All-to-All Broadcast in WDM Linear Array with 3-length extension

K. MANOHARAN[1], M. SABRIGIRIRAJ[2]
[1]Department of ECE, SNS College of Technology, Coimbatore, 641 035, Tamilnadu, INDIA
[2] Department of ECE, SVS College of Engineering, Coimbatore, 642 109, Tamilnadu, INDIA

Abstract: - All-to-all broadcast communication, distributing messages from each node to every other node, is a dense communication pattern and finds numerous applications in advanced computing and communication networks from the control plane to datacenters. In this article, a linear array is extended by directly linking all nodes which are separated by two intermediate nodes with additional fibers and this network is referred as linear array with 3-length extension. The wavelength allotment methods are proposed to realize all-to-all broadcast over WDM optical linear array with 3-length extension under multiple unicast routing model and the wavelength number needed atmost to establish all-to-all broadcast is determined. The wavelength number needed atmost to establish all-to-all broadcast in a linear array with 3-length extension is reduced by a minimum of 61% and a maximum of 66% when compared to a basic linear array. Similarly, the wavelength number needed atmost to establish all-to-all broadcast is reduced by a minimum of 24% and a maximum of 33% when compared to linear array with 2-length extension.

Key-Words: - All-to-All Broadcast, WDM Optical Network, Linear Array, Wavelength Assignment, RWA, Modified Linear Array

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1 Introduction

Wavelength Division Multiplexing (WDM) technology working over optical networks is proven to be a successful to cater the tremendous bandwidth demand of emerging high performance and computing applications. A WDM optical network employs numerous optical nodes and nodes are interconnected using optical fibers in some fashion. WDM technology permits the passage for multiple wavelength optical signals through the same fiber and thus provides abundant bandwidth. Each optical node employs required optical sources (Ex: laser diodes) at the transmitter section to modulate the input electrical signals and required optical detectors (Ex: photo diodes) at the receiver section to demodulate the received signal and extract the input signal that was fed at the transmitter. Though the same fiber can be used for signal transmission in both forward and reverse directions, it is normally assumed that each optical link is a set of two fibers, with one fiber dedicated to forward transmission and another one for reverse transmission. An optical connection (lightpath) (m, n) corresponds to the establishment of an optical path for transfer of a packet from source m to destination n on a unique wavelength. In the absence of wavelength converters at the intermediate optical nodes, each lightpath needs to be on the same wavelength from source to destination.

All-to-all broadcast communication, distributing messages from each node to every other node, is a dense communication pattern and finds numerous applications from network control plane to datacenters [1-3]. In general, all-to-all broadcast is employed for numerous applications in advanced computing and communication systems which employ WDM optical networks comprising hundreds of optical nodes at the backbone and involving huge number of operating wavelengths [4-12]. Wavelength need to be assigned for various lightpaths in such a way that no two lightpaths are established using the same wavelength, if they share any common link along entire route. Wavelengths being a scarce and costly resource, its use need to be restricted to reduce the complexity and cost of the network [13]. Linear array topology, due to its small node degree and regularity finds application in interconnection networks [14-19]. However, research on identifying new topologies with better properties for interconnection networks is an interesting and challenging area [20-21]. Researches have also been carried out in modifying/extending, few of the popular topologies namely linear array [22] and ring [20-21, 23] towards reducing the hop count, wavelength requirements and network survivability. On the same way, in this paper, linear array topology is extended in another fashion and all-to-all broadcast communication is investigated for the same and the reduction in wavelength requirements is identified. In addition, the results obtained in this paper will be useful to analyse practical long-haul backbone networks, as such networks may be partitioned into multiple extended linear array and/or linear array networks. Section 2 gives an overview of the basics needed to understand the investigation done in this paper. Wavelength number required atmost to establish all-to-all broadcast and its associated link load is obtained in section 3. Then, section 4 discusses about the significance of the results got from this paper. Finally, section 5 completes the paper highlighting future research avenues.

2 Preliminaries

Fig.1 shows a 16-node linear array network with 3-length extension. A linear array is extended by additionally linking two nodes which are separated by two intermediate nodes as in [22] and similar to that done for a ring network [20-21, 23]. This network is referred as linear array with 3-length extension. Here, in addition to immediate neighbouring node, every node is additionally linked to one another node on its right if feasible and one additional node on its left, if feasible. As a result, data can move from node x to node $x+1$ and node x to node $x+3$ if such nodes exist. This arrangement provides additional paths and reduces the hop count for various lightpaths and also wavelength number required to establish various lightpaths.

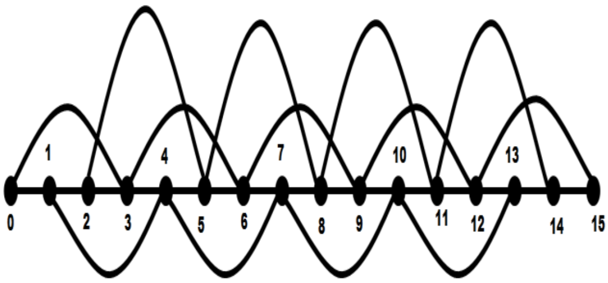

Fig. 1: A 16-node linear array with 3-length extension

The following concepts are essential to prove the main results of this work.

Definition 1: An optical connection (lightpath) (m, n) corresponds to the establishment of an optical path under a prescribed routing method for transfer of a packet from source m to destination n on a unique wavelength.

Definition 2: An l length connection is one, whose difference between the destination node index 'y' and the source node index 'x' is l. For example, in Fig. 1, the length of the connection that connects node 1 to node 4 is 3.

Definition 3: "A link that directly joins two adjacent nodes whose difference of index equals to unity is said to be a 'shorter link'. For example, in Fig. 1, the link that joins node 1 with node 2 is said to be a shorter link." [22]

Definition 4: "A link that directly joins two nodes whose difference of index equals to 3 is said to be a 'longer link'. For example, Fig. 1, the link that joins node 3 with node 6 is said to be a longer link." [22]

Definition 5: "A connection that selects the longest link among all the available links at the source node and at each of the intermediate nodes to reach the destination node is said to follow 'longest link first routing'. For example, in Fig. 1, under longest link first algorithm, a connection from node 1 to node 8 selects the links joining the nodes 1 with 4, then nodes 4 with 7 and nodes 7 with 8." [22]

Definition 6: For a connection (x, y), if $x < y$, then the connection is termed as rightward connection. Else, if $y < x$, then the connection is termed as leftward connection.

Example: An example of wavelength allotment for a 16-node linear array with 3-length extension under longest link first algorithm to establish all-to-all broadcast is described below.

Consider the 16-node linear array with 3-length extension shown in Fig. 1. As rightward connections and leftward connections do not share any fiber because they use different sets of fibers, they can be assigned same set of wavelengths. Hence, only all connections going in rightward direction of all-to-all broadcast are listed below:

$(0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (8,9),$
$(9,10), (10,11), (11,12), (12,13), (13,14), (14,15)$
$(0,2), (1,3), (2,4), (3,5), (4,6), (5,7), (6,8), (7,9),$
$(8,10), (9,11), (10,12), (11,13), (12,14), (13,15)$
$(0,3), (1,4), (2,5), (3,6), (4,7), (5,8), (6,9), (7,10),$
$(8,11), (9,12), (10,13), (11,14), (12,15)$
$(0,4), (1,5), (2,6), (3,7), (4,8), (5,9), (6,10),$
$(7,11), (8,12), (9,13), (10,14), (11,15)$
$(0,5), (1,6), (2,7), (3,8), (4,9), (5,10), (6,11),$
$(7,12), (8,13), (9,14), (10,15)$
$(0,6), (1,7), (2,8), (3,9), (4,10), (5,11), (6,12),$
$(7,13), (8,14), (9,15)$
$(0,7), (1,8), (2,9), (3,10), (4,11), (5,12), (6,13),$
$(7,14), (8,15)$
$(0,8), (1,9), (2,10), (3,11), (4,12), (5,13), (6,14),$
$(7,15)$
$(0,9), (1,10), (2,11), (3,12), (4,13), (5,14), (6,15)$

$(0,10),(1,11),(2,12),(3,13),(4,14),(5,15)$
$(0,11),(1,12),(2,13),(3,14),(4,15)$
$(0,12),(1,13),(2,14),(3,15)$
$(0,13),(1,14),(2,15)$
$(0,14),(1,15)$
$(0,15)$

The above list of connections is partitioned into multiple groups. All connections inside each group are nonoverlpping with each other and are allotted a common wavelength and is given below:

$$\left\{\begin{matrix}(0,1),(1,2),(2,3),(3,4),(4,5),(5,6),\\ (6,7),(7,8),(8,9),(9,10),(10,11),\\ (11,12),(12,13),(13,14),(14,15)\\ (0,3),(1,4),(2,5),(3,6),(4,7),(5,8),\\ (6,9),(7,10),(8,11),(9,12),(10,13),\\ (11,14),(12,15)\end{matrix}\right\} - \lambda_1$$

$$\left\{\begin{matrix}(0,2),(2,4),(4,6),(6,8),(8,10),(10,12),\\ (12,14),(0,6),(1,7),(2,8),(6,12),(7,13),\\ (8,14)\end{matrix}\right\} - \lambda_2$$

$$\left\{\begin{matrix}(1,3),(3,5),(5,7),(7,9),(9,11),(11,13),\\ (13,15)(3,9),(4,10),(5,11),(9,15)\end{matrix}\right\} - \lambda_3$$

$$\left\{\begin{matrix}(0,4),(1,5),(2,6),(3,7),(4,8),(5,9),\\ (6,10),(7,11),(8,12),(9,13),(10,14),\\ (11,15)\end{matrix}\right\} - \lambda_4$$

