Hidden Coexisting Firing Patterns in a Hindmarsh-Rose Neuron Model with the Simplest Memristor

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Abstract: - This work presents an improved three-dimensional Hindmarsh-Rose neuron model that takes into account the impact of electromagnetic induction. By employing magnetic flux to characterize this influence, the model demonstrates how electromagnetic induction produces membrane potential through a feedback memristive current. For this reason, the simplest memristor model, which has been reported in the literature, has been used. Interestingly the proposed neuron model does not possess equilibrium points and can exhibit hidden coexisting firing patterns. Numerical simulations have been conducted, unveiling the system's dynamics and confirming that the proposed neuron model exhibits hidden coexisting firing patterns.

Key-Words: - Chaos, Coexisting attractors, Firing patterns, Hidden attractors, Hindmarsh-Rose neuron model, Memristor.

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1 Introduction

The nervous system consists of countless biological neurons that act as the primary units for processing and integrating information, [1]. Hence, the dynamic characteristics of these neurons are necessary in defining the behavior of nervous systems, [2]. As a result, the modeling of biological neurons and the investigation of their dynamics have become key research areas, attracting substantial interest from researchers.

Numerous dynamical systems describing neuronal behavior have been documented in the literature, in order to characterize various types of biological neurons. These models are generally classified into two primary categories: discrete-time maps, [3], [4] and continuous-time neuron models, [5], [6], [7], [8], [9]. Both types of models are capable of accurately simulating various firing patterns as reactions to alterations in the electrophysiological environment. However, the effect of electromagnetic induction has recently garnered significant attention from researchers, as it can profoundly influence neurons' dynamics, [10], [11], as well as the behavior of neural networks [12].

The effect of electromagnetic induction occurs, when ions traverse the membrane of a neuron, generating ion channel currents alongside electromagnetic induction currents, which collectively influence the membrane potential. As a result, memristors especially flux-controlled are employed in well-known neuron models to illustrate.

The dynamic interplay between magnetic flux and the potential of the neuron's membrane, [13]. Therefore, flux-controlled memristors have been incorporated into various well-known neuron models, including the Izhikevich [14], FitzHugh-Nagumo [15], Hodgkin-Huxley [16], [16], [17], [18], as well as both three-dimensional (3D) [19], [20], [21] and two-dimensional (2D) [22], [23] Hindmarsh-Rose neuron models, to illustrate the effect of the electromagnetic induction. Furthermore, memristors enable advancements in neuromorphic computing, artificial neural networks [24], brain-machine interfaces [25], cognitive computing, biomedical signal processing [26], spiking neural networks, and low-power edge AI, enhancing efficiency, adaptability, and energy-conscious processing.

In this work, the Hindmarsh-Rose neuron model, a simplified neural model grounded in dynamic assumptions, has been adopted, [9]. This model employs the most basic flux-controlled memristor documented in the literature to simulate the phenomenon of electromagnetic induction. The system's numerical simulation results, by using bifurcation and continuation diagrams, as well as phase portraits and variables' time series, reveal interesting phenomena. As such phenomena, hidden coexisting firing patterns and routes to chaos have been revealed.

This paper is organized as follows. Section 2 presents the simple memristor, along with the proposed 3D neuron model. Section 3 provides the results of the numerical simulation, by using well-known tools from nonlinear theory, obtained by solving the proposed 3D neuron model. Finally, Section 4 presents the conclusions based on the simulation outcomes, along with some recommendations for future research.

2 Mathematical Description of the Proposed Neuron Model

A comprehensive mathematical explanation of the proposed 3D memristive neuron model is presented in this section.

2.1 The Memristor Model

To highlight the role of electromagnetic induction in the 2D Hindmarsh-Rose neuron model, the simplest flux-controlled memristor with a quadratic polynomial inductance function is used. The mathematical description of the proposed memristor model is provided as follows:

$$\begin{aligned} i_M &= W(\varphi)\upsilon_M = \varphi^2\upsilon_M \\ \frac{d\varphi}{dt} &= \upsilon_M \end{aligned} \tag{1}$$

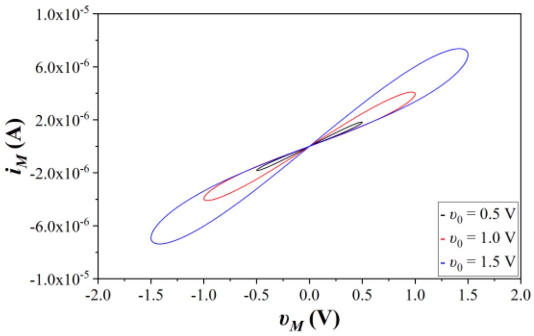
where i_M and υ_M represent the current and the voltage in the memristor respectively, while φ is the magnetic flux, which plays the role of the memristor's inner state variable. Furthermore $W(\varphi)$ is the quadratic memductance function.

