# Controllability subspaces of multi-agent linear dynamical systems.

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*Abstract:* This work addresses the controllability subspaces of a class of multi-agent linear systems that are interconnected via communication channels. Multiagent systems have attracted much attention because they have great applicability in multiple areas. Recently has taken an interest to analyze the control properties as consensus controllability of multi-agent dynamical systems motivated by the fact that the architecture of communication network in engineering multi-agent systems is usually adjustable. In this paper, the concept of invariant subspaces and controllability subspaces is reviewed and generalized to multi-agent systems. Finally, the consensus controllability subspaces are analyzed in the case of multiagent linear systems having all agents the same dynamics described as $\dot{x}^i = A_i x^i + B_i u^i$, $i = 0, 1, \dots, k$.

*Key–Words:* Controllability, invariant subspaces, controlled invariant subspaces, multiagent systems.

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## 1 Introduction

Controllability is a basic concept of control system theory. It is particularly important for practical implementations ([1], [5], [6], [7], [10]). Roughly speaking the controllability character can be defined as follows: If we want to do whatever with the given dynamic system under control input, necessarily the system must be controllable.

From a geometric point of view, many problems in the control theory of time-invariant linear systems can be played by controlled invariant subspaces and controllability subspaces. Controlled invariant subspaces are a generalization of invariant subspaces under a linear map, ([2], [3], [13]). The importance of the study of controllability subspaces of the system $\dot{x} = Ax + Bu$ derives from the fact that the restriction of the system $\dot{x} = (A + BF)x + Bu$ obtained by means of state feedback F to the original system, to an $(A + BF)$-invariant controllable subspace can be assigned an arbitrary spectrum by suitable choice of $F$, ([17]).

In recent years has grown the interest in the study of control multi-agent systems, as well as the increasing interest in distributed control and coordination of networks consisting of multiple autonomous agents. It is due to that they appear in different areas, and there are an amount of bibliography as [11], [12], [14], [16].

In this work the controllability subspaces of multiagent systems consisting of $k$ agents having linear dynamic modes, with dynamics

$$\left.\dot{x}^i = A_i x^i + B_i u^i \right\} \quad i = 1, \dots, k \tag{1}$$

with $A_i \in M_{n_i}(\mathbb{R})$ and $B_i \in M_{n_i \times m_i}(\mathbb{R})$, $x^i \in \mathbb{R}^{n_i}$, and $u^i \in \mathbb{R}^{m_i}$, are analyzed under geometrical point of view.

## 2 Preliminaries

We are interested in multi-agent linear systems that they are interconnected via communication channels, then we need to know the communication topology among agents of the system. The topology is defined by means an indirect graph. It should be noted that graph models are commonly used in network representations.

In this particular setup, we consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order $k$ with the set of vertices $\mathcal{V} = \{1, \dots, k\}$ and the set of edges $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge $(i, j)$ $i$ is called the parent node and $j$ is called the child node and $j$ is in the neighbor of $i$, concretely we define the neighbor of $i$ and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph it can be consider the Laplacian matrix of the graph defined in the following manner

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

**Remark 1.** *The following properties are verified.*

*i) If the graph is undirected then the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix $P$ such that $P\mathcal{L}P^t = \mathcal{D}$.*

*ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $(1, \dots, 1)^t$ is the associated eigenvector.*

*iii) If the graph is undirected and connected the eigenvalue 0 is simple.*

For more information on graph theory, see [15].

About matrices, we need to remember Kronecker product of matrices because it will be useful in our study.

Given a couple o matrices $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$, remember that the Kronecker product is defined as follows.