$\{(0,5),(2,7),(4,9),(6,11),(8,13),(10,15)\} - \lambda_5$
$\{(1,6),(3,8),(5,10),(7,12),(9,14)\} - \lambda_6$
$\{(0,7),(1,8),(2,9),(6,13),(7,14),(8,15)\} - \lambda_7$
$\{(3,10),(4,11),(5,12),\} - \lambda_8$
$\{(0,8),(2,10),(4,12),(6,14)\} - \lambda_9$
$\{(1,9),(3,11),(5,13),(7,15)\} - \lambda_{10}$
$\{(0,9),(1,10),(2,11)\} - \lambda_{11}$
$\{(3,12),(4,13),(5,14)\} - \lambda_{12}$
$\{(6,15)\} - \lambda_{13}$
$\{(0,10),(1,11),(2,12)\} - \lambda_{14}$
$\{(3,13),(4,14),(5,15)\} - \lambda_{15}$
$\{(0,11),(2,13),(4,15)\} - \lambda_{16}$
$\{(1,12),(3,14)\} - \lambda_{17}$
$\{(0,12),(1,13),(2,14)\} - \lambda_{18}$
$\{(3,15)\} - \lambda_{19}$
$\{(0,13),(1,14),(2,15)\} - \lambda_{20}$
$\{(0,14)\} - \lambda_{21}$
$\{(1,15)\} - \lambda_{22}$
$\{(0,15)\} - \lambda_{23}$

Thus, 23 wavelengths are needed atmost for a 16-node linear array with 3-length extension to establish all-to-all broadcast.

3 Main Results

Let the total number of nodes in a linear array with 3-length extension be represented by N.

Let m, l, s, t, u, v, j be positive integers.

Let the wavelength number wanted atmost under longest link first routing to establish all-to-all broadcast be represented by Ws.

Lemma 1a: Let $N = 6m$, where m is a positive integer. Then for each j, such that $3 \le j \le \frac{N}{2}$, the wavelength number wanted atmost to establish all connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if (j mod 3) = 0, $\frac{j-1}{3}$ if (j mod 3) = 1, $\frac{j+1}{3}$ if(j mod6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2.

Proof: First, all connections of length j in a network with N nodes are listed as follows: $(0,j),(1,j+1),(2,j+2),\ldots,(m,j+m)$where m is such that $j + m = N - 1$. Then, the number of connections of length j is $m + 1 = N - j$. Then the listed connections are partitioned into two or more groups such that no two connections in any group overlap with each other. It is shown below:

Case i) j mod 3 = 0

$$\left\{\begin{matrix}(0,j),(j,2j),(2j,3j),(3j,4j),\ldots,(z_0-j,z_0),\\ (1,j+1),(j+1,2j+1),(2j+1,3j+1),\\ (3j+1,4j+1),\ldots,(z_1-j,z_1),\\ (2,j+2),(j+2,2j+2),(2j+2,3j+2),\\ (3j+2,4j+2),\ldots,(z_2-j,z_2)\end{matrix}\right\}$$

$$\left\{\begin{matrix}(3,j+3),(j+3,2j+3),(2j+3,3j+3),\\ (3j+3,4j+3),\ldots,(z_3-j,z_3)\\ (4,j+4),(j+4,2j+4),(2j+4,3j+4),\\ (3j+4,4j+4),\ldots,(z_4-j,z_4),\\ (5,j+5),(j+5,2j+5),(2j+5,3j+5),\\ (3j+5,4j+5),\ldots,(z_5-j,z_5)\end{matrix}\right\}$$

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$$\left\{\begin{matrix}(j-3,2j-3),(2j-3,3j-3),(3j-3,4j-3),\\ (4j-3,5j-3),\ldots,\left(z_{j-3}-j,z_{j-3}\right),\\ (j-2,2j-2),(2j-2,3j-2),(3j-2,4j-2),\\ (4j-2,5j-2),\ldots,\left(z_{j-2}-j,z_{j-2}\right),\\ (j-1,2j-1),(2j-1,3j-1),(3j-1,4j-1),\\ (4j-1,5j-1),\ldots,\left(z_{j-1}-j,z_{j-1}\right)\end{matrix}\right\}$$

for $0 \le i \le j-1$, $z_i \in \{N-1, N-2, N-3,\ldots,N-j)\}$ and $z_a \ne z_b$ if $a \ne b$ where $a, b \in \{0,1,2,3,\ldots,j-1\}$

By assigning a unique wavelength to the connections of every group, $\frac{j}{3}$ wavelengths are wanted to establish all j length connections.

Case ii) j mod 3 = 1

$$\left\{\begin{array}{c}(0,j),(j-1,2j-1),(2j-2,3j-2),\\(3j-3,4j-3),\dots,(z_0-j,z_0),\\(1,j+1),(j,2j),(2j-1,3j-1),\\(3j-2,4j-2),\dots,(z_1-j,z_1),\\(2,j+2),(j+1,2j+1),(2j,3j),\\(3j-1,4j-1),\dots,(z_2-j,z_2),\end{array}\right\}$$

$$\left\{\begin{array}{c}(3,j+3),(j+2,2j+2),(2j+1,3j+1),\\(3j,4j),\dots,(z_3-j,z_3),\\(4,j+4),(j+3,2j+3),(2j+2,3j+2),\\(3j+1,4j+1),\dots,(z_4-j,z_4),\\(5,j+5),(j+4,2j+4),(2j+3,3j+3),\\(3j+2,4j+2),\dots,(z_5-j,z_5)\end{array}\right\}$$

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$$\left\{\begin{array}{c}(j-4,2j-4),(2j-5,3j-5),(3j-6,4j-6),\\(4j-7,5j-7),\dots,(z_{j-4}-j,z_{j-4}),\\(j-3,2j-3),(2j-4,3j-4),(3j-5,4j-5),\\(4j-6,5j-6),\dots,(z_{j-3}-j,z_{j-3}),\\(j-2,2j-2),(2j-3,3j-3),(3j-4,4j-4),\\(4j-5,5j-5),\dots,(z_{j-2}-j,z_{j-2})\end{array}\right\}$$

for $0 \le i \le j-2$, $z_i \in \{N-1, N-2, N-3, \dots, N-(j-1)\}$ and $z_a \ne z_b$ if $a \ne b$ where $a, b \in \{0,1,2,3,\dots,j-2\}$
By assigning a unique wavelength to the connections of every group, $\frac{j-1}{3}$ wavelengths are wanted to establish all j length connections.

Case iii) j mod 3 = 2

$$\left\{\begin{array}{c}(0,j),(j+1,2j+1),(2j+2,3j+2),\\(3j+3,4j+3),\dots,(z_0-j,z_0),\\(2,j+2),(j+3,2j+3),(2j+4,3j+4),\\(3j+5,4j+5),\dots,(z_2-j,z_2),\\(4,j+4),(j+5,2j+5),(2j+6,3j+6),\\(3j+7,4j+7),\dots,(z_4-j,z_4),\end{array}\right\}$$

$$\left\{\begin{array}{c}(1,j+1),(j+2,2j+2),(2j+3,3j+3),\\(3j+4,4j+4),\dots,(z_1-j,z_1),\\(3,j+3),(j+4,2j+4),(2j+5,3j+5),\\(3j+6,4j+6),\dots,(z_3-j,z_3),\\(5,j+5),(j+6,2j+6),(2j+7,3j+7),\\(3j+8,4j+8),\dots,(z_5-j,z_5),\end{array}\right\}$$

$$\left\{\begin{array}{c}(6,j+6),(j+7,2j+7),(2j+8,3j+8),\\(3j+9,4j+9),\dots,(z_6-j,z_6),\\(8,j+8),(j+9,2j+9),(2j+10,3j+10),\\(3j+11,4j+11),\dots,(z_8-j,z_8),\\(10,j+10),(j+11,2j+11),(2j+12,3j+12),\\(3j+13,4j+13),\dots,(z_{10}-j,z_{10}),\end{array}\right\}$$

$$\left\{\begin{array}{c}(7,j+7),(j+8,2j+8),(2j+9,3j+9),\\(3j+10,4j+10),\dots,(z_7-j,z_7),\\(9,j+9),(j+9,2j+9),(2j+11,3j+11),\\(3j+12,4j+12),\dots,(z_9-j,z_9),\\(11,j+11),(j+11,2j+10),(2j+13,3j+13),\\(3j+14,4j+14),\dots,(z_{11}-j,z_{11}),\end{array}\right\}$$

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$$\left\{\begin{array}{c}(j-5,2j-5),(2j-4,3j-4),(3j-3,4j-3),\\(4j-2,5j-2),\dots,(z_{j-5}-j,z_{j-5}),\\(j-3,2j-3),(2j-2,3j-2),(3j-1,4j-1),\\(4j,5j),\dots,(z_{j-3}-j,z_{j-3}),\\(j-1,2j-1),(2j,3),(3j+1,4j+1),\\(4j+2,5j+2),\dots,(z_{j-1}-j,z_{j-1}),\end{array}\right\}$$

$$\left\{\begin{array}{c}(j-4,2j-4),(2j-3,3j-3),(3j-2,4j-2),\\(4j-1,5j-1),\dots,(z_{j-4}-j,z_{j-4}),\\(j-2,2j-2),(2j-1,3j-1),(3j,4j),\\(4j+1,5j+1),\dots,(z_{j-2}-j,z_{j-2}),\\(j,2j),(2j+1,3j+1),(3j+2,4j+2),\\(4j+3,5j+3),\dots,(z_j-j,z_j),\end{array}\right\}$$

for $0 \le i \le j$, $z_i \in \{N-1, N-2, N-3, \dots, N-(j+1)\}$ and $z_a \ne z_b$ if $a \ne b$ where $a, b \in \{0,1,2,3,\dots,j\}$

By assigning a unique wavelength to the connections of every group $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2 wavelengths are wanted to establish all j length connections.

