An Extended Solution to the Equations Describing a 3-Conductor Transmission Line

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Abstract: - The full derivation of the generalized and extended solution to the equations describing three-conductor transmission line is given in this paper; the brief results are presented in a previous paper. The considerations proceed from the C. Paul formulation of lossless transmission lines terminated by linear loads. In contrast to C. Paul, the conjoint interaction between the two lines is considered here and the influence of the receptor line is not neglected, that is the weak-coupling approximation is not applied. In result, an extended and generalized mathematical model compared the original model of C. Paul is obtained. In particular, a mixed problem for the hyperbolic system describing the three-conductor transmission line is formulated. It is shown that the formulated mixed problem is equivalent to an initial value problem for a functional system on the boundary of hyperbolic system's domain with voltages and currents as the unknown functions in this system are the lines'. The system of functional equations can be resolved by a fixed-point method that enables us to find an approximated but explicit solution. The method elaborated in this paper might be applied also for linear as well as nonlinear boundary conditions.

Key-Words: three-conductor transmission line, electromagnetic compatibility, fixed point method, linear hyperbolic system, initial-boundary problem, mixed problem for hyperbolic system

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1 Introduction

VLSI systems and their electromagnetic compatibility (EMC) aspects have attracted a lot of research interest (cf. [1]-[8]). In this paper an EMC model of a 3-conductor transmission line is considered taking into account the results of Clayton R. Paul [9]. In contrast to the considerations in [9], a generalized approach for solving the above problem is proposed. It is also proved that the weak coupling assumptions introduced in [9] are a particular case of the more general handling.

We obtain a general solution of the system by modeling pairwise the interacting 3-conductor transmission line introduced in [9] by keeping to the methods in [11]-[13] that were also used in other solutions such as [14], [15]. Our starting circuit is given in Fig. 1 (cf. [9]). The reference conductor for the line voltages is denoted by the ground symbol. Although it could represent an infinite ground plane, a wire, an overall shield, in our setup it is a land of a printed circuit board. The rest conductors are also lands of a printed circuit board, nevertheless they could be other objects as well. We presume the line to be an uniform and lossless line (cf. [7], [8]).

The top circuit is the generator circuit. It is terminated by a resistive load R_L and it is driven by a voltage source with open-circuit voltage $U_S(t)$ and source resistance R_S. The bottom circuit is the receptor circuit. It is terminated by a resistive load R_{NE} at the near end and by a resistive load R_{FE} at the far end. At the terminals of the receptor circuit, the electric and magnetic fields originating by the voltage and current of the generator circuit, interact with the receptor circuit producing crosstalk voltages.

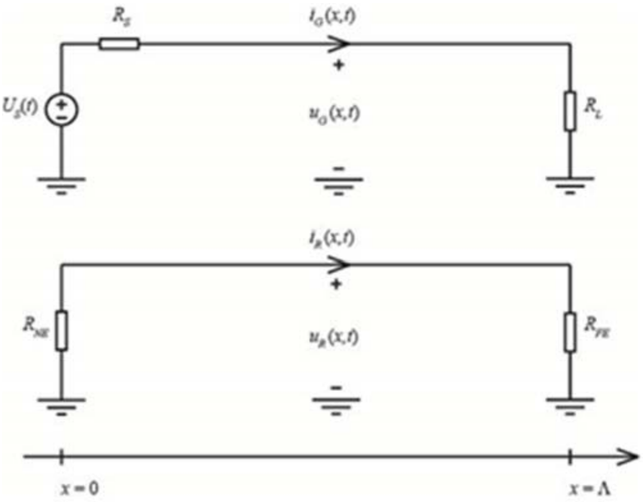

Fig. 1. *Circuit of the 3-conductor transmission line*.

We aim at finding a solution for the crosstalk voltages based on a system that is more than the one in [9]. That is, we proceed from the hyperbolic system (1) obtained in accordance to the TEM mode of propagation (cf. [1]-[8]). The voltages with respect to the reference conductor $u_k\,(x,\,t)$ (k = 1; 2) and the currents of each circuit $i_k\,(x,\,t)$ (k = 1, 2) are functions of position x and time t.

$$\left|\begin{array}{l}
\dfrac{\partial u_G(x,t)}{\partial x}+L_G\dfrac{\partial i_G(x,t)}{\partial t}=-L_m\dfrac{\partial i_R(x,t)}{\partial t}\\
\dfrac{\partial i_G(x,t)}{\partial x}+(C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t}=C_m\dfrac{\partial u_R(x,t)}{\partial t}\\
\dfrac{\partial u_R(x,t)}{\partial x}+L_R\dfrac{\partial i_R(x,t)}{\partial t}=-L_m\dfrac{\partial i_G(x,t)}{\partial t}\\
\dfrac{\partial i_R(x,t)}{\partial x}+(C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t}=C_m\dfrac{\partial u_G(x,t)}{\partial t}
\end{array}\right. \qquad (1)$$

with the following boundary

$$\begin{array}{ll}
u_G(0,t)=U_S(t)-R_S i_G(0,t), & U_{NE}=u_R(0,t)=-R_{NE}i_R(0,t)\\
u_G(\Lambda,t)=R_\Lambda i_G(\Lambda,t), & U_{FE}=u_R(\Lambda,t)=R_{FE}i_R(\Lambda,t)
\end{array} \qquad (2)$$

and initial conditions:

$$\begin{array}{ll}
u_G(x,0)=u_{G0}(x), & u_R(x,0)=u_{R0}(x)\\
i_G(x,0)=i_{G0}(x), & i_R(x,0)=i_{R0}(x),\ x\in[0,\Lambda]
\end{array} \qquad (3)$$

Before going further, we would stress upon the fact that in our considerations, we do not apply the weak coupling assumption as opposite to [9] where this assumption is applied. This means that we do not neglect the right-hand side of (1). Therefore, our method is a more general case of (1).