**Definition 2.** *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of $A$ and $B$, write $A \otimes B$, is the matrix*

$$A \otimes B = (a^i_j B) \in M_{np\times mq}(\mathbb{C})$$

Kronecker product verifies the following properties

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$
2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$
3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
4) $(A \otimes B)^t = A^t \otimes B^t$
5) If $A \in Gl(n; \mathbb{C})$ and $B \in Gl(p; \mathbb{C}))$, then $A \otimes B \in Gl(np; \mathbb{C}))$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
6) If the products $AC$ and $BD$ are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

**Corollary 3.** *The vector $\mathbf{1}_k \otimes v$ is an eigenvector corresponding to the zero eignevalue of $\mathcal{L} \otimes I_n$.*

*Proof.*

$$(\mathcal{L} \otimes I_n)(\mathbf{1}_k \otimes v) = \mathcal{L}\mathbf{1}_k \otimes v = 0 \otimes v = 0$$

□

Consequently, if $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{C}^n$, then $\mathbf{1}_k \otimes e_i$ is a basis for the nullspace of $\mathcal{L} \otimes I_n$.

Associated to the Kronecker product, can be defined the vectorizing operator that transforms any matrix $A$ into a column vector, by placing the columns in the matrix one after another,

**Definition 4.** *Let $X = (x^i_j) \in M_{n\times m}(\mathbb{C})$ be a matrix, and we denote $x_i = (x^1_i, \dots, x^n_i)^t$ for $1 \le i \le m$ the $i$-th column of the matrix $X$. We define the vectorizing operator $vec$, as*

$$\begin{aligned} vec : M_{n\times m}(\mathbb{C}) &\longrightarrow M_{nm\times 1}(\mathbb{C}) \\ X &\longrightarrow \begin{pmatrix} x^t_1 & x^t_2 & \dots & x^t_m \end{pmatrix}^t \end{aligned}$$

*Obviously, $vec$ is an isomorphism.*

See [8] for more information and properties.

**Example 1.** *Let $X = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$, Then*

$$vec(X) = \left( \begin{pmatrix} 1 \\ 2 \end{pmatrix}^t \begin{pmatrix} 2 \\ 4 \end{pmatrix}^t \begin{pmatrix} 1 \\ 3 \end{pmatrix}^t \right)^t = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \\ 1 \\ 3 \end{pmatrix}$$

# 3 Control Properties

The character of controllability is one of the most important properties of dynamical systems. A system is controllable if we can drive the state variables from an initial to any desired values within a finite period with properly selected inputs, more concretely:

**Definition 5.** *The dynamical system $\dot{x} = Ax + Bu$ is said to be controllable if for every initial condition $x(0)$ and every vector $x_1 \in \mathbb{R}^n$, there exist a finite time $t_1$ and control $u(t) \in \mathbb{R}^m$, $t \in [0, t_1]$, such that $x(t_1) = x_1$.*

This definition requires only that any initial state $x(0)$ can be steered to any final state $x_1$ at time $t_1$. However, the trajectory of the dynamical system between 0 and $t_1$ is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

For simplicity and if confusion is not possible, we will write $(A, B)$ for dynamical system $\dot{x} = Ax + Bu$.

To formulate easily computable algebraic controllability criteria we use the so-called controllability matrix $\mathcal{C}$, which is well-known as Kalman matrix and defined in the following manner:

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$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}. \quad (3)$$

We want to emphasize that the controllability matrix $\mathcal{C}$ is a $n \times nm$-dimensional constant matrix and depends only on the parameters of the system.

and we have the following result:

**Theorem 6.** *The dynamical system $\dot{x} = Ax + Bu$ is controllable if and only if* $\operatorname{rank} \mathcal{C} = n$.

**Corollary 7.** *The dynamical system $\dot{x} = Ax + Bu$ is controllable if and only if the $n$-dimensional symmetric matrix $\mathcal{C}\mathcal{C}^t$ is nonsingular.*

As we says, controllability of the dynamical system $\dot{x} = Ax + Bu$ implies that each initial state can be steered to 0 on a finite time-interval. If only is required that this to happen asymptotically for $t \to \infty$, we have the following concept.