The following lemmas 1b through 1f, can be proved similar to lemma 1a.

Lemma 1b: Let $N = 6m + 1$, where m is a positive integer. Then for each j, such that $3 \leq j \leq \frac{N-1}{2}$, wavelength number wanted atmost to establish all connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if (j mod 3) = 0, $\frac{j-1}{3}$ if (j mod 3) = 1, $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2.

Lemma 1c: Let $N = 6m + 2$, where m is a positive integer. Then for each j, such that $3 \leq j \leq \frac{N}{2}$, wavelength number needed atmost to establish all connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if (j mod 3) = 0, $\frac{j-1}{3}$ if(j mod 3) = 1, $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2 under longest link first routing.

Lemma 1d: Let $N = 6m + 3$, where m is a positive integer. Then for each j, suchthat $3 \leq j \leq \frac{N-1}{2}$ wavelength number wanted atmost to establish connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if (j mod 3) = 0, $\frac{j-1}{3}$ if (j mod 3) = 1, $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2.

Lemma 1e: Let $N = 6m + 4$, where m is a positive integer. Then for each j, such that $3 \leq j \leq \frac{N}{2}$ wavelength number wanted atmost to establish connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if (j mod 3) = 0, $\frac{j-1}{3}$ if (j mod 3) = 1, $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if (j mod 6) = 2.

Lemma 1f: Let $N = 6m + 5$, where m is a positive integer. Then for each j, such that $3 \leq j \leq \frac{N-1}{2}$, wavelength number wanted atmost to establish all connections of length j, in a N node linear array with 3-length extension is $\frac{j}{3}$ if(j mod 3) = 0, $\frac{j-1}{3}$ if (j mod 3) = 1, $\frac{j+1}{3}$ if (j mod 6) = 5 and $\frac{j-2}{3}$ if(j mod 6) = 2.

Lemma 2: Let N be a positive integer, then for j =1 and j = 3, one wavelength is sufficient to establish all connections of length j =1 and j = 3 in a N node linear array with 3-length extension.

Proof: It is easy to observe that all connections of length 1 namely $(0,1),(1,2),(2,3),\ldots,(N-2,N-1)$ use only shorter link. Similarly, all connections of length 3, namely $(0,3),(1,4),(2,5),(3,6),(4,7),(5,8)\ \ldots\ ,\ (N-5,N-2)$, $(N-4,N-1)$ all use only longer links because of longest link first algorithm. It is also observed that all connections of length 1 uses only one shorter link and connections of length 3 uses only one longer link and hence they do not share any common link. So, one wavelength is sufficient to route all connections of length 1 and 3.

Lemma 3: Let N be a positive integer, then for $j = 2$ and $j = 6$, two wavelengths are sufficient to establish all connections of length $j = 2$ and $j = 6$ in a N node linear array with 3-length extension.

Proof: It is easy to observe that all connections of length 2 namely $(0,2),(1,3),(2,4),\ldots,(N-3,N-1)$ use two shorter links alone for all connections.Similarly all connections of length 6, namely $(0,6)(1,7)(2,8),\ldots,(N-8,N-2),(N-7,N-1)$ all uses two longer links alone for all connection in longest link first algorithm and hence they do not share any common link. So, two wavelengths are sufficient to route all connections of length 2 and 6.

Lemma 4a: Let $N = 6m$, where m is a positive integer. Then for each j, such that $1 \leq j \leq \frac{N}{2} - 1$, wavelength number wanted atmost to establish connections of length $\frac{N}{2} + j$, in a N node linear array with 3-length extension is $\frac{N-2j}{6}$ if$\left(\left(\frac{N}{2} + j\right) \bmod 3\right) = 0$, $\frac{N-2j+2}{6}$ if$\left(\left(\frac{N}{2} + j\right) \bmod 3\right) = 1$, $\frac{N-2j+4}{6}$ if$\left(\left(\frac{N}{2} + j\right) \bmod 3\right) = 2$.

Proof: First, all connections of length $\frac{N}{2} + j$ in a network with N nodes are listed as follows,$\left(0,\frac{N}{2} + j\right),\left(1,\frac{N}{2} + j + 1\right),\left(2,\frac{N}{2} + j + 2\right),\ldots,\left(m,\frac{N}{2} + j + m\right)$ where m is such that $\frac{N}{2} + j + m = N - 1$. Then, the number of connections of length $\frac{N}{2} + j$ is $\frac{N}{2} - j$. Then the listed connections are partitioned into two or more groups such that no two connections in any group overlap with each other. It is shown below:

Case i) $\left(\frac{N}{2} + j\right)$mod 3 = 0

$$\left\{\left(0,\frac{N}{2} + j\right),\left(1,\frac{N}{2} + j + 1\right),\left(2,\frac{N}{2} + j + 2\right)\right\}$$

$\left\{\left(3,\frac{N}{2}+j+3\right),\left(4,\frac{N}{2}+j+4\right),\left(5,\frac{N}{2}+j+5\right)\right\}$

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$$\left\{\begin{matrix}\left(m-2,\frac{N}{2}+j+m-2\right),\left(m-1,\frac{N}{2}+j+m-1\right),\\ \left(m,\frac{N}{2}+j+m\right)\end{matrix}\right\}$$

where $m=\frac{N}{2}-1-j$, for each sub collection, a unique wavelength can be associated. The number of connections of length $\frac{N}{2}+j$ is $m+1=\frac{N}{2}-j$. Therefore, atleast $\frac{\frac{N}{2}-j}{3}=\frac{N-2j}{6}$ wavelengths are needed to establish all connections of length $\frac{N}{2}+j$.

Case ii) $\left(\frac{N}{2}+j\right)$ mod 3 = 1

$\left\{\left(0,\frac{N}{2}+j\right),\left(1,\frac{N}{2}+j+1\right),\left(2,\frac{N}{2}+j+2\right)\right\}$

$\left\{\left(3,\frac{N}{2}+j+3\right),\left(4,\frac{N}{2}+j+4\right),\left(5,\frac{N}{2}+j+5\right)\right\}$

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$$\left\{\begin{matrix}\left(m-2,\frac{N}{2}+j+m-2\right),\\ \left(m-1,\frac{N}{2}+j+m-1\right),\left(m,\frac{N}{2}+j+m\right)\end{matrix}\right\}$$

where $m=\frac{N}{2}-1-j$, for each sub collection, a unique wavelength can be associated. The number of connections of length $\frac{N}{2}+j$ is $m+1=\frac{N}{2}-j$. Therefore, atleast $\frac{\frac{N}{2}-j-1}{3}=\frac{N-2j-2}{6}$ wavelengths are needed to establish all connections of length $\frac{N}{2}+j$.

Case iii) $\frac{N}{2}+j$ mod 3 = 2

$\left\{\left(0,\frac{N}{2}+j\right),\left(2,\frac{N}{2}+j+2\right),\left(4,\frac{N}{2}+j+4\right)\right\}$

$\left\{\left(1,\frac{N}{2}+j+1\right),\left(3,\frac{N}{2}+j+3\right),\left(5,\frac{N}{2}+j+5\right)\right\}$

$\left\{\left(6,\frac{N}{2}+j+6\right),\left(8,\frac{N}{2}+j+8\right),\left(10,\frac{N}{2}+j+10\right)\right\}$

$\left\{\left(7,\frac{N}{2}+j+7\right),\left(9,\frac{N}{2}+j+8\right),\left(11,\frac{N}{2}+j+11\right)\right\}$

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$$\left\{\begin{matrix}\left(m-5,\frac{N}{2}+j+m-5\right),\\ \left(m-3,\frac{N}{2}+j+m-3\right),\\ \left(m-1,\frac{N}{2}+j+m-1\right)\end{matrix}\right\}$$

$$\left\{\begin{matrix}\left(m-4,\frac{N}{2}+j+m-4\right),\\ \left(m-2,\frac{N}{2}+j+m-2\right),\\ \left(m,\frac{N}{2}+j+m\right)\end{matrix}\right\}$$

where $m=\frac{N}{2}-1-j$, for each sub collection, a unique wavelength can be associated. The total number of sub such of connections of length $\frac{N}{2}+j$ is $m+1=\frac{N}{2}-j$. Therefore, atleast $\frac{\frac{N}{2}-j+2}{3}=\frac{N+4-2j}{6}$ wavelengths are needed to establish all connections of length $\frac{N}{2}+j$.

The following lemmas 4b through 4f can be proved similar to lemma 4a.

Lemma 4b: Let $N=6m+1$, where m is a positive integer. Then for each j, such that $1\le j\le \frac{N-1}{2}$, wavelength number wanted atmost to establish connections of length $\frac{N-1}{2}+j$, in a N node linear array with 3-length extension is $\frac{N+1-2j}{6}$ if $\left(\left(\frac{N-1}{2}+j\right) \text{ mod } 3\right)=1$, $\frac{N+9-2j}{6}$ if $\left(\left(\frac{N-1}{2}+j\right) \text{ mod } 6\right)=2$, $\frac{N+3-2j}{6}$ if $\left(\left(\frac{N-1}{2}+j\right) \text{ mod } 6\right)=5$ and $\frac{N+5-2j}{6}$ if $\left(\left(\frac{N-1}{2}+j\right) \text{ mod } 3\right)=0$.