Rewrite the above system (1) in the form

$$\left|\begin{array}{l}
(C_G+C_m)\dfrac{\partial u_G(x,t)}{\partial t}-C_m\dfrac{\partial u_R(x,t)}{\partial t}+\dfrac{\partial i_G(x,t)}{\partial x}=0\\
-C_m\dfrac{\partial u_G(x,t)}{\partial t}+(C_R+C_m)\dfrac{\partial u_R(x,t)}{\partial t}+\dfrac{\partial i_R(x,t)}{\partial x}=0\\
L_G\dfrac{\partial i_G(x,t)}{\partial t}+L_m\dfrac{\partial i_R(x,t)}{\partial t}+\dfrac{\partial u_G(x,t)}{\partial x}=0\\
L_m\dfrac{\partial i_G(x,t)}{\partial t}+L_R\dfrac{\partial i_R(x,t)}{\partial t}+\dfrac{\partial u_R(x,t)}{\partial x}=0
\end{array}\right.$$

and introduce denotations

$$u_1(x,t)=u_G(x,t);\quad u_2(x,t)=u_R(x,t);$$
$$i_1(x,t)=i_G(x,t);\quad i_2(x,t)=i_R(x,t)$$

$$C_{11}=C_G+C_m,\ C_{12}=C_{21}=-C_m,\ C_{22}=C_R+C_m$$
$$L_{11}=L_G,\ L_{12}=L_{21}=L_m,\ L_{22}=L_R$$

$$C=\begin{pmatrix}C_{11} & C_{12}\\ C_{12} & C_{22}\end{pmatrix},\ L=\begin{pmatrix}L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}$$

Then we reach the following mixed problem in the new denotations

$$\begin{array}{l}
C_{11}\dfrac{\partial u_1(x,t)}{\partial t}+C_{12}\dfrac{\partial u_2(x,t)}{\partial t}+\dfrac{\partial i_1(x,t)}{\partial x}=0,\\
C_{12}\dfrac{\partial u_1(x,t)}{\partial t}+C_{22}\dfrac{\partial u_2(x,t)}{\partial t}+\dfrac{\partial i_2(x,t)}{\partial x}=0,\\
L_{11}\dfrac{\partial i_1(x,t)}{\partial t}+L_{12}\dfrac{\partial i_2(x,t)}{\partial t}+\dfrac{\partial u_1(x,t)}{\partial x}=0,\\
L_{12}\dfrac{\partial i_1(x,t)}{\partial t}+L_{22}\dfrac{\partial i_2(x,t)}{\partial t}+\dfrac{\partial u_2(x,t)}{\partial x}=0
\end{array}$$

$$\begin{array}{ll}
u_1(0,t)=U_S(t)-R_S i_1(0,t), & U_{NE}=u_2(0,t)=-R_{NE}i_2(0,t),\\
u_1(\Lambda,t)=R_L i_1(\Lambda,t), & U_{FE}=u_2(\Lambda,t)=R_{FE}i_2(\Lambda,t)
\end{array} \qquad (4)$$

$$\begin{array}{ll}
u_1(x,0)=u_{10}(x), & u_2(x,0)=u_{20}(x),\ x\in[0,\Lambda]\\
i_1(x,0)=i_{10}(x), & i_2(x,0)=i_{20}(x),\ x\in[0,\Lambda]
\end{array}$$

2 Hyperbolic system transformation

In a matrix form the above system (4) is

$$\begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22} \end{bmatrix} \begin{bmatrix} \partial u_1/\partial t \\ \partial u_2/\partial t \\ \partial i_1/\partial t \\ \partial i_2/\partial t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \\ \partial i_1/\partial x \\ \partial i_2/\partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

Since

$$\Delta_C = C_{11}C_{22} - C_{12}^2 = (C_G + C_m)(C_R + C_m) - C_m^2 = = C_G C_R + C_G C_m + C_R C_m > 0$$

we have to assume

Assumption (L): $\Delta_L = L_G L_R - L_m^2 = L_{11}L_{22} - L_{12}^2 \neq 0$.

This implies

$$|A| = \begin{vmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22} \end{vmatrix} = \Delta_C \Delta_L \neq 0$$

and therefore A^{-1} does exist:

$$A^{-1} = \frac{1}{\Delta_C \Delta_L} \begin{bmatrix} C_{22}\Delta_L & -C_{12}\Delta_L & 0 & 0 \\ -C_{12}\Delta_L & C_{11}\Delta_L & 0 & 0 \\ 0 & 0 & L_{22}\Delta_C & -L_{12}\Delta_C \\ 0 & 0 & -L_{12}\Delta_C & L_{11}\Delta_C \end{bmatrix} = \begin{bmatrix} C_{22}/\Delta_C & -C_{12}/\Delta_C & 0 & 0 \\ -C_{12}/\Delta_C & C_{11}/\Delta_C & 0 & 0 \\ 0 & 0 & L_{22}/\Delta_L & -L_{12}/\Delta_L \\ 0 & 0 & -L_{12}/\Delta_L & L_{11}/\Delta_L \end{bmatrix}$$