**Definition 8.** *The system $\dot{x} = Ax+Bu$ is called stabilizable if for each initial state $x(0) \in \mathbb{R}^n$ there exists a (piece-wise continuous) control input $u : [0, \infty) \longrightarrow \mathbb{R}^m$ such that the state-response with $x(0)$ verifies*

$$\lim_{t\to\infty} x(t) = 0.$$

**Remark 9.** *i) All controllable systems are stabilizable but the converse is false.*

*ii) If the matrix $A$ in the system $\dot{x} = Ax + Bu$ is Hurwitz then, the system is stabilizable.*

It is important the following result

**Theorem 10.** *The system $\dot{x} = Ax+Bu$ is stabilizable if and only if there exists some feedback $F$ such that $\dot{x} = (A - BF)x$ is asymptotically stable.*

The controllability and stabilizable characters are preserved under feedback

**Definition 11.** *Two systems $(A_1, B_1)$ and $(A_2, B_2)$ are feedback equivalent, if and only if, there exist $P \in Gl(n, \mathbb{R})$, $Q \in Gl(m, \mathbb{R})$ and $F \in M_{m\times n}(\mathbb{R})$ such that*

$$(A_2, B_2) = (P^{-1}A_1P + P^{-1}B_1F, P^{-1}B_1Q)$$

**Proposition 12.** *Let $(A_1, B_1)$ and $(A_2, B_2)$ feedback equivalent systems, then*

*i) $(A_1, B_1)$ is controllable if and only if $(A_2, B_2)$ is*

*i) $(A_1, B_1)$ is stabilizable if and only if $(A_2, B_2)$ is*

# 4 Controled invariant $(A, B)$-subspaces

In this section we remember the definition of invariant subspace under $(A, B)$-map.

**Definition 13.** *A subspace $G \subset \mathbb{C}^n$ is controled invariant or invariant under $(A, B)$ if and only if*

$$AG \subset G + \operatorname{Im} B \quad (4)$$

Notice that if $B = 0$, this definition coincides with the definition of $A$-invariant subspace.

We can construct invariant subspaces in the following manner. Let $H \subset \mathbb{C}^n$ be a subspace, we define

$$G_{k+1} = H \cap \{x \in \mathbb{C}^n \mid Ax \in G_k + \operatorname{Im} B\},\ G_0 = H,$$

limit of recursion exists and we will denote by $G(H)$. This subspace is the supremal $(A, B)$-invariant subspace contained in $H$. Taking $H = \mathbb{C}^n$, we will write it as $G^*$.

**Example 2.** *Let $(A, B)$ be the pair $A = \begin{pmatrix} 0 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $H = \{(x, y, z) \mid z = 0\}$,*

*Computation of $G_1$:*

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ y+z \end{pmatrix} = \begin{pmatrix} \mu \\ \nu \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$[(x, y, -y)] \cap H = [(x, 0, 0)] = G_1$.

*Computation of $G_2$:*

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$[(x, 0, 0)] \cap H = [(1, 0, 0)] = G_2 = G_1$. *Then $G = G_1$.*

*Obviously $AG \subset G + Im\, B$.*

**Proposition 14.** *Let $(A, B)$ be a pair of matrices. A subspace $G \subset \mathbb{C}^n$ is invariant under $(A, B)$ if and only if is invariant under $(A+BF, B)$ for all feedback $F \in M_{m\times n}(\mathbb{C})$.*

*Proof.* Suppose that $AG \subset G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $Ax = y + Bw$ so, for any $F \in M_{m\times n}(\mathbb{C})$, we have

$$\begin{gathered} Ax + BFx - BFx = y + Bw \\ (A + BF)x = y + B(Fx + w). \end{gathered}$$

Consequently, for all $x \in G$, $(A + BF)G \subset G + \operatorname{Im} B$.