Lemma 4c: Let $N=6m+2$, where m is a positive integer. Then for each j, such that $1\le j\le \frac{N}{2}-1$, wavelength number wanted atmost to establish connections of length $\frac{N}{2}+j$, in a N node linear array with 3-length extension is $\frac{N-2j+2}{6}$ if $\left(\left(\frac{N}{2}+j\right) \text{ mod } 3\right)=0$, $\frac{N-2j+4}{6}$ if $\left(\left(\frac{N}{2}+j\right) \text{ mod } 3\right)=2$, $\frac{N-2j+6}{6}$ if $\left(\left(\frac{N}{2}+j\right) \text{mod } 6\right)=1$ and $\frac{N-2j}{6}$ if $\left(\left(\frac{N}{2}+j\right) \text{mod } 6\right)=4$.

Lemma 4d: Let $N=6m+3$, where m is a positive integer. Then for each j, such that $1\le j\le \frac{N-1}{2}$ wavelength number wanted atmost to establish connections of length $\frac{N-1}{2}+j$, in a N node linear array with 3-length extension is $\frac{N+3-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 3\right) = 0$, $\frac{N+5-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 3\right) = 1$, $\frac{N+1-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 3\right) = 2$.

Lemma 4e: Let $N = 6m+4$, where m is a positive integer. Then for each j, such that $1 \le j \le \frac{N}{2}-1$, wavelength number wanted atmost to establish connections of length $\frac{N}{2}+j$, in a N node linear array with 3-length extension is $\frac{N+4-2j}{6}$ if$\left(\left(\frac{N}{2}+j\right) \bmod 3\right) = 1$, $\frac{N-2j}{6}$ if $\left(\left(\frac{N}{2}+j\right) \bmod 3\right) = 2$, $\frac{N-2j+2}{6}$ if$\left(\left(\frac{N}{2}+j\right) \bmod 6\right) = 0$ and $\frac{N-2j+8}{6}$ if $\left(\left(\frac{N}{2}+j\right) \bmod 6\right) = 3$.

Lemma 4f: Let $N = 6m+5$, where m is a positive integer. Then for each j, such that $1 \le j \le \frac{N-1}{2}$, wavelength number wanted atmost to establish connections of length $\frac{N-1}{2}+j$, in a N node linear array with 3-length extension is $\frac{N+3-2j}{6}$if$\left(\left(\frac{N-1}{2}+j\right) \bmod 3\right) = 1$, $\frac{N+5-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 3\right) = 2$, $\frac{N+7-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 6\right) = 3$ and $\frac{N+1-2j}{6}$ if$\left(\left(\frac{N-1}{2}+j\right) \bmod 6\right) = 0$.

Theorem 1: If $N = 12m$, then $W_s = \frac{N^2+N}{12}$.
Proof: By Lemma 1a, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N}{2}-2} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}-1} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N}{2}-4} \frac{j-2}{3}$$

Substitute $s = \frac{j}{3}$, $t = \frac{j-1}{3}$, $u = \frac{j+1}{6}$ and $v = \frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively,

$$= \sum_{s=1}^{\frac{N}{6}} s + \sum_{t=1}^{\frac{N-6}{6}} t + \sum_{u=1}^{\frac{N}{12}} 2u + \sum_{v=1}^{\frac{N-12}{12}} 2v$$

$$= \frac{N(N+6)}{72} + \frac{(N-6)N}{72} + \frac{2N(N+12)}{288} + 2\frac{(N-12)N}{288}$$

$$= \frac{N^2}{36} + \frac{N^2}{72} = \frac{3N^2}{72}$$

$$= \frac{N^2}{24} \quad (1)$$

By Lemma 4a, the wavelength number wanted to establish all connections of length $\frac{N}{2}+j$, where $1 \le j \le \frac{N}{2}-1$ is

$$\sum_{j=3,6,9\ldots,}^{\frac{N}{2}-3} \frac{N-2j}{6} + \sum_{j=1,4,7,\ldots,}^{\frac{N}{2}-2} \frac{N-2j+2}{6} + \sum_{j=2,5,8\ldots,}^{\frac{N}{2}-1} \frac{N-2j+4}{6}$$

Substitute $s = \frac{j}{3}$, $t = \frac{j+2}{3}$ and $u = \frac{j+1}{3}$ in the first term, second term and third term respectively,

$$= \sum_{s=1}^{\frac{N-6}{6}} \frac{N-6s}{6} + \sum_{t=1}^{\frac{N}{6}} \frac{N+6-6t}{6} + \sum_{u=1}^{\frac{N}{6}} \frac{N+6-6u}{6}$$

$$= \left\{\left(\frac{(N-6)(N)}{36}\right) - \left(\frac{(N-6)(N)}{72}\right)\right\} + \left\{\left(\frac{(N+6)(N)}{36}\right) - \left(\frac{(N)(N+6)}{72}\right)\right\} + \left\{\left(\frac{N(N+6)}{36}\right) - \left(\frac{(N)(N+6)}{72}\right)\right\}$$

$$= \left(\frac{N^2-6N}{72}\right) + \left(\frac{N^2+6N}{72}\right) + \left(\frac{N^2+6N}{72}\right)$$

$$= \frac{N^2+2N}{24} \quad (2)$$

By combining (1) and (2),

$$W_s = \frac{N^2}{24} + \frac{N^2+2N}{24}$$

$$W_s = \frac{N^2+N}{12}$$

Theorem 2: If $N = 12m+1$, then $W_s = \frac{N^2+2N-3}{12}$.
Proof: By Lemma 1b, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-1}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N-5}{2}} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N-3}{2}} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N-9}{2}} \frac{j-2}{3}$$

Substitute $s = \frac{j}{3},\ t = \frac{j-1}{3},\ u = \frac{j+1}{6}$ and $v = \frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-1}{6}} s + \sum_{t=1}^{\frac{N-7}{6}} t + \sum_{u=1}^{\frac{N-1}{12}} 2\mathrm{u} + \sum_{v=1}^{\frac{N-13}{12}} 2\mathrm{v}$$

$$= \frac{(N-1)(N+5)}{72} + \frac{(N-7)(N-1)}{72} + \frac{2(N-1)(N+11)}{288} + \frac{2(N-13)(N-1)}{288}$$

$$= \frac{2N^2-4N+2}{72} + \frac{N^2-2N+1}{72}$$

$$= \frac{N^2-2N+1}{24} \qquad (3)$$

By Lemma 4b, the wavelength number wanted to establish all connections of length $\frac{N-1}{2} + j$, where $1 \le j \le \frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-5}{2}} \frac{N-2j+1}{6} + \sum_{j=2,8,14\ldots,}^{\frac{N-9}{2}} \frac{N-2j+9}{6} + \sum_{j=5,11,17\ldots,}^{\frac{N-3}{2}} \frac{N-2j+3}{6} + \sum_{j=3,6,9\ldots,}^{\frac{N-1}{2}} \frac{N-2j+5}{6}$$

Substitute $s = \frac{j+2}{3},\ t = \frac{j+4}{6},\ u = \frac{j+1}{6}$ and $v = \frac{j}{3}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-1}{6}} \frac{N+5-6s}{6} + \sum_{t=1}^{\frac{N-1}{12}} \frac{N+17-12t}{6} + \sum_{u=1}^{\frac{N-1}{12}} \frac{N+5-12u}{6} + \sum_{v=1}^{\frac{N-1}{6}} \frac{N+5-6v}{6}$$

$$= \left\{\left(\frac{(N+5)(N-1)}{36}\right) - \left(\frac{(N-1)(N+5)}{72}\right)\right\} + \left\{\left(\frac{(N+17)(N-1)}{72}\right) - \left(\frac{2(N-1)(N+11)}{288}\right)\right\} + \left\{\left(\frac{(N+5)(N-1)}{72}\right) - \left(\frac{2(N-1)(N+11)}{288}\right)\right\} + \left\{\left(\frac{(N+5)(N-1)}{36}\right) - \left(\frac{(N-1)(N+5)}{72}\right)\right\}$$

$$= \left(\frac{N^2+4N-5}{72}\right) + \left(\frac{N^2+22N-23}{144}\right) + \left(\frac{N^2-2N+1}{144}\right) + \left(\frac{N^2+4N-5}{72}\right)$$

$$= \frac{N^2+6N-7}{24} \qquad (4)$$

By combining (3) and (4),

$$W_s = \frac{N^2-2N+1}{24} + \frac{N^2+6N-7}{24}$$

$$W_s = \frac{N^2+2N-3}{12}$$

Theorem 3: If $N = 12m + 2$, then $W_s = \frac{N^2+2N-8}{12}$.