Multiplying (5) from the left by A^{-1} and in view of

$$B = \begin{bmatrix} \frac{C_{22}}{\Delta_C} & \frac{-C_{12}}{\Delta_C} & 0 & 0 \\ \frac{-C_{12}}{\Delta_C} & \frac{C_{11}}{\Delta_C} & 0 & 0 \\ 0 & 0 & \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} \\ 0 & 0 & \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{C_{22}}{\Delta_C} & \frac{-C_{12}}{\Delta_C} \\ 0 & 0 & \frac{-C_{12}}{\Delta_C} & \frac{C_{11}}{\Delta_C} \\ \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} & 0 & 0 \\ \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & 0 \end{bmatrix}$$

we obtain

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial i_1}{\partial t} \\ \frac{\partial i_2}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{C_{22}}{\Delta_C} & \frac{-C_{12}}{\Delta_C} \\ 0 & 0 & \frac{-C_{12}}{\Delta_C} & \frac{C_{11}}{\Delta_C} \\ \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} & 0 & 0 \\ \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \frac{\partial i_1}{\partial x} \\ \frac{\partial i_2}{\partial x} \end{bmatrix} = 0 \quad (6)$$

Rewrite (6) in a matrix form

$$\frac{\partial U(x,t)}{\partial t} + B\frac{\partial U(x,t)}{\partial x} = 0 \quad (7)$$

Substitute $U(x,t) = H\,Z(x,t)$ in (7):

$$H\frac{\partial Z(x,t)}{\partial t} + BH\frac{\partial Z(x,t)}{\partial x} = 0$$

and multiplying by H^{-1} we obtain

$$\frac{\partial Z(x,t)}{\partial t} + H^{-1}BH\frac{\partial Z(x,t)}{\partial x} = 0$$

We have to find H such that $H^{-1}BH = B^{\mathrm{can}}$, where

$$B^{\mathrm{can}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

and $\lambda_k (k = 1,2,3,4)$ are the eigen values of B, i.e. the roots of the equation

$$|B - \lambda I| = \begin{vmatrix} -\lambda & 0 & \frac{C_{22}}{\Delta_C} & \frac{-C_{12}}{\Delta_C} \\ 0 & -\lambda & \frac{-C_{12}}{\Delta_C} & \frac{C_{11}}{\Delta_C} \\ \frac{L_{22}}{\Delta_L} & \frac{-L_{12}}{\Delta_L} & -\lambda & 0 \\ \frac{-L_{12}}{\Delta_L} & \frac{L_{11}}{\Delta_L} & 0 & -\lambda \end{vmatrix} =$$

$$= \lambda^4 - \frac{L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}}{\Delta_C \Delta_L}\lambda^2 + \frac{1}{\Delta_C \Delta_L} =$$
$$= \frac{\Delta_C \Delta_L \lambda^4 - \left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right)\lambda^2 + 1}{\Delta_C \Delta_L} = 0$$

We suppose

Assumption (D):

$$D = \left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right)^2 - \\ -4\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right) > 0$$

For the characteristic roots we obtain

$$\lambda_1 = \sqrt{\frac{\left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right) + \sqrt{D}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}},$$

$$\lambda_2 = \sqrt{\frac{\left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right) - \sqrt{D}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}},$$

$$\lambda_3 = -\sqrt{\frac{\left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right) + \sqrt{D}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}},$$

$$\lambda_4 = -\sqrt{\frac{\left(L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}\right) - \sqrt{D}}{2\left(C_{11}C_{22} - C_{12}^2\right)\left(L_{11}L_{22} - L_{12}^2\right)}}.$$

Remark. For the sake of simplicity, we could find the eigenvectors of

$$\left(B^{-1} - \mu_k I\right)H^{(k)} = 0; \\ \mu_k = 1/\lambda_k, \quad H^{(k)} = \left(\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k}\right)^T \tag{8}$$

(instead of $\left(B - \lambda_k I\right)H^{(k)} = 0$) because

$$B^{-1} = \begin{pmatrix} 0 & 0 & L_{11} & L_{12} \\ 0 & 0 & L_{12} & L_{22} \\ C_{11} & C_{12} & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \end{pmatrix}$$

has a simpler form than B.

Now we have to solve the following systems in order to obtain 4 eigenvectors $H^{(k)}(k = 1,2,3,4)$

$$\left(B - \lambda_1 I\right)H^{(1)} = 0, \\ \left(B - \lambda_2 I\right)H^{(2)} = 0, \\ \left(B - \lambda_3 I\right)H^{(3)} = 0, \\ \left(B - \lambda_4 I\right)H^{(4)} = 0 \tag{9}$$

corresponding to eigenvector

$$H^{(k)} = \left(\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{4k}\right)^T; \quad (k = 1,2,3,4).$$

To solve (9) we have to assume:

$$L_{12}C_{11} + L_{22}C_{12} = L_m\left(C_G + C_m\right) - L_R C_m \neq 0$$

and

$$L_{12}C_{22} + L_{11}C_{12} = L_m\left(C_R + C_m\right) - L_G C_m \neq 0$$

Therefore

$$\xi_{1k} = \frac{\lambda_k^2 \Delta_L C_{12} + L_{12}}{\lambda_k\left(L_{12}C_{11} + L_{22}C_{12}\right)}, \\ \xi_{2k} = \frac{L_{22} - \lambda_k^2 \Delta_L C_{11}}{\lambda_k\left(L_{12}C_{11} + L_{22}C_{12}\right)}, \\ \xi_{3k} = 1, \\ \xi_{4k} = \frac{L_{12}C_{12} + L_{22}C_{22} - \lambda_k^2 \Delta_L \Delta_C}{\left(L_{12}C_{22} + L_{22}C_{12}\right)}.$$

