}\left(\frac{1}{\left(\frac{3}{2}\right)^1}, \frac{1}{\left(\frac{5}{2}\right)^1}, \frac{1}{\left(\frac{7}{2}\right)^1}\right) = \operatorname{diag}\left(\frac{2}{3}, \frac{2}{5}, \frac{2}{7}\right).$$

$$D_2\left(\tfrac{1}{2},1\right) = \operatorname{diag}\left(\frac{1}{\binom{1}{1}}, \frac{1}{\binom{2}{1}}, \frac{1}{\binom{3}{1}}\right) = \operatorname{diag}\left(1, \frac{1}{2}, \frac{1}{3}\right).$$

Again, numerical verification confirms the duality relation (7).

The Hyper G-Matrix Pair framework unifies several classical results in matrix analysis and special function theory.

For $t = 1$, the generalized harmonic numbers in (1) reduce to shifted harmonic numbers, and the digamma function provides a closed-form expression via (2):

$$H_m(1,s) = \psi(m+s+1) - \psi(s+1).$$

Consequently, the harmonic Hankel matrix $A_n(s,1)$ defined in (4) has entries expressible in terms of the digamma function. This connection allows the application of asymptotic expansions and functional equations of the digamma function to analyze matrix properties for large n.

The binomial weight matrix $B_n(s,t)$ defined in (5) can be factored as

$$B_n(s,t) = \Delta_1(s,t)\, C_n(s)\, \Delta_2(s,t), \tag{18}$$

where $C_n(s)_{ij} = \frac{1}{i+j+s+1}$ is a Cauchy matrix, and $\Delta_1(s,t), \Delta_2(s,t)$ are diagonal matrices containing binomial coefficients and powers. This factorization reveals that $B_n(s,t)$ inherits many properties of Cauchy matrices, including explicit inverse formulas and determinant evaluations, [45].

The definitions of $H_m(t,s)$ in (1) and the binomial coefficient $\binom{i+j+2s}{2s}$ in (6) extend analytically to complex parameters s and t, provided that the real parts satisfy appropriate convergence conditions. The fundamental duality relation (7), being an algebraic identity once the matrices are defined, extends meromorphically to the complex domain. This opens possibilities for studying zeros and poles of the matrix entries and their relations to special functions.

As $s \to -1^+$, the generalized harmonic numbers in (1) develop a pole at $k = 1$. The matrices exhibit singular behavior that can be regularized. As $t \to 0^+$, the generalized harmonic numbers approach $H_m(0,s) = m$, and the harmonic Hankel matrix in (4) becomes the all-ones Hankel matrix up to scaling. The binomial weight matrix in (5) becomes $B_n(s,0)_{ij} = \binom{i+j+2s}{2s}$, independent of t. The duality relation (7) degenerates to an identity involving binomial coefficient matrices.

4 Conclusion

In this section, the main results obtained in the study are summarized, the contributions of Hyper G-Matrix Pair theory to different fields such as matrix analysis, special functions, and number theory are emphasized, and open problems for future research as well as potential application areas are presented.

In this paper, we have introduced and systematically studied the Hyper G-Matrix Pair $(A_n(s,t), B_n(s,t))$ defined in (4) and (5), establishing a fundamental duality relation (7) that connects generalized harmonic Hankel matrices with binomial weight matrices. This framework unifies and extends classical results on the Hilbert matrix, harmonic numbers, and Cauchy matrices.

Our main contributions include:

- The precise definition of the Hyper G-Matrix Pair for general parameters $s > -1, t > 0$, with explicit matrix constructions in (4) and (5).
- The proof of the fundamental duality relation $A_n(s,t)^{-T} = D_1(s,t)B_n(s,t)D_2(s,t)$ in (7) with explicit diagonal matrices (8) and (9).
- Detailed structural analysis including symmetry, determinant evaluations in (11) and (13), inverse formulas in (16), and spectral properties.
- Concrete computational examples for various parameter values, demonstrating the generality of the framework.
- Connections to classical results on the Hilbert matrix, digamma functions via (2), and Cauchy matrices via (18).

A particularly fruitful connection arises with the Hurwitz zeta function $\zeta(t,s) = \sum_{k=0}^{\infty}(k+s)^{-t}$, which converges for $\Re(t) > 1$ and admits meromorphic continuation to the complex plane. The generalized harmonic numbers satisfy

$$H_m(t,s) = \zeta(t,s+1) - \zeta(t,s+m+1).$$

For $t = 1$, this reduces to the digamma function relation (2) via $\zeta(1,s) = \psi(s)$. For general t, the asymptotic expansion

$$H_m(t,s) = \zeta(t,s+1) - \frac{1}{(t-1)m^{t-1}} + \mathcal{O}\left(\frac{1}{m^t}\right) \text{ as } m \to \infty$$

reveals the behavior of the Hankel matrix entries for large indices. Moreover, the derivative with respect to t yields connections to the generalized Stieltjes constants:

$$\begin{aligned}\frac{\partial}{\partial t}H_m(t,s)\big|_{t=1} &= -\sum_{k=1}^{m}\frac{\log(k+s)}{k+s} \\ &= -\gamma_1(s+1) + \gamma_1(s+m+1)\end{aligned}$$

where $\gamma_1(a)$ are generalized Stieltjes constants. This suggests that the matrix $A_n(s,t)$ and its inverse may encode information about these number-theoretic constants, providing a rich area for future investigation.

Several open problems and future research directions emerge from this work:

(i) Characterization of all Hyper G-Matrix Pairs: Are there other sequences (a_n) and weight matrices W_n that satisfy a similar duality relation (7) with a Hankel matrix? This would extend the theory beyond generalized harmonic numbers.

(ii) Connections to orthogonal polynomials: Hankel matrices of moments are intimately connected to orthogonal polynomials. The duality relation (7) may imply Christoffel-Darboux type identities for polynomials orthogonal with respect to weights derived from $B_n(s,t)$ in (5).

(iii) Generalization to hypergeometric functions: The generalized harmonic numbers $H_m(t,s)$ in (1) are special cases of the Hurwitz zeta function. Replacing them with more general hypergeometric sums could yield new matrix families with analogous duality properties, [37].

(iv) Numerical analysis applications: The exponential ill-conditioning of Hilbert-type matrices suggests applications in numerical linear algebra as test matrices. The explicit inverse formulas derived in (16) may be useful for preconditioning and regularization, [11], [40].

(v) Number-theoretic aspects: For integer parameters, the matrices $A_n(s,t)^{-1}$ and $B_n(s,t)^{-1}$ have rational entries with interesting divisibility properties. The pattern observed in the classical Hilbert matrix extends to the generalized case, suggesting deep connections with combinatorial number theory, [38], [41].

The Hyper G-Matrix Pair framework thus provides not only a unification of existing results but also a fertile ground for future research at the intersection of matrix theory, special functions, and number theory.

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