Proof: By Lemma 1c, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}-1} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N}{2}} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}-2} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N}{2}-5} \frac{j-2}{3}$$

Substitute $s = \frac{j}{3}$, $t = \frac{j-1}{3}$, $u = \frac{j+1}{6}$ and $v = \frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-2}{6}} s + \sum_{t=1}^{\frac{N-2}{6}} t + \sum_{u=1}^{\frac{N-2}{12}} 2\mathrm{u} + \sum_{v=1}^{\frac{N-14}{12}} 2\mathrm{v}$$

$$= \frac{(N-2)(N+4)}{72} + \frac{(N-2)(N+4)}{72} + \frac{2(N-2)(N+10)}{288} + \frac{2(N-14)(N-2)}{288}$$

$$= \frac{N^2+2N-8}{72} + \frac{N^2+2N-8}{72} + \frac{N^2+8N-20}{144} + \frac{N^2-16N+28}{144}$$
$$= \frac{2N^2+4N-16}{72} + \frac{N^2-4N+4}{72}$$
$$= \frac{N^2-4}{24} \quad (5)$$

By Lemma 4c, the wavelength number wanted to establish all connections of length $\frac{N}{2}+j$, where $1 \le j \le \frac{N}{2}-1$ is

$$\sum_{j=3,6,9\ldots,}^{\frac{N}{2}-1} \frac{N-2j+2}{6} + \sum_{j=2,5,8,\ldots,}^{\frac{N}{2}-2} \frac{N-2j+4}{6} + \sum_{j=1,7,13\ldots,}^{\frac{N}{2}-6} \frac{N-2j+6}{6} + \sum_{j=4,10,16,\ldots,}^{\frac{N}{2}-3} \frac{N-2j}{6}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j+1}{3}$, $u=\frac{j+5}{6}$ and $v=\frac{j+2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-2}{6}} \frac{N+2-6s}{6} + \sum_{t=1}^{\frac{N-2}{6}} \frac{N+6-6t}{6} + \sum_{u=1}^{\frac{N-2}{12}} \frac{N+16-12u}{6} + \sum_{u=1}^{\frac{N-2}{12}} \frac{N+4-12v}{6}$$

$$= \left\{\left(\frac{(N+2)(N-2)}{36}\right) - \left(\frac{(N-2)(N+4)}{72}\right)\right\} + \left\{\left(\frac{(N+6)(N-2)}{36}\right) - \left(\frac{(N-2)(N+4)}{72}\right)\right\} + \left\{\left(\frac{(N+16)(N-2)}{72}\right) - \left(\frac{2(N-2)(N+10)}{288}\right)\right\} + \left\{\left(\frac{(N+4)(N-2)}{72}\right) - \left(\frac{2(N-2)(N+10)}{288}\right)\right\}$$

$$= \left(\frac{N^2-2N}{72}\right) + \left(\frac{N^2+6N-16}{72}\right) + \left(\frac{N^2+20N-44}{144}\right) + \left(\frac{N^2-4N+4}{144}\right)$$
$$= \frac{N^2+4N-12}{24} \quad (6)$$

By combining (5) and (6),

$$W_s = \frac{N^2-4}{24} + \frac{N^2+4N-12}{24}$$

$$W_s = \frac{N^2+2N-8}{12}$$

Theorem 4: If $N=12m+3$, then $W_s = \frac{N^2+N-12}{12}$.

Proof: By Lemma 1d, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-3}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N-1}{2}} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N-5}{2}} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N-11}{2}} \frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-3}{6}} s + \sum_{t=1}^{\frac{N-3}{6}} t + \sum_{u=1}^{\frac{N-3}{12}} 2\text{u} + \sum_{v=1}^{\frac{N-15}{12}} 2\text{v}$$
$$= \frac{(N-3)(N+3)}{72} + \frac{(N-3)(N+3)}{72} + \frac{2(N-3)(N+9)}{288} + \frac{2(N-15)(N-3)}{288}$$
$$= \frac{2N^2-18}{72} + \frac{N^2-6N-36}{72}$$
$$= \frac{N^2-2N-27}{24} \quad (7)$$

By Lemma 4d, the wavelength number wanted to establish all connections of length $\frac{N-1}{2}+j$, where $1 \le j \le \frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-1}{2}} \frac{N-2j+5}{6} + \sum_{j=2,5,8\ldots,}^{\frac{N-5}{2}} \frac{N-2j+1}{6} + \sum_{j=3,6,9\ldots,}^{\frac{N-3}{2}} \frac{N-2j+3}{6}$$

Substitute $s=\frac{j+2}{3}$, $t=\frac{j+1}{3}$ and $u=\frac{j}{3}$in the first term, second term and third term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N+3}{6}} \frac{N+9-6s}{6} + \sum_{t=1}^{\frac{N-3}{6}} \frac{N+3-6t}{6} + \sum_{u=1}^{\frac{N-3}{6}} \frac{N+3-6u}{6}$$

$$= \left\{\left(\frac{(N+9)(N+3)}{36}\right) - \left(\frac{(N+3)(N+9)}{72}\right)\right\} + \left\{\left(\frac{(N+3)(N-3)}{36}\right) - \left(\frac{(N-3)(N+3)}{72}\right)\right\} + \left\{\left(\frac{(N+3)(N-3)}{36}\right) - \left(\frac{(N-3)(N+3)}{72}\right)\right\}$$

$$= \left(\frac{N^2+12N+27}{72}\right) + \left(\frac{N^2-9}{72}\right) + \left(\frac{N^2-9}{72}\right)$$

$$= \frac{N^2+4N+3}{24} \qquad (8)$$

By combining (7) and (8),

$$W_s = \frac{N^2-2N-27}{24} + \frac{N^2+4N+3}{24}$$

$$W_s = \frac{N^2+N-12}{12}$$

Theorem 5: If $N = 12m + 4$, then $W_s = \frac{N^2+2N-12}{12}$.

Proof: By Lemma 1e, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}-2} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N}{2}-1} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}-3} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N}{2}} \frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-4}{6}} s + \sum_{t=1}^{\frac{N-4}{6}} t + \sum_{u=1}^{\frac{N-4}{12}} 2\mathrm{u} + \sum_{v=1}^{\frac{N-4}{12}} 2\mathrm{v}$$

$$= \frac{(N-4)(N+2)}{72} + \frac{(N-4)(N+2)}{72} + \frac{2(N-4)(N+8)}{288} + \frac{2(N-4)(N+8)}{288}$$

$$= \frac{N^2-2N-8}{72} + \frac{N^2-2N-8}{72} + \frac{N^2+4N-32}{144} + \frac{N^2+4N-32}{144}$$

$$= \frac{2N^2-4N-16}{72} + \frac{N^2+4N-32}{72}$$

$$= \frac{N^2-16}{24} \qquad (9)$$

By Lemma 4e, the wavelength number wanted to establish all connections of length$\frac{N}{2}+j$, where $1 \le j \le \frac{N}{2}-1$ is

$$\sum_{j=1,4,7\ldots,}^{\frac{N}{2}-1} \frac{N+4-2j}{6} + \sum_{j=2,5,8,\ldots,}^{\frac{N}{2}-3} \frac{N-2j}{6} + \sum_{j=3,9,15,,,\ldots,}^{\frac{N}{2}-5} \frac{N+8-2j}{6} + \sum_{j=6,12,18,\ldots,}^{\frac{N}{2}-2} \frac{N+2-2j}{6}$$

Substitute $s=\frac{j+2}{3}$, $t=\frac{j+1}{3}$, $u=\frac{j+3}{6}$ and $v=\frac{j}{6}$in the first term, second term and third term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N+2}{6}} \frac{N+8-6s}{6} + \sum_{t=1}^{\frac{N-4}{6}} \frac{N+2-6t}{6} + \sum_{u=1}^{\frac{N-4}{12}} \frac{N+14-12u}{6} + \sum_{u=1}^{\frac{N-4}{12}} \frac{N+2-12v}{6}$$

$$=\left\{\left(\frac{(N+8)(N+2)}{36}\right)-\left(\frac{(N+2)(N+8)}{72}\right)\right\}+\left\{\left(\frac{(N+2)(N-4)}{36}\right)-\left(\frac{(N-4)(N+2)}{72}\right)\right\}+\left\{\left(\frac{(N+14)(N-4)}{72}\right)-\left(\frac{2(N-4)(N+8)}{288}\right)\right\}+\left\{\left(\frac{(N+2)(N-4)}{72}\right)-\left(\frac{2(N-4)(N+8)}{288}\right)\right\}$$

$$=\left(\frac{N^2+10N+16}{72}\right)+\left(\frac{N^2-2N-8}{72}\right)+\left(\frac{N^2+16N-80}{144}\right)+\left(\frac{N^2-8N+16}{144}\right)$$

$$=\frac{N^2+4N-8}{24} \qquad (10)$$

By combining (9) and (10),

$$W_s=\frac{N^2-16}{24}+\frac{N^2+4N-8}{24}$$

$$W_s=\frac{N^2+2N-12}{12}$$

Theorem 6: If $N=12m+5$, then $W_s=\frac{N^2+2N-11}{12}$.