Introduce denotations:

$$\gamma_k = \frac{\left(1/\lambda_k^2\right) - \left(L_{11}C_{11} + L_{12}C_{12}\right)}{L_{12}C_{11} + L_{22}C_{12}} = \\ = \frac{1 - \lambda_k^2\left(L_{11}C_{11} + L_{12}C_{12}\right)}{\lambda_k^2\left(L_{12}C_{11} + L_{22}C_{12}\right)} \quad (k=1,2).$$

Note that

$$\lambda_1 > \lambda_2 > 0;\ \lambda_3 = -\lambda_1;\ \lambda_4 = -\lambda_2; \\ \sqrt{D} \Rightarrow \gamma_2 - \gamma_1 = \frac{\sqrt{D}}{L_{12}C_{11} + L_{22}C_{12}}.$$

Then we obtain the following eigenvectors:

$$H^{(1)} = \left(p_1, q_1, 1, \gamma_1\right)^T, \quad H^{(2)} = \left(p_2, q_2, 1, \gamma_2\right)^T,$$

$$H^{(2)} = \left(-p_1, -q_1, 1, \gamma_1\right)^T, \quad H^{(1)} = \left(-p_2, -q_2, 1, \gamma_2\right)^T$$

where

$$p_k = \frac{L_{12} + \lambda_k^2 \Delta_L C_{12}}{\lambda_k\left(L_{12}C_{11} + L_{22}C_{12}\right)} = \lambda_k\left(L_{11} + L_{12}\gamma_k\right)$$

$$q_k = \frac{L_{22} - \lambda_k^2 \Delta_L C_{11}}{\lambda_k\left(L_{12}C_{11} + L_{22}C_{12}\right)} = \lambda_k\left(L_{12} + L_{22}\gamma_k\right), \quad (k = 1,2)$$

Thus transformation matrix becomes

$$H = \begin{bmatrix} p_1 & p_2 & -p_1 & -p_2 \\ q_1 & q_2 & -q_1 & -q_2 \\ 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_1 & \gamma_2 \end{bmatrix}.$$

Since $|H| = 4\sqrt{\Delta_L / \Delta_C}\left(\gamma_2 - \gamma_1\right)^2 \neq 0$

for the inverse matrix we obtain

$$H^{-1} =$$

$$= \frac{1}{2(\gamma_2 - \gamma_1)} \begin{bmatrix} q_2\sqrt{\Delta_L / \Delta_C} & -p_2\sqrt{\Delta_L / \Delta_C} & \gamma_2 & -1 \\ -q_1\sqrt{\Delta_L / \Delta_C} & p_1\sqrt{\Delta_L / \Delta_C} & -\gamma_1 & 1 \\ -q_2\sqrt{\Delta_L / \Delta_C} & p_2\sqrt{\Delta_L / \Delta_C} & \gamma_2 & -1 \\ q_1\sqrt{\Delta_L / \Delta_C} & -p_1\sqrt{\Delta_L / \Delta_C} & -\gamma_1 & 1 \end{bmatrix}$$

because

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \lambda_1 \lambda_2 \left| \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix} \right| = \sqrt{\Delta_L / \Delta_C}\left(\gamma_2 - \gamma_1\right).$$

3 Boundary conditions derivation with respect to the new variables

Introduce new variables $U = HZ$ and $Z = H^{-1}U$, where $U = \left(u_1, u_2, i_1, i_2\right)^T$, $Z = \left(I_1, I_2, I_3, I_4\right)^T$.

Then

$$u_1(x,t) = p_1 I_1(x,t) + p_2 I_2(x,t) - p_1 I_3(x,t) - p_2 I_4(x,t)$$
$$u_2(x,t) = q_1 I_1(x,t) + q_2 I_2(x,t) - q_1 I_3(x,t) - q_2 I_4(x,t)$$
$$i_1(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t)$$
$$i_2(x,t) = \gamma_1 I_1(x,t) + \gamma_2 I_2(x,t) + \gamma_1 I_3(x,t) + \gamma_2 I_4(x,t)$$

and

$$I_1(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2\sqrt{\Delta_C / \Delta_L}\, u_1(x,t) - \\ -p_2\sqrt{\Delta_C / \Delta_L}\, u_2(x,t) + \\ +\gamma_2 i_1(x,t) - i_2(x,t) \end{pmatrix}$$

$$I_2(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_1\sqrt{\Delta_C / \Delta_L}\, u_1(x,t) + \\ +p_1\sqrt{\Delta_C / \Delta_L}\, u_2(x,t) - \\ -\gamma_1 i_1(x,t) + i_2(x,t) \end{pmatrix}$$

$$I_3(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_2\sqrt{\Delta_C / \Delta_L}\, u_1(x,t) + \\ +p_2\sqrt{\Delta_C / \Delta_L}\, u_2(x,t) + \\ +\gamma_2 i_1(x,t) - i_2(x,t) \end{pmatrix}$$

$$I_4(x,t) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_1\sqrt{\Delta_C / \Delta_L}\, u_1(x,t) - \\ -p_1\sqrt{\Delta_C / \Delta_L}\, u_2(x,t) - \\ -\gamma_1 i_1(x,t) + i_2(x,t) \end{pmatrix}$$