Reciprocally, suppose that $(A + BF)G \subset G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $(A + BF)x = y + Bw$ so, $Ax = y - BFxBw$ and $Ax = y + B(-Fx + w)$. Then, for all $x \in G$ we have $AG \subset G + \operatorname{Im} B$. $\square$

**Proposition 15.** *Let $(A_1, B_1)$ and $(A_2, B_2)$ be two equivalent pairs under equivalence defined in 11. Then $G \subset \mathbb{C}^n$ is an invariant subspace under $(A_1, B_1)$ if and only if $P^{-1}G$ is invariant under $(A_2, B_2)$.*

*Proof.* Suppose that $A_1G \subset G + \operatorname{Im} B$. Then $A_2P^{-1}G = (P^{-1}A_1P + P^{-1}B_1F)P^{-1}G = P^{-1}(A_1G + B_1FP^{-1}G) \subset P^{-1}(G + \operatorname{Im} B_1) = P^{-1}G + \operatorname{Im} PB_2Q^{-1}) = P^{-1}G + P\operatorname{Im} B_2Q^{-1}) = P^{-1}G + \operatorname{Im}B_2R^{-1}) \subset (P^{-1}G + \operatorname{Im} B_2$ □

## 5 Controllability subspaces

In this section we are going to study a particular case of invariant subspaces. First of all we observe the following result.

**Proposition 16.** *Let $(A, B)$ be a pair of matrices. Then*

$$G = [B, AB, \dots, A^{n-1}B]$$

*is a $(A, B)$-invariant subspace.*

*Proof.*

$$\begin{aligned} AG &= A[B, AB, \dots, A^{n-1}B] \\ &= [AB, A^2B, \dots, A^nB] \end{aligned}$$

Now, it suffices to apply the Cayley-Hamilton theorem that states states that every square matrix $A$ over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

Then, $A^nB = \sum_{i=0}^{n-1} a_iA^iB$, and

$$\begin{array}{l} [AB, A^2B, \dots, A^nB] = \\ [AB, A^2B, \dots, \sum_{i=0}^{n-1} a_iA^iB] = \\ [AB, A^2B, \dots, A^{n-1}B] \subset \\ [B, AB, \dots, A^{n-1}B]. \end{array}$$

□

Now, we consider the following sequence of matrices called $r$-controllability matrices

$C_1 = (B) \in M_{n\times m}$

$C_2 = \begin{pmatrix} I & B & \\ A & 0 & B \end{pmatrix} \in M_{n\cdot 2\times n\cdot 1+m\cdot 2}$

$$C_r = \begin{pmatrix} I & & & & B & & & \\ A & I & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & I & & & & B & \\ & & & A & & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

**Theorem 17.** *Let $C_r$ be the $r$-controllability matrix. Suppose $r$ being the least such that* $\operatorname{rank} C_r < (n(r-1) + mr)$*, and let $\begin{pmatrix} v_1^t & \dots & v_{r-1}^t & w_1^t & \dots & w_r^t \end{pmatrix}^t \in \operatorname{Ker} C_r$ ($v_i$ are vectors in $\mathbb{C}^n$ and $w_i$ vectors in $\mathbb{C}^m$). Then $G = [v_1, \dots, v_{r-1}]$ is a $(A, B)$-invariant subspace.*

*Proof.* Taking into account that $\begin{pmatrix} v_1^t & \dots & v_{r-1}^t & w_1^t & \dots & w_r^t \end{pmatrix}^t \in \operatorname{Ker} C_r$ we have that

$$\begin{pmatrix} I & & & & B & & & \\ A & I & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & I & & & & B & \\ & & & A & & & & & B \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{r-1} \\ w_1 \\ \vdots \\ w_r \end{pmatrix} =$$

$$\begin{pmatrix} v_1 + Bw_1 \\ Av_1 + v_2 + Bw_2 \\ \vdots \\ Av_{r-2} + v_{r-1} + Bw_{r-1} \\ Av_{r-1} + Bw_r \end{pmatrix} = 0$$

Now we consider $v = \lambda_1v_1 + \lambda_2v_2 + \dots + \lambda_{r-2}v_{r-2} + \lambda_{r-1}v_{r-1}$ a vector in $[v_1, \dots, v_{r-1}]$. So,