By applying a sinusoidal voltage stimuli to the proposed memristor model, as defined in Eq. (1) the three key characteristics, which are crucial for identifying memristors [27], are depicted. Figure 1 shows the pinched hysteresis loops for different amplitude (Figure 1(a)) and frequency (Figure 1(b)) values of the sinusoidal voltage signals.

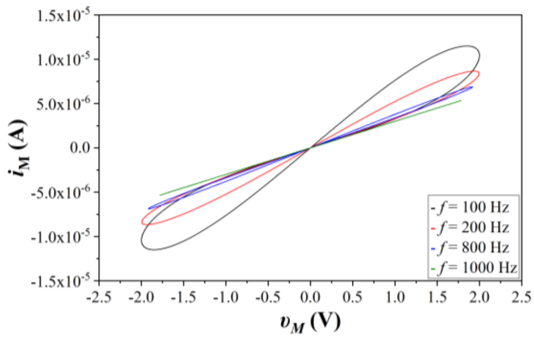
2.2 The Memristive Hindmarsh–Rose Neuron Model

To examine the impact of electromagnetic induction on a neuron, the aforementioned memristor model of Eq. (1) is incorporated into the well-known 2D Hindmarsh-Rose neuron model, [4]. Consequently, this results in a novel 3D memristive Hindmarsh-Rose neuron model, which is described mathematically as follows:

$$\begin{aligned} \frac{dx}{dt} &= y - ax^3 + bx^2 + I + k\varphi^2 x \\ \frac{dy}{dt} &= c - dx^2 - y \\ \frac{d\varphi}{dt} &= x \end{aligned} \tag{2}$$



(a)



(b)

Fig. 1: Pinched hysteresis loops of the memristor model of Eq. (1), driven by different sinusoidal voltage stimuli $\upsilon_M = V_m\sin(2\pi ft)$, by using (a) f = 100 Hz, with different values of voltage amplitudes V_m, and (b) V_m = 2 V, with different values of frequencies f

where x and y represent the membrane potential and the recovery variable respectively, while a, b, c, and d are four adjustable parameters in the original Hindmarsh-Rose model. Additionally, I stands for the externally applied stimulus, and k is the coupling factor of electromagnetic induction.

In general by setting the left sides of system (2) equal to 0, one can obtain the equilibrium points.

However, in the case of the system (2) there is no solution. Hence, the proposed system falls into the category of dynamical systems with hidden attractors, [28]. In biological systems, hidden attractors are crucial for understanding the complex, nonlinear dynamics of neuronal activity, including phenomena such as irregular spiking, bursting, and chaotic oscillations observed in the brain, [29]. The presence of hidden attractors in neuronal models indicates that brain activity is governed by intricate dynamical structures, which shape processes, such as cognition, perception, and motor control in ways that are not readily apparent through traditional equilibrium-based analyses.

3 Numerical Simulation Results

This section utilizes the externally applied current I as a bifurcation parameter, with numerical simulations carried out via the 4th-order Runge-Kutta algorithm, and with initial values $(x_0, y_0, \varphi_0) = (0, 0, 0.1)$. The remaining parameters of the system (2) are selected as $(a, b, c, d) = (1, 2, 1, 5)$, while the coupling factor k has taken the values of 0.02 and 0.03.