Then the mixed problem (1)-(3) becomes as follows: to find a solution of the system

$$\begin{aligned} &\frac{\partial I_1(x,t)}{\partial t} + \lambda_1 \frac{\partial I_1(x,t)}{\partial x} = 0, \\ &\frac{\partial I_2(x,t)}{\partial t} + \lambda_2 \frac{\partial I_2(x,t)}{\partial x} = 0, \\ &\frac{\partial I_3(x,t)}{\partial t} - \lambda_1 \frac{\partial I_3(x,t)}{\partial x} = 0, \\ &\frac{\partial I_4(x,t)}{\partial t} - \lambda_2 \frac{\partial I_4(x,t)}{\partial x} = 0 \end{aligned} \tag{10}$$

with initial conditions and boundary conditions in the new variables:

$$I_1(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2\sqrt{\Delta_C / \Delta_L}\, u_{10}(x) - \\ -p_2\sqrt{\Delta_C / \Delta_L}\, u_{20}(x) + \\ +\gamma_2 i_{10}(x) - i_{20}(x) \end{pmatrix} \equiv I_{10}(x)$$

$$I_2(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} -q_1\sqrt{\Delta_C / \Delta_L}\, u_{10}(x) + \\ +p_1\sqrt{\Delta_C / \Delta_L}\, u_{20}(x) - \\ -\gamma_1 i_{10}(x) + i_{20}(x) \end{pmatrix} \equiv I_{20}(x)$$

$$I_3(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_2\sqrt{\Delta_C / \Delta_L}\, u_{10}(x) + \\ +p_2\sqrt{\Delta_C / \Delta_L}\, u_{20}(x) + \\ +\gamma_2 i_{10}(x) - i_{20}(x) \end{pmatrix} \equiv I_{30}(x)$$

$$I_4(x,0) = \frac{1}{2(\gamma_2 - \gamma_1)} \begin{pmatrix} q_1\sqrt{\Delta_C / \Delta_L}\, u_{10}(x) - \\ -p_1\sqrt{\Delta_C / \Delta_L}\, u_{20}(x) - \\ -\gamma_1 i_{10}(x) + i_{20}(x) \end{pmatrix} \equiv I_{40}(x) \tag{11}$$

To obtain the boundary conditions

$$u_1(0,t) = U_S(t) - R_S i_1(0,t), \quad u_1(\Lambda,t) = R_\Lambda i_1(\Lambda,t),$$
$$U_{NE} = u_2(0,t) = -R_{NE} i_2(0,t), \; U_{FE} = u_2(\Lambda,t) = R_{FE} i_2(\Lambda,t)$$

with respect to the new variables we take into account

$$\begin{aligned}
&u_1(0,t)=p_1 I_1(0,t)+p_2 I_2(0,t)-p_1 I_3(0,t)-p_2 I_4(0,t)\\
&u_2(0,t)=q_1 I_1(0,t)+q_2 I_2(0,t)-q_1 I_3(0,t)-q_2 I_4(0,t)\\
&i_1(0,t)=I_1(0,t)+I_2(0,t)+I_3(0,t)+I_4(0,t)\\
&i_2(0,t)=\gamma_1 I_1(0,t)+\gamma_2\ I_2(0,t)+\gamma_1 I_3(0,t)+\gamma_2 I_4(0,t)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&u_1(\Lambda,t)=p_1 I_1(\Lambda,t)+p_2 I_2(\Lambda,t)-p_1 I_3(\Lambda,t)-p_2 I_4(\Lambda,t)\\
&u_2(\Lambda,t)=q_1 I_1(\Lambda,t)+q_2 I_2(\Lambda,t)-q_1 I_3(\Lambda,t)-q_2 I_4(\Lambda,t)\\
&i_1(\Lambda,t)=I_1(\Lambda,t)+I_2(\Lambda,t)+I_3(\Lambda,t)+I_4(\Lambda,t)\\
&i_2(\Lambda,t)=\gamma_1 I_1(\Lambda,t)+\gamma_2\ I_2(\Lambda,t)+\gamma_1 I_3(\Lambda,t)+\gamma_2 I_4(\Lambda,t)
\end{aligned}$$

Consequently

$$\begin{aligned}
&p_1 I_1(0,t)+p_2 I_2(0,t)-p_1 I_3(0,t)-p_2 I_4(0,t)=\\
&\quad=U_S(t)-R_S\left[I_1(0,t)+I_2(0,t)+I_3(0,t)+I_4(0,t)\right]\\
&q_1 I_1(0,t)+q_2 I_2(0,t)-q_1 I_3(0,t)-q_2 I_4(0,t)=\\
&\quad=-R_{NE}\left[\gamma_1 I_1(0,t)+\gamma_2\ I_2(0,t)+\gamma_1 I_3(0,t)+\gamma_2 I_4(0,t)\right]\\
&p_1 I_1(\Lambda,t)+p_2 I_2(\Lambda,t)-p_1 I_3(\Lambda,t)-p_2 I_4(\Lambda,t)=\\
&\quad=R_\Lambda\left[\ I_1(\Lambda,t)+I_2(\Lambda,t)+I_3(\Lambda,t)+I_4(\Lambda,t)\right]\\
&q_1 I_1(\Lambda,t)+q_2 I_2(\Lambda,t)-q_1 I_3(\Lambda,t)-q_2 I_4(\Lambda,t)=\\
&\quad=R_{FE}\left[\gamma_1 I_1(\Lambda,t)+\gamma_2\ I_2(\Lambda,t)+\gamma_1 I_3(\Lambda,t)+\gamma_2 I_4(\Lambda,t)\right]
\end{aligned} \tag{14}$$