$Av = \lambda_1Av_1 + \lambda_2Av_2 + \dots + \lambda_{r-2}Av_{r-2} + \lambda_{r-1}Av_{r-1} = \lambda_1(-v_2 - Bw_2) + \lambda_2(-v_3 - Bw_3) + \dots + \lambda_{r-2}(-v_{r-1} - Bw_{r-1}) - \lambda_{r-1}Bw_{r+} = (\lambda_1v_2 - \lambda_2v_3 - \dots - \lambda_{r-1}v_r) + B(-\lambda_1w_2 - \lambda_2w_3 - \dots - \lambda_{r-2}w_{r-1} - \lambda_{r-1}w_r) \in G + \operatorname{Im} B$. Then, the subspace $[v_1, \dots, v_{r-1}]$ is $(A, B)$-invariant. □

**Definition 18.** *The space sum of all spaces $G$ in theorem before is a invariant subspace that we will call controllability subspace and we will denote it by $\mathcal{C}(A, B)$.*

Notice that $\mathcal{C}(A, B)$ is the set of states in which the system is controllable.

**Corollary 19.** *Let $(A, B)$ be a pair of matrices. In this case the invariant subspace $G$ obtained in the above theorem, coincides with the controllability $(A, B)$-invariant subspaces $[B, AB, \dots, A^{r-1}B]$.*

*Proof.* Making block-row elemental transformations to the matrix $C_r$ we obtain the equivalent matrix

$$\widetilde{C}_r = P{\cdot}C_r = \begin{pmatrix} I & & & B & & & \\ 0 & I & & -AB & B & & \\ & \ddots & \ddots & & & \ddots & \\ & & I & (-1)^{r-2}A^{r-2}B & & -AB & B \\ & & 0 & (-1)^{r-1}A^{r-1}B & & & -AB & B \end{pmatrix}.$$

with $P \in Gl(n \cdot r; \mathbb{C})$. Then $\begin{pmatrix} v_1^t & \dots & v_{r-1}^t & w_1^t & \dots & w_r^t \end{pmatrix}^t \in \operatorname{Ker} C_r$ if and only if $\begin{pmatrix} v_1^t & \dots & v_{r-1}^t & w_1^t & \dots & w_r^t \end{pmatrix}^t \in \operatorname{Ker} \widetilde{C}_r$. So, $v_1 = -Bw_1 \in [B]$, $v_2 = ABw_1 - Bw_2 \in [BAB]$, ..., and $v_{r-1} = (-1)^r A^{r-2} w_1 + \dots - Bw_{r-1} \in [BAB \dots A^{r-2}B]$. □

### 5.1 Controllability subspaces of multiagent systems

Writing

$$\mathcal{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathcal{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix},$$

$$\mathcal{U}(t) = \begin{pmatrix} u^1(t) \\ \vdots \\ u^k(t) \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

Following this notation we can describe the multisystem as a system:

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{U}(t) \tag{5}$$

Clearly, we have the following result

**Proposition 20.** *The system 5 is controllable if and only if each subsystem is controllable, and, in this case, there exist a feedback in which we obtain the desired solution.*

We consider the vector space $\mathbb{R}^{n_1} \times \overset{k}{\dots} \times \mathbb{R}^{n_k}$ and a subspace $\mathcal{H} = H_1 \times \ldots \times H_k$ a subspace.

(Observe that the decomposition of $\mathcal{H}$ in product of subspaces $H_i$ in each factor $\mathbb{R}^{n_i}$ is unique).

With these notations we have

$$\mathcal{A}\mathcal{H} = \begin{pmatrix} A_1 H_1 & & \\ & \ddots & \\ & & A_k H_k \end{pmatrix}$$

and

$$\mathcal{H} + \operatorname{Im} \mathcal{B} = \begin{pmatrix} H_1 + \operatorname{Im} B_1 & & \\ & \ddots & \\ & & H_k + \operatorname{Im} B_k \end{pmatrix}$$

So, we have the following proposition.

**Proposition 21.** *The subspace $\mathcal{H}$ is $(\mathcal{A}, \mathcal{B})$-invariant, if and only if each $H_i$ is $(A_i, B_i)$-invariant.*

In the particular case where $A_1 = \ldots = A_k$, we generate subspaces $(\mathcal{A}, \mathcal{B})$- invariants by making the product of $k$ subspaces $(A, B)$-invariants, equal or not.