Proof: By Lemma 1f, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-5}{2}}\frac{j}{3}+\sum_{j=4,7,10,\ldots,}^{\frac{N-3}{2}}\frac{j-1}{3}+\sum_{j=5,11,17,,\ldots,}^{\frac{N-7}{2}}\frac{j+1}{3}+\sum_{j=8,14,20,\ldots,}^{\frac{N-1}{2}}\frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$=\sum_{s=1}^{\frac{N-5}{6}}s+\sum_{t=1}^{\frac{N-5}{6}}t+\sum_{u=1}^{\frac{N-5}{12}}2\mathrm{u}+\sum_{v=1}^{\frac{N-5}{12}}2\mathrm{v}$$

$$=\frac{(N-5)(N+1)}{72}+\frac{(N-5)(N+1)}{72}+\frac{2(N-5)(N+7)}{288}+\frac{2(N-5)(N+7)}{288}$$

$$=\frac{2N^2-8N-10}{72}+\frac{N^2+2N-35}{72}$$

$$=\frac{N^2-2N-15}{24} \qquad (11)$$

By Lemma 4f, the wavelength number wanted to establish all connections of length $\frac{N-1}{2}+j$, where $1\le j\le\frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-3}{2}}\frac{N-2j+3}{6}+\sum_{j=2,5,8,\ldots,}^{\frac{N-1}{2}}\frac{N-2j+5}{6}+\sum_{j=3,9,15\ldots,}^{\frac{N-11}{2}}\frac{N-2j+7}{6}+\sum_{j=6,12,18\ldots,}^{\frac{N-5}{2}}\frac{N-2j+1}{6}$$

Substitute $s=\frac{j+2}{3}$, $t=\frac{j+1}{3}$, $u=\frac{j+3}{6}$ and $v=\frac{j}{6}$ in the first term, second term and third term respectively,

$$=\sum_{s=1}^{\frac{N+1}{6}}\frac{N+7-6s}{6}+\sum_{t=1}^{\frac{N+1}{6}}\frac{N+7-6t}{6}+\sum_{u=1}^{\frac{N-5}{12}}\frac{N+13-12u}{6}+\sum_{v=1}^{\frac{N-5}{12}}\frac{N+1-12v}{6}$$

$$= \left\{\left(\frac{(N+7)(N+1)}{36}\right) - \left(\frac{(N+1)(N+7)}{72}\right)\right\} + \left\{\left(\frac{(N+7)(N+1)}{36}\right) - \left(\frac{(N+1)(N+7)}{72}\right)\right\} + \left\{\left(\frac{(N+13)(N-5)}{72}\right) - \left(\frac{2(N-5)(N+7)}{288}\right)\right\} + \left\{\left(\frac{(N+1)(N-5)}{72}\right) - \left(\frac{2(N-5)(N+7)}{288}\right)\right\}$$

$$= \left(\frac{N^2+8N+7}{72}\right) + \left(\frac{N^2+8N+7}{72}\right) + \left(\frac{N^2+14N-95}{72}\right) + \left(\frac{N^2-10N+25}{72}\right)$$

$$= \frac{N^2+6N-7}{24} \qquad (12)$$

By combining (11) and (12),

$$W_s = \frac{N^2-2N-15}{24} + \frac{N^2+6N-7}{24}$$

$$W_s = \frac{N^2+2N-11}{12}$$

Theorem 7: If $N = 12m + 6$, then $W_s = \frac{N^2+N-6}{12}$.

Proof: By Lemma 1a, Lemma 2 and Lemma3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N}{2}-2} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}-4} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N}{2}-1} \frac{j-2}{3}$$

Substitute $s = \frac{j}{3}$, $t = \frac{j-1}{3}$, $u = \frac{j+1}{6}$ and $v = \frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N}{6}} s + \sum_{t=1}^{\frac{N-6}{6}} t + \sum_{u=1}^{\frac{N-6}{12}} 2\mathrm{u} + \sum_{v=1}^{\frac{N-6}{12}} 2\mathrm{v}$$

$$= \frac{N(N+6)}{72} + \frac{(N-6)N}{72} + \frac{2(N-6)(N+6)}{288} + \frac{2(N-6)(N+6)}{288}$$

$$= \frac{2N^2}{72} + \frac{N^2-36}{72}$$

$$= \frac{N^2-12}{24} \qquad (13)$$

By Lemma 4a, the wavelength number wanted to establish all connections of length $\frac{N}{2} + j$, where $1 \leq j \leq \frac{N}{2} - 1$ is

$$\sum_{j=3,6,9\ldots,}^{\frac{N}{2}-3} \frac{N-2j}{6} + \sum_{j=1,4,7,\ldots,}^{\frac{N}{2}-2} \frac{N-2j+2}{6} + \sum_{j=2,5,8\ldots,}^{\frac{N}{2}-1} \frac{N-2j+4}{6}$$

Substitute $s = \frac{j}{3}, t = \frac{j+2}{3}$ and $u = \frac{j+1}{3}$ in the first term, second term and third term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-6}{6}} \frac{N-6s}{6} + \sum_{t=1}^{\frac{N}{6}} \frac{N+6-6t}{6} + \sum_{u=1}^{\frac{N}{6}} \frac{N+6-6u}{6}$$

$$= \left\{\left(\frac{(N-6)(N)}{36}\right) - \left(\frac{(N-6)(N)}{72}\right)\right\} + \left\{\left(\frac{(N+6)(N)}{36}\right) - \left(\frac{(N)(N+6)}{72}\right)\right\} + \left\{\left(\frac{N(N+6)}{36}\right) - \left(\frac{(N)(N+6)}{72}\right)\right\}$$

$$= \left(\frac{N^2-6N}{72}\right) + \left(\frac{N^2+6N}{72}\right) + \left(\frac{N^2+6N}{72}\right)$$

$$= \frac{N^2+2N}{24} \qquad (14)$$

By combining (13) and (14),

$$= \frac{N^2-12}{24} + \frac{N^2+2N}{24}$$

$$= \frac{N^2+N-6}{12}$$

Theorem 8: If $N = 12m + 7$, then $W_s = \frac{N^2+2N-3}{12}$.

Proof: By Lemma 1b, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-1}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N-5}{2}} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N-9}{2}} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N-3}{2}} \frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$=\sum_{s=1}^{\frac{N-1}{6}} s+\sum_{t=1}^{\frac{N-7}{6}} t+\sum_{u=1}^{\frac{N-7}{12}} 2u+\sum_{v=1}^{\frac{N-7}{12}} 2v$$

$$=\frac{(N-1)(N+5)}{72}+\frac{(N-7)(N-1)}{72}+\frac{2(N-7)(N+5)}{288}+\frac{2(N-7)(N+5)}{288}$$

$$=\frac{2N^2-4N+2}{72}+\frac{N^2-2N-35}{72}$$

$$=\frac{N^2-2N-11}{24} \quad (15)$$

By Lemma 4b, the wavelength number wanted to establish all connections of length $\frac{N-1}{2}+j$, where $1\le j\le\frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-5}{2}} \frac{N-2j+1}{6}+\sum_{j=2,8,14\ldots,}^{\frac{N-3}{2}} \frac{N-2j+9}{6}+\sum_{j=5,11,17\ldots,}^{\frac{N-9}{2}} \frac{N-2j+3}{6}+\sum_{j=3,6,9\ldots,}^{\frac{N-1}{2}} \frac{N-2j+5}{6}$$

Substitute $s=\frac{j+2}{3}$, $t=\frac{j+4}{6}$, $u=\frac{j+1}{6}$ and $v=\frac{j}{3}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$=\sum_{s=1}^{\frac{N-1}{6}} \frac{N+5-6s}{6}+\sum_{t=1}^{\frac{N+5}{12}} \frac{N+17-12t}{6}+\sum_{u=1}^{\frac{N-7}{12}} \frac{N+5-12u}{6}+\sum_{v=1}^{\frac{N-1}{6}} \frac{N+5-6v}{6}$$

$$=\left\{\left(\frac{(N+5)(N-1)}{36}\right)-\left(\frac{(N-1)(N+5)}{72}\right)\right\}+\left\{\left(\frac{(N+17)(N+5)}{72}\right)-\left(\frac{2(N+5)(N+17)}{288}\right)\right\}$$
$$+\left\{\left(\frac{(N+5)(N-7)}{72}\right)-\left(\frac{2(N-7)(N+5)}{288}\right)\right\}+\left\{\left(\frac{(N+5)(N-1)}{36}\right)-\left(\frac{(N-1)(N+5)}{72}\right)\right\}$$

$$=\left(\frac{N^2+4N-5}{72}\right)+\left(\frac{N^2+22N+85}{144}\right)+\left(\frac{N^2-2N-35}{144}\right)+\left(\frac{N^2+4N-5}{72}\right)$$

$$=\frac{N^2+6N+5}{24} \quad (16)$$

By combining (15) and (16),

$$W_s=\frac{N^2-2N-11}{24}+\frac{N^2+6N+5}{24}$$

$$W_s=\frac{N^2+2N-3}{12}$$

Theorem 9: If $N=12m+8$, then $W_s=\frac{N^2+2N-8}{12}$.

Proof: By Lemma 1c, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}-1} \frac{j}{3}+\sum_{j=4,7,10,\ldots,}^{\frac{N}{2}} \frac{j-1}{3}+\sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}-5} \frac{j+1}{3}+\sum_{j=8,14,20,\ldots,}^{\frac{N}{2}-2} \frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$=\sum_{s=1}^{\frac{N-2}{6}} s+\sum_{t=1}^{\frac{N-2}{6}} t+\sum_{u=1}^{\frac{N-8}{12}} 2u+\sum_{v=1}^{\frac{N-8}{12}} 2v$$

$$=\frac{(N-2)(N+4)}{72}+\frac{(N-2)(N+4)}{72}+\frac{2(N-8)(N+4)}{288}+\frac{2(N-8)(N+4)}{288}$$

$$=\frac{N^2+2N-8}{72}+\frac{N^2+2N-8}{72}+\frac{N^2-4N-32}{144}+\frac{N^2-4N-32}{144}$$

$$=\frac{2N^2+4N-16}{72}+\frac{N^2-4N-32}{72}$$

$$=\frac{N^2-16}{24} \tag{17}$$

By Lemma 4c, the wavelength number wanted to establish all connections of length $\frac{N}{2}+j$, where $1 \leq j \leq \frac{N}{2}-1$ is

$$\sum_{j=3,6,9\ldots,}^{\frac{N}{2}-1} \frac{N-2j+2}{6} + \sum_{j=2,5,8,\ldots,}^{\frac{N}{2}-2} \frac{N-2j+4}{6} + \sum_{j=1,7,13\ldots,}^{\frac{N}{2}-3} \frac{N-2j+6}{6} + \sum_{j=4,10,16,\ldots,}^{\frac{N}{2}-6} \frac{N-2j}{6}$$