4 Obtaining functional equations equivalent to the mixed problem

Proceeding as in [11], we find the characteristics of the system (10) which are forming four families of curves

$$\frac{dx}{dt}=\lambda_1,\quad \frac{dx}{dt}=\lambda_2,\quad \frac{dx}{dt}=-\lambda_1,\quad \frac{dx}{dt}=-\lambda_2 \tag{15}$$

Through each point $(x,t)\in\Pi=\{(x,t)\in[0,\Lambda]\times[0,T]\}$ there are 4 characteristics: C_1, C_2 with positive slopes and C_3, C_4 with negative slopes. A characteristic C_k (k = 1,2) through a point $(0,\hat{t}_k)$ intersects the boundary x=□ at some point $(\Lambda,\hat{t}_k+T_k)$ where T_k can be found by integration of $\frac{dx}{dt}=\lambda_k$. Since the characteristic C_k is $x-\lambda_k T$ = const, then the straight line through $(0,\hat{t}_k)$ is

$$x-\lambda_k t=-\lambda_k\hat{t}_k \Rightarrow t=\frac{x}{\lambda_k}+\hat{t}_k\,.$$

Setting $x=\Lambda$ and $t=\hat{t}_k+T_k$ we obtain

$$\Lambda-\lambda_k\left(\hat{t}+T_k\right)=-\lambda_k\hat{t}\ \Rightarrow\ T_k=\Lambda/\lambda_k\,.$$

Similarly, a characteristic C_p (p = 3,4) is $x+\lambda_p T$ = const (with $\lambda_3=-\lambda_1$, $\lambda_4=-\lambda_2$) and the straight line through a point $(\Lambda,\hat{t}_p)$ is $x+\lambda_p t=\Lambda+\lambda_p\hat{t}_p$. It intersects $x=0$ at a point $(0,\hat{t}_p+T_p)$. Therefore

$$\lambda_p\left(\hat{t}_p+T_p\right)=\Lambda+\lambda_p\hat{t}_p \Rightarrow T_p=\Lambda/\lambda_p\,(p=3,4)\,, \tag{13}$$

i.e. $T_3=T_1$, $T_4=T_2$.

Introduce directional derivatives

$$D_k=\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\frac{d}{dt}=\frac{\partial}{\partial t}-\lambda_k\frac{\partial}{\partial x}\left(k=1,2,3,4\right).$$

Then system (10) can be written in the form:

$$D_k I_k=0\ (k=1,2) \tag{16}$$

$$D_k I_k=0\ (k=3,4) \tag{17}$$

Integrating each equation from (16) along the characteristic C_k from $(0,t)$ to $(\Lambda, t+T_k)$ (where the integration is a line integral along C_k), we obtain

$$I_k(\Lambda,t+T_k)=I_k(0,t)\left(t\ge 0\right).$$

In the same way, by integrating in (17) from $(0, t+T_k)$ to (Λ,t), we get

$$I_k(\Lambda,t)=I_k(0,t+T_k)\left(t\ge 0\right).$$

Present (12) in the form:

$$\begin{aligned}
&\left(p_1+R_S\right)I_1(0,t)+\left(p_2+R_S\right)I_2(0,t)=\\
&\quad=U_S(t)+\left(p_2-R_S\right)I_3(0,t)+\left(p_1-R_S\right)I_4(0,t)\\
&\left(q_1+R_{NE}\gamma_1\right)I_1(0,t)+\left(q_2+\gamma_2 R_{NE}\right)I_2(0,t)=\\
&\quad=\left(q_2-R_{NE}\gamma_2\right)I_3(0,t)+\left(q_1-R_{NE}\gamma_1\right)I_4(0,t)\\
&\left(R_\Lambda+p_2\right)I_3(\Lambda,t)+\left(p_1+R_L\right)I_4(\Lambda,t)=\\
&\quad=\left(p_1-R_\Lambda\right)I_1(\Lambda,t)+\left(p_2-R_L\right)I_2(\Lambda,t)\\
&\left(q_2+R_{FE}\gamma_2\right)I_3(\Lambda,t)+\left(q_1+R_{FE}\gamma_1\right)I_4(\Lambda,t)=\\
&\quad=\left(q_1-R_{FE}\gamma_1\right)I_1(\Lambda,t)+\left(q_2-R_{FE}\gamma_2\right)I_2(\Lambda,t)
\end{aligned} \tag{18}$$

Since

$$\begin{aligned}
\Delta_{12}&=\begin{vmatrix} p_1+R_S & p_2+R_S \\ q_1+R_{NE}\gamma_1 & q_2+R_{NE}\gamma_2 \end{vmatrix}=\\
&=\left(p_1 q_2-p_2 q_1\right)+\left(\gamma_2-\gamma_1\right)R_S R_{NE}+\\
&\quad+\left(p_1\gamma_2-p_2\gamma_1\right)R_{NE}+\left(q_2-q_1\right)R_S=
\end{aligned}$$

$$=\left(\gamma_2-\gamma_1\right)\sqrt{\Delta_L/\Delta_C}+\left(\gamma_2-\gamma_1\right)R_S R_{NE}+$$
$$+\left(\lambda_1-\lambda_2\right)\frac{C_{22}\Delta_L+L_{11}\sqrt{\Delta_L\Delta_C}}{L_{12}C_{11}+L_{22}C_{12}}R_{NE}+\left(q_2-q_1\right)R_S=$$
$$=\frac{\sqrt{D}\left(\sqrt{\Delta_L/\Delta_C}+R_S R_{NE}\right)}{L_{12}C_{11}+L_{22}C_{12}}+\frac{\left(\lambda_1-\lambda_2\right)}{\left(L_{12}C_{11}+L_{22}C_{12}\right)}\times$$
$$\times\left(\begin{array}{l}\left(C_{22}\Delta_L+L_{11}\sqrt{\Delta_L\Delta_C}\right)R_{NE}+\\ +\dfrac{\left(L_{22}+\lambda_1\lambda_2\Delta_L C_{11}\right)}{\lambda_1\lambda_2}R_S\end{array}\right)\neq$$
$$\neq 0$$