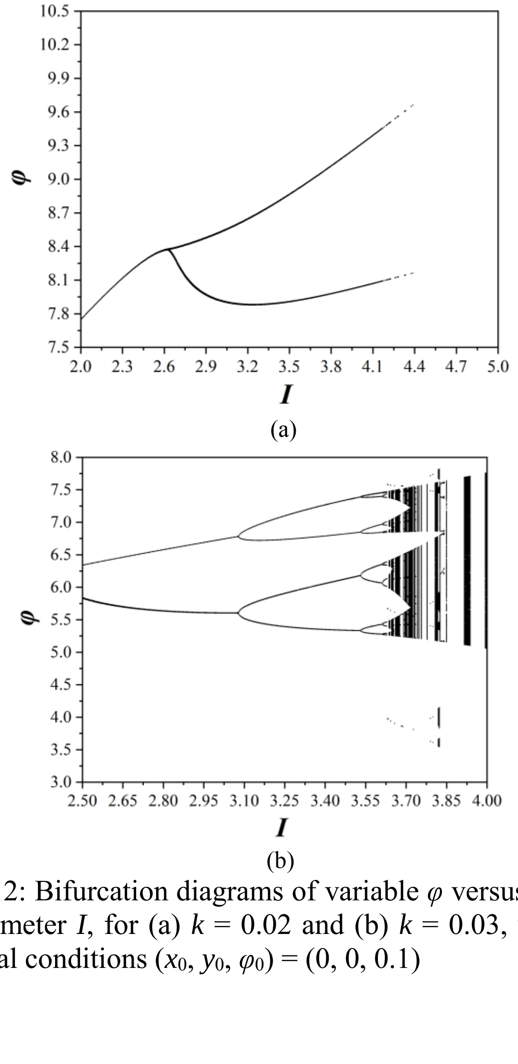


Fig. 2: Bifurcation diagrams of variable φ versus the parameter I, for (a) $k = 0.02$ and (b) $k = 0.03$, with initial conditions $(x_0, y_0, \varphi_0) = (0, 0, 0.1)$

Thus, Figure 2 displays the bifurcation diagrams of φ, for the previously mentioned values of the coupling factor k, as the parameter I increases with a small step. From Figure 2(a) the route from period-1 to period-2, as the parameter I increases, is observed. However, from the value of I equal to 4.4 the system diverges. In the second case of Figure 2(b), the system is driven to chaos through a period-doubling sequence. However, in this case, there are windows inside the chaotic region, in which the system also diverges. This behavior was the trigger for investigating the phenomenon of coexisting attractors. In this direction, the respective continuation diagrams (Figure 3) of the bifurcation diagrams of Figure 2 are produced, In both cases continuation diagrams.

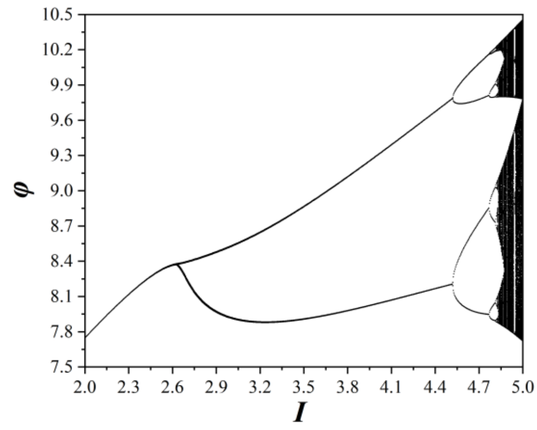
In Figure 3 the route to chaos through a period-doubling is revealed. Figure 4 (Appendix) depicts the time series of x, as well as the phase portraits in the y-φ plane, for four values of parameter I (with $k = 0.03$), extracted from Figure 3(b) in order to demonstrate the neuron's activation sequences emerging from the proposed 3D neuron model.

Also, in the ranges of values of I, where the system diverges, as demonstrated by the bifurcation diagrams of Figure 2, the system presents, according to the continuation diagrams other types of attractors (periodic and/or chaotic). In Figure 5 (Appendix), for $k = 0.03$ and $I = 3.79$, two coexisting attractors (periodic and chaotic), for different sets of initial conditions, are presented.