Consider now, the feedback matrices in the form

$$\mathcal{F} = \begin{pmatrix} F_1 & & \\ & \ddots & \\ & & F_k \end{pmatrix},$$

with $F_i \in M_{n_i \times m_i}(\mathbb{C})$,

**Corollary 22.** *Let $\mathcal{H}$ a $(\mathcal{A}, \mathcal{B})$-invariant subspace. Then, for all feedback $\mathcal{F}$, $\mathcal{H}$ is a $(\mathcal{A} + \mathcal{B}\mathcal{F}, \mathcal{B})$-invariant subspace.*

## 6 Consensus

We are interested in take the output of the system to a reference value and keep it there, we can ensure that if the system is controllable.

Roughly speaking, we can define the consensus as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. More concretely, the consensus problem is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

**Definition 23.** *Consider the system 1 having all systems identical linear dynamic mode. We say that the consensus is achieved using local information if there is a state feedback $u^i = K \sum_{j \in \mathcal{N}_i} (x^i - x^j)$ such that*

$$\lim_{t \to \infty} \|x^i - x^j\| = 0, \ 1 \le i, j \le k.$$

The closed-loop system obtained under this feedback is as follows

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{K}\mathcal{Z}$$

where $\mathcal{X}$, $\dot{\mathcal{X}}$, $\mathcal{A}$, $\mathcal{B}$ are as before and

$$\mathcal{K} = \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix}, \ \mathcal{Z} = \begin{pmatrix} \sum_{j \in \mathcal{N}_1} x^1 - x^j \\ \vdots \\ \sum_{j \in \mathcal{N}_k} x^k - x^j \end{pmatrix}.$$

Following this notation we can conclude the following.

**Proposition 24.** *The closed-loop system can be described as*

$$\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))\mathcal{X} \quad (6)$$

Calling $\mathcal{A}_1 = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))$ the system is written as $\dot{\mathcal{X}} = \mathcal{A}_1\mathcal{X}$.

Assuming $\mathcal{X}(0) = 0$, the equation 6 can be solved as

$$\mathcal{X}(t) = \int_0^t e^{((I_k\otimes A)+(I_k\otimes BK)(\mathcal{L}\otimes I_n))(t-s)}\mathcal{X}(s)dsds. \quad (7)$$

In our particular setup, we have that there exists an orthogonal matrix $P \in Gl(k,\mathbb{R})$ such that $P\mathcal{L}P^t = \mathcal{D} = \text{diag}\,(\lambda_1,\ldots,\lambda_k)$, $(\lambda_1 \geq \ldots \geq \lambda_k)$.

**Corollary 25.** *The closed-loop system can be described in terms of the matrices $A$, $B$, the feedback $K$, the output injection $W$ and the eigenvalues of $\mathcal{L}$.*

*Proof.* Following properties of Kronecker product we have that

$$\begin{array}{l} (P \otimes I_n)(I_k \otimes A)(P^t \otimes I_n) = (I_k \otimes A) \\ (P \otimes I_n)(I_k \otimes BK)(P^t \otimes I_n) = \\ (I_k \otimes BK) \\ (P \otimes I_n)(\mathcal{L} \otimes I_n)(P^t \otimes I_n) = (\mathcal{D} \otimes I_n) \end{array}$$

and calling $\widehat{\mathcal{X}} = (P \otimes I_n)\mathcal{X}$, we have

$$\dot{\widehat{\mathcal{X}}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{D} \otimes I_n))\widehat{\mathcal{X}}.$$

Equivalently,

$$\dot{\widehat{\mathcal{X}}} = \begin{pmatrix} A+\lambda_1 BK & & \\ & \ddots & \\ & & A+\lambda_k BK \end{pmatrix} \widehat{X}. \quad (8)$$

□

Calling $\mathcal{A}_2$ the matrix

$$\begin{pmatrix} A+\lambda_1 BK & & \\ & \ddots & \\ & & A+\lambda_k BK \end{pmatrix},$$

the system is written as

$$\dot{\widehat{\mathcal{X}}} = \mathcal{A}_2\widehat{\mathcal{X}}.$$

Now let $\mathcal{H}$ be a $\mathcal{A}_1$-invariant subspace, i.e. $\mathcal{A}_1\mathcal{H} \subset \mathcal{H}$, we have the following proposition.