Substitute $s=\frac{j}{3},\ t=\frac{j+1}{3},\ u=\frac{j+5}{6}$ and $v=\frac{j+2}{6}$ in the first term, second term, third term and fourth term respectively,

$$=\sum_{s=1}^{\frac{N-2}{6}} \frac{N+2-6s}{6} + \sum_{t=1}^{\frac{N-2}{6}} \frac{N+6-6t}{6} + \sum_{u=1}^{\frac{N+4}{12}} \frac{N+16-12u}{6} + \sum_{u=1}^{\frac{N-8}{12}} \frac{N+4-12v}{6}$$

$$=\left\{\left(\frac{(N+2)(N-2)}{36}\right)-\left(\frac{(N-2)(N+4)}{72}\right)\right\} + \left\{\left(\frac{(N+6)(N-2)}{36}\right) - \left(\frac{(N-2)(N+4)}{72}\right)\right\} + \left\{\left(\frac{(N+16)(N+4)}{72}\right)-\left(\frac{2(N+4)(N+16)}{288}\right)\right\} + \left\{\left(\frac{(N+4)(N-8)}{72}\right)-\left(\frac{2(N-8)(N+4)}{288}\right)\right\}$$

$$=\left(\frac{N^2-2N}{72}\right)+\left(\frac{N^2+6N-16}{72}\right) + \left(\frac{N^2+20N+64}{144}\right) + \left(\frac{N^2-4N-32}{144}\right)$$

$$=\frac{N^2+4N}{24} \tag{18}$$

By combining (17) and (18),

$$W_s=\frac{N^2-16}{24}+\frac{N^2+4N}{24}$$

$$W_s=\frac{N^2+2N-8}{12}$$

Theorem 10: If $N=12m+9$, then $W_s=\frac{N^2+N-6}{12}$.

Proof: By Lemma 1d, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-3}{2}} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N-1}{2}} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N-11}{2}} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N-5}{2}} \frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$=\sum_{s=1}^{\frac{N-3}{6}} s + \sum_{t=1}^{\frac{N-3}{6}} t + \sum_{u=1}^{\frac{N-9}{12}} 2\mathrm{u} + \sum_{v=1}^{\frac{N-9}{12}} 2\mathrm{v}$$

$$=\frac{(N-3)(N+3)}{72}+\frac{(N-3)(N+3)}{72} + \frac{2(N-9)(N+3)}{288} + \frac{2(N-9)(N+3)}{288}$$

$$=\frac{2N^2-18}{72}+\frac{N^2-6N-27}{72}$$

$$=\frac{N^2-2N-15}{24} \tag{19}$$

By Lemma 4d, the wavelength number wanted to establish all connections of length $\frac{N-1}{2}+j$, where $1 \leq j \leq \frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-1}{2}} \frac{N-2j+5}{6} + \sum_{j=2,5,8\ldots,}^{\frac{N-5}{2}} \frac{N-2j+1}{6} + \sum_{j=3,6,9\ldots,}^{\frac{N-3}{2}} \frac{N-2j+3}{6}$$

Substitute $s=\frac{j+2}{3}, t=\frac{j+1}{3}$ and $u=\frac{j}{3}$ in the first term, second term and third term respectively,

$$= \sum_{s=1}^{\frac{N+3}{6}} \frac{N+9-6s}{6} + \sum_{t=1}^{\frac{N-3}{6}} \frac{N+3-6t}{6} + \sum_{u=1}^{\frac{N-3}{6}} \frac{N+3-6u}{6}$$

$$= \left\{\left(\frac{(N+9)(N+3)}{36}\right) - \left(\frac{(N+3)(N+9)}{72}\right)\right\} + \left\{\left(\frac{(N+3)(N-3)}{36}\right) - \left(\frac{(N-3)(N+3)}{72}\right)\right\} + \left\{\left(\frac{(N+3)(N-3)}{36}\right) - \left(\frac{(N-3)(N+3)}{72}\right)\right\}$$

$$= \left(\frac{N^2+12N+27}{72}\right) + \left(\frac{N^2-9}{72}\right) + \left(\frac{N^2-9}{72}\right)$$

$$= \frac{N^2+4N+3}{24} \quad (20)$$

By combining (19) and (20),

$$W_s = \frac{N^2-2N-15}{24} + \frac{N^2+4N+3}{24}$$

$$W_s = \frac{N^2+N-6}{12}$$

Theorem 11: If $N = 12m + 10$, then $W_s = \frac{N^2+2N}{12}$

Proof: By Lemma 1e, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N}{2}-2} \frac{j}{3} + \sum_{j=4,7,10,\ldots,}^{\frac{N}{2}-1} \frac{j-1}{3} + \sum_{j=5,11,17,,\ldots,}^{\frac{N}{2}} \frac{j+1}{3} + \sum_{j=8,14,20,\ldots,}^{\frac{N}{2}-3} \frac{j-2}{3}$$

Substitute $s = \frac{j}{3}$, $t = \frac{j-1}{3}$, $u = \frac{j+1}{6}$ and $v = \frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively in the above expression,

$$= \sum_{s=1}^{\frac{N-4}{6}} s + \sum_{t=1}^{\frac{N-4}{6}} t + \sum_{u=1}^{\frac{N+2}{12}} 2\text{u} + \sum_{v=1}^{\frac{N-10}{12}} 2\text{v}$$

$$= \frac{(N-4)(N+2)}{72} + \frac{(N-4)(N+2)}{72} + \frac{2(N+2)(N+14)}{288} + \frac{2(N-10)(N+2)}{288}$$

$$= \frac{N^2-2N-8}{72} + \frac{N^2-2N-8}{72} + \frac{N^2+16N+28}{144} + \frac{N^2-8N-20}{144}$$

$$= \frac{2N^2-4N-16}{72} + \frac{N^2+4N+4}{72}$$

$$= \frac{N^2-4}{24} \quad (21)$$

By Lemma 4e, the wavelength number wanted to establish all connections of length $\frac{N}{2} + j$, where $1 \leq j \leq \frac{N}{2} - 1$ is

$$\sum_{j=1,4,7\ldots,}^{\frac{N}{2}-1} \frac{N+4-2j}{6} + \sum_{j=2,5,8,\ldots,}^{\frac{N}{2}-3} \frac{N-2j}{6} + \sum_{j=3,9,15,,,\ldots,}^{\frac{N}{2}-2} \frac{N+8-2j}{6} + \sum_{j=6,12,18,\ldots,}^{\frac{N}{2}-5} \frac{N+2-2j}{6}$$

Substitute $s = \frac{j+2}{3}$, $t = \frac{j+1}{3}$, $u = \frac{j+3}{6}$ and $v = \frac{j}{6}$ in the first term, second term, third term and fourth term respectively,

$$= \sum_{s=1}^{\frac{N+2}{6}} \frac{N+8-6s}{6} + \sum_{t=1}^{\frac{N-4}{6}} \frac{N+2-6t}{6} + \sum_{u=1}^{\frac{N+2}{12}} \frac{N+14-12u}{6} + \sum_{u=1}^{\frac{N-10}{12}} \frac{N+2-12v}{6}$$

$$= \left\{\left(\frac{(N+8)(N+2)}{36}\right) - \left(\frac{(N+2)(N+8)}{72}\right)\right\} + \left\{\left(\frac{(N+2)(N-4)}{36}\right) - \left(\frac{(N-4)(N+2)}{72}\right)\right\} + \left\{\left(\frac{(N+2)(N+14)}{72}\right) - \left(\frac{2(N+2)(N+14)}{288}\right)\right\} + \left\{\left(\frac{(N+2)(N-10)}{72}\right) - \left(\frac{2(N-10)(N+2)}{288}\right)\right\}$$

$$= \left(\frac{N^2+10N+16}{72}\right) + \left(\frac{N^2-2N-8}{72}\right) + \left(\frac{N^2+16N+28}{144}\right) + \left(\frac{N^2-8N-20}{144}\right)$$

$$=\frac{N^2+4N+4}{24} \quad (22)$$

By combining (21) and (22),

$$W_s=\frac{N^2-4}{24}+\frac{N^2+4N+4}{24}$$

$$W_s=\frac{N^2+2N}{12}$$

Theorem 12: If $N=12m+11$, then $W_s=\frac{N^2+2N+1}{12}$

Proof: By Lemma 1f, Lemma 2 and Lemma 3, the wavelength number wanted to establish all connections of length less than or equal to $\frac{N-1}{2}$ is

$$\sum_{j=3,6,9,\ldots,}^{\frac{N-5}{2}}\frac{j}{3}+\sum_{j=4,7,10,\ldots,}^{\frac{N-3}{2}}\frac{j-1}{3}+\sum_{j=5,11,17,,\ldots,}^{\frac{N-1}{2}}\frac{j+1}{3}+\sum_{j=8,14,20,\ldots,}^{\frac{N-7}{2}}\frac{j-2}{3}$$