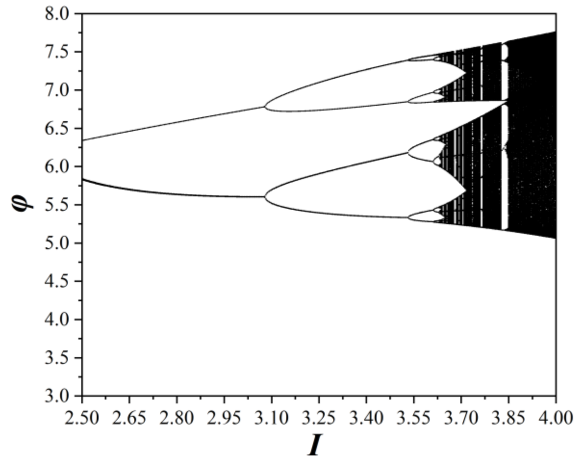
4 Conclusion

This research introduces a 3D memristive Hindmarsh-Rose dynamical system that integrates the influence of electromagnetic induction to thesystem. For this reason, the simplest ideal memristor model, with a memductance function based on a quadratic polynomial, was adopted. The improved neuron model presented interesting dynamical behavior. As concluded the improved neuron model lacked an equilibrium point and also it exhibited hidden coexisting firing patterns, which were investigated through a numerical simulation process, by using bifurcation and continuation diagrams.

In future works the implementation of an analog electronic circuit based on the op-amp approach, for emulating the proposed memristive neuron model, which was based on the Hindmarsh-Rose system, has been planned. Furthermore, other ideal or non-ideal memristors' models with various nonlinear memductance functions, in the Hindmarsh-Rose neuron model, will be used.



(a)



(b)

Fig. 3: Continuation diagrams of variable φ versus the parameter I, for (a) k = 0.02 and (b) k = 0.03, with starting initial conditions (x_0, y_0, φ_0) = (0, 0, 0.1)

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and they have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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APPENDIX