Hence, we can solve first two equations with respect to $I_1(0,t),\, I_2(0,t)$ which leads to

$$I_1(0,t)=A_{10}(t)+A_{11}I_3(0,t)+A_{12}I_4(0,t),$$

where

$$A_{10}(t)=\frac{(q_2+R_{NE}\gamma_2)U_S(t)}{\Delta_{12}};$$
$$A_{11}=\frac{2\left(p_2\gamma_2R_{NE}-q_2R_S\right)}{\Delta_{12}};$$
$$A_{12}=\frac{p_1q_2-p_2q_1-\left(q_2+q_1\right)R_S}{\Delta_{12}}+$$
$$+\frac{\left(p_1\gamma_2+p_2\gamma_1\right)R_{NE}+\left(\gamma_1-\gamma_2\right)R_S R_{NE}}{\Delta_{12}}$$

$$I_2(0,t)=A_{20}(t)+A_{21}I_3(0,t)+A_{22}I_4(0,t),$$

where

$$A_{20}(t)=-\frac{(q_1+R_{NE}\gamma_1)U_S(t)}{\Delta_{12}};$$
$$A_{21}=\frac{p_1q_2-p_2q_1+\left(q_2+q_1\right)R_S}{\Delta_{12}}-$$
$$-\frac{\left(p_1\gamma_2+p_2\gamma_1\right)R_{NE}+\left(\gamma_1-\gamma_2\right)R_S R_{NE}}{\Delta_{12}};$$
$$A_{22}=\frac{-2p_1\gamma_1+2q_1R_S}{\Delta_{12}}$$

Similarly

$$\Delta_{34}=\begin{vmatrix}p_2+R_L & p_1+R_L\\ q_2+\gamma_2R_{FE} & q_1+\gamma_1R_{FE}\end{vmatrix}=$$
$$=\left(p_2+R_L\right)\left(q_1+\gamma_1R_{FE}\right)-\left(q_2+\gamma_2R_{FE}\right)\left(p_1+R_L\right)=$$
$$=-\frac{\sqrt{D}\left(\sqrt{\Delta_L/\Delta_C}+R_{FE}R_L\right)}{L_{12}C_{11}+L_{22}C_{12}}-\frac{\lambda_1-\lambda_2}{L_{12}C_{11}+L_{22}C_{12}}\times$$
$$\times\left(\left(C_{22}\Delta_L+L_{11}\sqrt{\Delta_L\Delta_C}\right)R_{FE}+\frac{\left(L_{22}+\lambda_1\lambda_2\Delta_L C_{11}\right)}{\lambda_1\lambda_2}R_L\right)\neq$$
$$\neq 0$$

Consequently

$$I_3(\Lambda,t)=B_{11}I_1(\Lambda,t)+B_{12}I_2(\Lambda,t),$$

where

$$B_{11}=\frac{2p_1\gamma_1R_{FE}-2q_1R_L}{\Delta_{34}};$$
$$B_{12}=\frac{\left(p_2q_1-p_1q_2\right)-\left(q_1+q_2\right)R_L}{\Delta_{34}}+$$
$$+\frac{\left(p_2\gamma_1+p_1\gamma_2\right)R_{FE}+\left(\gamma_2-\gamma_1\right)R_L R_{FE}}{\Delta_{34}}$$

$$I_4(\Lambda,t)=B_{21}I_1(\Lambda,t)+B_{22}I_2(\Lambda,t),$$

where

$$B_{21}=\frac{\left(p_2q_1-p_1q_2\right)+\left(q_1+q_2\right)R_L}{\Delta_{34}}+$$
$$+\frac{\left(\gamma_2-\gamma_1\right)R_L R_{FE}-\left(p_2\gamma_1+p_1\gamma_2\right)R_{FE}}{\Delta_{34}}.$$
$$B_{22}=\frac{2q_2R_L-2p_2\gamma_2R_{FE}}{\Delta_{34}}$$

So, we have obtained a system of functional equations

$$\begin{array}{l}I_1(0,t)=A_{10}(t)+A_{11}I_3(0,t)+A_{12}I_4(0,t)\\ I_2(0,t)=A_{20}(t)+A_{21}I_3(0,t)+A_{22}I_4(0,t)\\ I_3(\Lambda,t)=B_{11}I_1(\Lambda,t)+B_{12}I_2(\Lambda,t)\\ I_4(\Lambda,t)=B_{21}I_1(\Lambda,t)+B_{22}I_2(\Lambda,t)\end{array}.$$

Taking into account $I_k(\Lambda,t-T_k)=I_k(0,t)$, $(k=3,4)$ we can rewrite the first two equations in the following way:

$$\begin{array}{l}I_1(0,t)=A_{10}(t)+A_{11}I_3(\Lambda,t-T_3)+A_{12}I_4(\Lambda,t-T_4)\\ I_2(0,t)=A_{20}(t)+A_{21}I_3(\Lambda,t-T_3)+A_{22}I_4(\Lambda,t-T_4)\end{array}$$