Substitute $s=\frac{j}{3}$, $t=\frac{j-1}{3}$, $u=\frac{j+1}{6}$ and $v=\frac{j-2}{6}$ in the first term, second term, third term and fourth term respectively,

$$=\sum_{s=1}^{\frac{N-5}{6}}s+\sum_{t=1}^{\frac{N-5}{6}}t+\sum_{u=1}^{\frac{N+1}{12}}2\text{u}+\sum_{v=1}^{\frac{N-11}{12}}2\text{v}$$

$$=\frac{(N-5)(N+1)}{72}+\frac{(N-5)(N+1)}{72}+\frac{2(N+1)(N+13)}{288}+\frac{2(N-11)(N+1)}{288}$$

$$=\frac{2N^2-8N-10}{72}+\frac{N^2+2N+1}{72}$$

$$=\frac{N^2-2N-3}{24} \quad (23)$$

By Lemma 4f, the wavelength number wanted to establish all connections of length $\frac{N-1}{2}+j$, where $1\le j\le\frac{N-1}{2}$ is

$$\sum_{j=1,4,7,\ldots,}^{\frac{N-3}{2}}\frac{N-2j+3}{6}+\sum_{j=2,5,8,\ldots,}^{\frac{N-1}{2}}\frac{N-2j+5}{6}+\sum_{j=3,9,15\ldots,}^{\frac{N-5}{2}}\frac{N-2j+7}{6}+\sum_{j=6,12,18\ldots,}^{\frac{N-11}{2}}\frac{N-2j+1}{6}$$

Substitute $s=\frac{j+2}{3}$, $t=\frac{j+1}{3}$, $u=\frac{j+3}{6}$ and $v=\frac{j}{6}$ in the first term, second term, third term and fourth term respectivelyin the above expression,

$$=\sum_{s=1}^{\frac{N+1}{6}}\frac{N+7-6s}{6}+\sum_{t=1}^{\frac{N+1}{6}}\frac{N+7-6t}{6}+\sum_{u=1}^{\frac{N+1}{12}}\frac{N+13-12u}{6}+\sum_{v=1}^{\frac{N-11}{12}}\frac{N+1-12v}{6}$$

$$=\left\{\left(\frac{(N+7)(N+1)}{36}\right)-\left(\frac{(N+1)(N+7)}{72}\right)\right\}+\left\{\left(\frac{(N+7)(N+1)}{36}\right)-\left(\frac{(N+1)(N+7)}{72}\right)\right\}$$

$$+\left\{\left(\frac{(N+13)(N+1)}{72}\right)-\left(\frac{2(N+1)(N+13)}{288}\right)\right\}+\left\{\left(\frac{(N+1)(N-11)}{72}\right)-\left(\frac{2(N-11)(N+1)}{288}\right)\right\}$$

$$=\left(\frac{N^2+8N+7}{72}\right)+\left(\frac{N^2+8N+7}{72}\right)+\left(\frac{N^2+14N+13}{144}\right)+\left(\frac{N^2-10N-11}{144}\right)$$

$$=\frac{N^2+6N+5}{24} \quad (24)$$

By combining (23) and (24),

$$W_s=\frac{N^2-2N-3}{24}+\frac{N^2+6N+5}{24}$$

$$W_s=\frac{N^2+2N+1}{12}$$

Now, the link load of the linear array with 3-length extension for all-to-all broadcast is derived. The link load of a network is the maximum number of lightpaths that share a common link. First, consider an arbitrary shorter link l_i,joining the nodes indexed i and $i+1$.It is easy to observe that $i+1$ number of connections share the link l_i to transmit the message to other nodes. Hence, the total number of connections that share any arbitrary shorter link is $i+1$. Next, consider the link load of an arbitrary longer link. For $0\le \text{i}\le \text{N}-4$, let l_i represent the longer link joining the nodes indexed

i and $i+3$.It is easy to observe that $\left\lceil \frac{i+1}{3} \right\rceil - 1$ number of nodes before node i shares the link l_i to transmit the message to $(N-(i+3))$ number of nodes after node i. Therefore, the number of connections sharing any link $l_i$is $max \left\lceil \frac{i+1}{3} \right\rceil (N-i-3)$. Hence, the link load of a longer link is $max\left\{\left\lceil \frac{i+1}{3} \right\rceil (N-i-3)\right\}$. As the link load of longer link is higher than that of shorter link, so the link load of a longer link is the link load of the network. Hence, the link load of the network is given by, $\pi = max \left\lceil \frac{i+1}{3} \right\rceil (N-i-3)$.

4 Results and Discussion

Table 1 shows the wavelength requirement for all-to-all broadcast in a linear array with 3-length extension and its associated link load. Table 2 shows the wavelength number wanted atmost to establish all-to-all broadcast and the associated link load for certain values of node number N in a linear array with 3-length extension. It can be noted that the value of link load is slightly greater than the wavelength number. Hence, the results derived in the previous section are either optimum or near optimum. The use of wavelength converter may reduce wavelength requirement to the minimum value but wavelength converter is very expensive. Table 3 shows the wavelength number wanted atmost to establish all-to-all broadcast for certain values of node number N in a linear array with 3-length extension, 2-length extension and a basic linear array. The results show that the wavelength number wanted atmost to establish all-to-all broadcast in a linear array with 3-length extension is reduced by a minimum of 61% and a maximum of 66% when compared to a basic linear array. Similarly, the wavelength number needed atmost to establish all-to-all broadcast is reduced by a minimum of 24% and a maximum of 33% when compared to linear array with 2-length extension.

Table 1. Wavelength number wanted atmost to establish all-to-all broadcast along with its link load in a linear array with 3-length extension.

Network Topology with *N* nodes	Wavelength number wanted atmost to establish all-to-all broadcast	Link Load
Linear array with 3-length extension	$\frac{N^2+N}{12}$ $when\ N = 12m$, $\frac{N^2+2N-3}{12}$ $when\ N = 12m+1$, $\frac{N^2+2N-8}{12}$ $when\ N = 12m+2$, $\frac{N^2+N-12}{12}$ $when\ N = 12m+3$, $\frac{N^2+2N-12}{12}$ $when\ N = 12m+4$, $\frac{N^2+2N-11}{12}$ $when\ N = 12m+5$, $\frac{N^2+N-6}{12}$ $when\ N = 12m+6$, $\frac{N^2+2N-3}{12}$ $when\ N = 12m+7$, $\frac{N^2+2N-8}{12}$ $whenN = 12m+8$ $\frac{N^2+N-6}{12}$ $when\ N = 12m+9$ $\frac{N^2+2N}{12}$ $when\ N = 12m+10$ $\frac{N^2+2N+1}{12}$ $when\ N = 12m+11$ m is a positive integer	$\pi = \max \left\lceil \frac{i+1}{3} \right\rceil (N-i-3)$ for $0 \le i \le N-4$

Table 2. Wavelength number wanted atmost to establish all-to-all broadcast along with its link load for certain values of node number *N* in a linear array with 3-length extension

Node number *N*	Wavelength number wanted atmost	Link load	Difference between wavelength number and link load
12	13	12	1
13	16	14	2
14	18	16	2
15	19	18	1
16	23	21	2
17	26	24	2
18	28	27	1
19	33	30	3
20	36	33	3
21	38	36	2

22	44	40	4
23	48	44	4
24	50	48	2
25	56	52	4
26	60	56	4
27	62	60	2
28	69	65	4
29	74	70	4
30	77	75	2
31	85	80	5
32	90	85	5
33	93	90	3
34	102	96	6
35	108	102	6
36	111	108	3
37	120	114	6
38	126	120	6
39	129	126	3
40	139	133	6
41	146	140	6
42	150	147	3
43	161	154	7
44	168	161	7
45	172	168	4
46	184	176	8
47	192	184	8
48	196	192	4
50	216	208	8
60	305	300	5
70	420	408	12
80	546	533	13
90	682	675	7
100	849	833	16

Table 3. Comparison of wavelength number wanted atmost to establish all-to-all broadcast for certain values of node number *N* in a linear array, linear array with 2-length extension and linear array with 3-length extension.

Node number *N*	Linear array [14]	Linear array with 2-length extension [22]	Linear array with 3-length extension
12	36	18	13
13	42	21	16
14	49	24	18
15	56	28	19
16	64	32	23
17	72	36	26
18	81	40	28
19	90	45	33
20	100	50	36
21	110	55	38
22	121	60	44
23	132	66	48
24	144	72	50
27	182	91	62
30	225	112	77
33	272	136	93
36	324	162	111
39	380	190	129
42	441	220	150
45	506	253	172
48	576	288	196
51	650	325	220
60	900	450	305
70	1225	612	420
80	1600	800	546
90	2025	1012	682
100	2500	1250	849
125	3906	1953	1322
150	5625	2812	1887
175	7656	3828	2581
200	10000	5000	3366
300	22500	11250	7525
500	62500	31250	20916
700	122500	61250	40949
1000	250000	125000	83499

5 Conclusion and Future Work

In this work, the wavelength number wanted atmost to establish all-to-all broadcast in a WDM optical linear array with 3-length extension is determined and the wavelength allotment technique is given. The wavelength number needed is nearly equal to the link load and so the results are near to optimum or optimum. The wavelength number wanted atmost to establish all-to-all broadcast in a linear array with 3-length extension is reduced by a minimum of 61% and a maximum of 66% when compared to a basic linear array. Similarly, the wavelength number wanted atmost to establish all-to-all broadcast is reduced by a minimum of 24% and a maximum of 33% when compared to linear array with 2-length extension. Wavelength number requirement needs to be investigated with still higher order extensions, to judge the rate of reduction in wavelength number requirements with increasing extension and is a challenging issue. Also, deriving a generalized expression for wavelength number requirement in a linear array/ring network with k-length extension (k is any positive integer and k < N where N is the total number of nodes in the network) is another interesting and challenging future work. Examining the effects of physical layer impairments and network survivability on routing and wavelength assignment are other research issues with these extended networks.

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