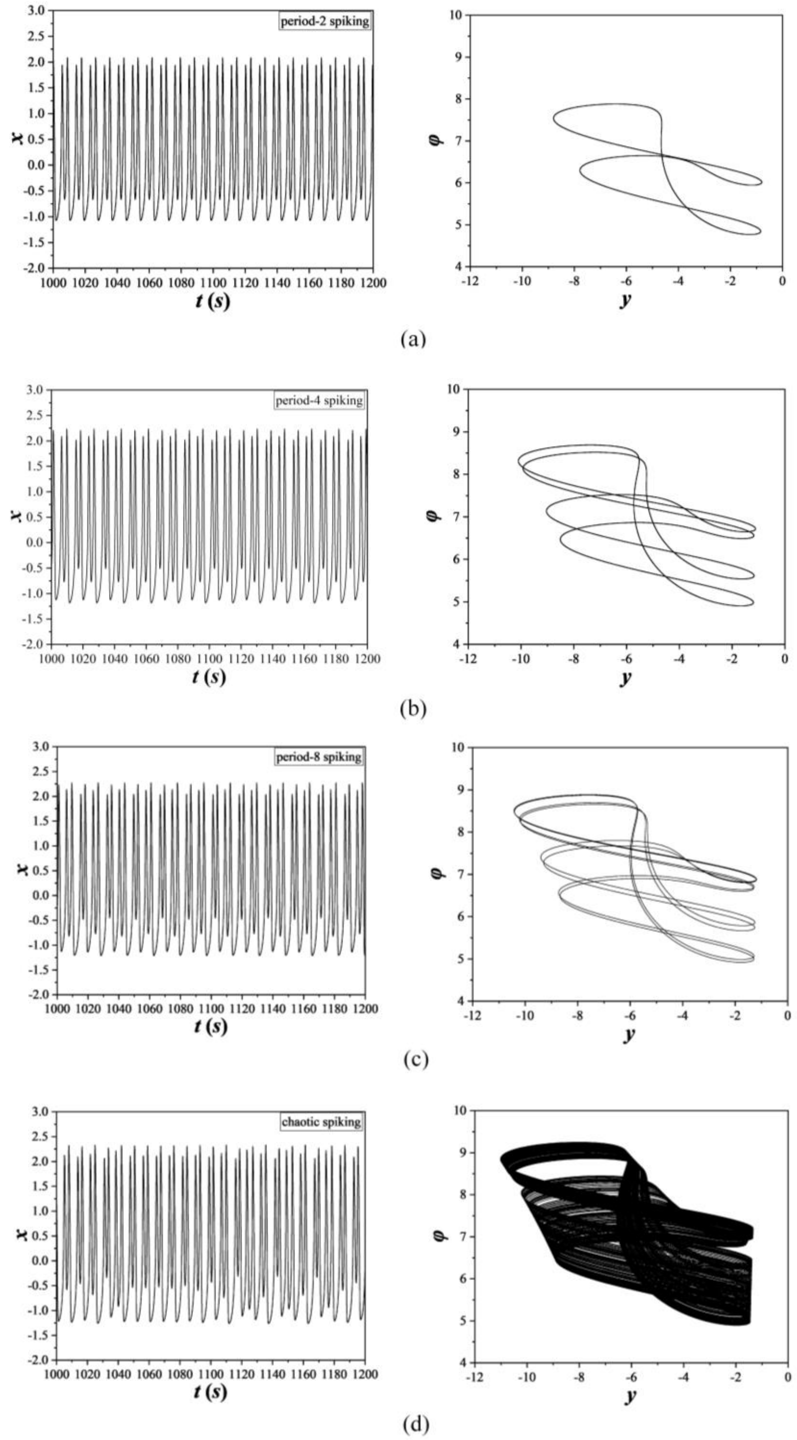

Fig. 4: Variable's x time series (left) and phase-portraits in the y-φ plane (right) with k = 0.03 for (a) I = 2.8 (period-2 spiking), (b) I = 3.4 (period-4 spiking), (c) I = 3.55 (period-8 spiking), and (d) I = 3.8 (chaotic spiking)

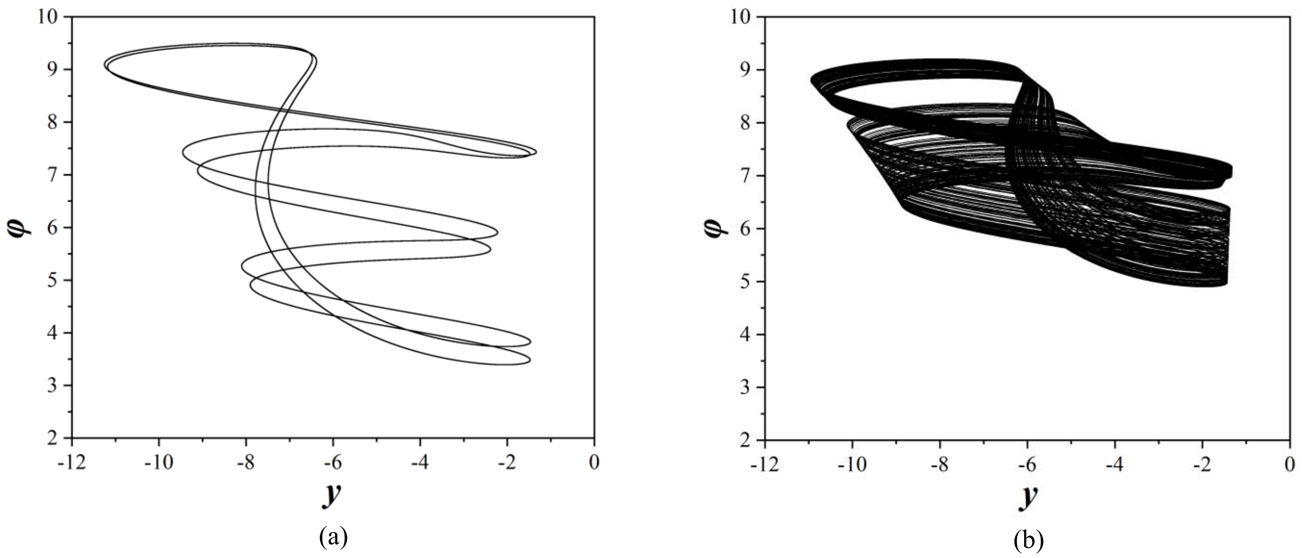

Fig. 5: Coexisting attractors (periodic and chaotic), with $k = 0.03$ and $I = 3.79$, for (a) $(x_0, y_0, \varphi_0) = (0, 0, 0.1)$, and (b) $(x_0, y_0, \varphi_0) = (0, 0, 0.5)$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Lazaros Laskaridis carried out the simulations and the optimization.
- Christos Volos has designed the memristive Hindmarsh-Rose neuron model.
- Nikolaos Bardis has organized the simulations of the proposed neuron model.

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No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.