**Proposition 26.** *The subspace $\mathcal{H}$ is $\mathcal{A}_1$-invariant if and only if $(P \otimes I_n)\mathcal{H}$ is $\mathcal{A}_2$ invariant.*

*Proof.* $\mathcal{A}_1\mathcal{H} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))\mathcal{H} \subset \mathcal{H}$.

Equivalently $(P \otimes I_n((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L} \otimes I_n))(P^t \otimes I_n)(P \otimes I_n)\mathcal{H} \subset (P \otimes I_n)\mathcal{H}$

That is to say $\mathcal{A}_2(P \otimes I_n)\mathcal{H} \subset (P \otimes I_n)\mathcal{H}$. □

Other properties.

**Corollary 27.** *The system 1 is consensus stabilizable if and only if the systems $A + \lambda_i BK$ are stable by means the same $K$.*

For more information about consensus stability see [4].

# 7 Conclusion

In this paper, a review of the concept of invariant subspaces and controllability subspaces is made. These concepts have been generalized to multi-agent systems and finally we have analyzed the consensus controllability subspaces in the case of multiagent linear systems having all agents the same dynamics.


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*References:*


[1] C. Chen, *Introduction to Linear System Theory*, Holt, Rinehart and Winston Inc., New York, 1970

[2] M.I. Garcıa-Planas. (2007). *Controllability subspaces of singular systems*. International Journal of Pure and Applied Mathematics, 35(4), pp. 485-490.

[3] M.I. García Planas. (2011). *Generalized controllability subspaces for time-invariant singular linear systems*. Proceedings of 5th International Scientific Conference on Physics and Control.

[4] M.I. García Planas. (2016). *Consensus stabilizability and exact consensus controllability of multi-agent linear systems*. WSEAS transactions on systems, 15, pp. 225–231.

[5] M.I. Garcia-Planas, S. Tarragona. (2016). *On stability and controllability of multi-agent linear systems*. Int. J. Complex Systems in Science, 6: 1, pp. 9–15.

[6] A. Heniche, I. Kamwa. (2002). *Using measures of controllability and observability for input and output selection*. IEEE International Conference on Control Applications. 2, pp. 1248—1251.

[7] P. Kundur, *Power System Stability and Control*. McGraw-Hill, New York, 1994.

[8] P. Lancaster, M. Tismenetsky, *The Thoery of Matrices*. Academic–Press. San Diego, 1985

[9] C. Lin. (1974). *Structural controllability*. IEEE Trans. Automat. Contr. 19, pp. 201–208.

[10] Y. Liu, J. Slotine, A. Barabási. (2011), *Controllability of complex networks*. Nature, 473: (7346). pp. 167-–173.

[11] R.O. Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Automat. Control*. 49, (9), 2004, pp. 1520–1533.

[12] C. Sun, G. Hu, and L. Xie. (2017). Controllability of multiagent networks with antagonistic interactions. IEEE transactions on automatic control, 62(10), 5457-5462.

[13] H.L. Trentelman, A.A., Stoorvogel, M., Hautus. (2012). Control theory for linear systems. Springer Science & Business Media.

[14] J. Wang, D. Cheng, X. Hu, Consensus of multiagent linear dynamics systems, *Asian Journal of Control* 10, (2), 2008, pp. 144–155.

[15] D. West *Introduction to Graph Theory* Prentice Hall (3rd Edition), 2007.

[16] G. Xie, L. Wang, Consensus control for a class of networks of dynamic agents: switching topology, *Proc. 2006 Amer. Contro. Conf.*, 2007, pp. 1382–1387.

[17] W.M. Wonham. (1979) Controllability Subspaces. In: Linear Multivariable Control: a Geometric Approach. Applications of Mathematics, vol 10. Springer, New York, NY.

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