Similarly in view of $I_k(\Lambda,t)=I_k(0,t-T_k)$ $(k=1,2)$ we obtain

$$\begin{array}{l}I_3(\Lambda,t)=B_{11}I_1(0,t-T_1)+B_{12}I_2(0,t-T_2)\\ I_4(\Lambda,t)=B_{21}I_1(0,t-T_1)+B_{22}I_2(0,t-T_2)\end{array}.$$

Denoting the unknown functions by

$$I_1(0,t) \equiv I_1(t), \quad I_2(0,t) \equiv I_2(t),$$
$$I_3(t) \equiv I_3(\Lambda,t), \quad I_4(t) \equiv I_4(\Lambda,t)$$

and taking into account $T_1 = T_3$, $T_2 = T_4$ we obtain the following system:

$$\begin{aligned} I_1(t) &= A_{10}(t) + A_{11}I_3(t-T_1) + A_{12}I_4(t-T_2) \\ I_2(t) &= A_{20}(t) + A_{21}I_3(t-T_1) + A_{22}I_4(t-T_2) \\ I_3(t) &= B_{11}I_1(t-T_1) + B_{12}I_2(t-T_2) \\ I_4(t) &= B_{21}I_1(t-T_1) + B_{22}I_2(t-T_2) \end{aligned}$$

To obtain initial conditions on the intervals $[-T_1,0],[-T_2,0]$ one can shifted the initial functions

$$u_{10}(x), \quad u_{20}(x), \quad i_{10}(x), \quad i_{20}(x)$$

from the interval $[0, \Lambda]$ along the characteristics to the intervals $[-T_1,0]$, $[-T_2,0]$ (cf. [12]).

The obtained functions after the above transformation on the boundary we denote by

$$I_{10}(t), \quad I_{20}(t), \quad I_{30}(t), \quad I_{40}(t).$$

If $u_{10}(x), u_{20}(x), i_{10}(x), i_{20}(x)$ are periodic functions then $I_{10}(t), I_{20}(t), I_{30}(t), I_{40}(t)$ are periodic functions too.

5 Operator presentation of the periodic problem

Introduce the sets

$$M_1 = \left\{ \begin{array}{l} I_1(t) \in C_{T_0}[0,\infty) : \ |I_1(t)| \le I_{01}e^{\mu(t-kT_0)}, \\ t \in [kT_0,(k+1)T_0]; \quad I_1(t) = I_{10}(t), t \in [-T_1,0] \end{array} \right\},$$

$$M_2 = \left\{ \begin{array}{l} I_2(t) \in C_{T_0}[0,\infty) : \ |I_2(t)| \le I_{02}e^{\mu(t-kT_0)}, \\ t \in [kT_0,(k+1)T_0]; \quad I_2(t) = I_{20}(t), t \in [-T_2,0] \end{array} \right\},$$

$$M_3 = \left\{ \begin{array}{l} I_3(t) \in C_{T_0}[0,\infty) : \ |I_3(t)| \le I_{03}e^{\mu(t-kT_0)}, \\ t \in [kT_0,(k+1)T_0]; \quad I_3(t) = I_{30}(t), t \in [-T_2,0] \end{array} \right\},$$

$$M_4 = \left\{ \begin{array}{l} I_4(t) \in C_{T_0}[0,\infty) : \ |I_4(t)| \le I_{04}e^{\mu(t-kT_0)}, \\ t \in [kT_0,(k+1)T_0]; \quad I_4(t) = I_{40}(t), t \in [-T_1,0] \end{array} \right\},$$

(k=0,1,2,…) $(k = 0,1,2,\ldots)$, where $C_{T_0}[0,\infty)$ is the set of all continuous T_0-periodic functions and I_{0k}, T_0, μ are positive constants and $\mu T_0 = \mu_0$ = const.

We use the technique of fixed point theory in uniform spaces (cf. [13]). For that purpose, we introduce a saturated family of pseudo-metrics in the Cartesian product

$$M = M_1 \times M_2 \times M_3 \times M_4 :$$

$$\rho^{(k)}(I_n, \overline{I}_n) = $$
$$= \max\left\{ \left| I_n(t) - \overline{I}_n(t) \right| e^{-\mu(t-kT_0)} : t \in [kT_0,(k+1)T_0] \right\}$$
$$(n = 1,2,3,4;\ k = 0,1,2,...),$$

where the index set of this family consists of all ordered fours

$$(p_1, p_2, p_3, p_4) \in N_0 \times N_0 \times N_0 \times N_0;\ N_0 = \{0,1,2,\ldots\}$$

corresponding to the initial points of the intervals

$$\begin{array}{l} [p_1T_0,(p_1+1)T_0] \times [p_2T_0,(p_2+1)T_0] \times \\ \times [p_3T_0,(p_3+1)T_0] \times [p_4T_0,(p_4+1)T_0] \end{array}.$$

Introduce maps $j_n(k): N_0 \to N_0$ $(n = 1,2)$ in the following way:

$$j_1 : N_0 \to N_0; [kT_0,(k+1)T_0] \to [kT_0 - T_1,(k+1)T_0 - T_1]$$

We suppose that $T_p = m_pT_0,\ (p = 1,2)$. Therefore

$$[kT_0 - T_p,\ (k+1)T_0 - T_p] = \left[\left(k - m_p\right)T_0, \left(k+1-m_p\right)T_0 \right]$$

and then $j_p(k): k \to k - m_p$ provided $k - m_p \ge 0$; $j_p{}^m(k) = j_p(j_p^{m-1}(k)),\ j_p{}^0(k) = k$. The definition of j_p implies that $j_p^m(k) \in N_0$ only for finite m.

Define

$$j(p_1,p_2,p_3,p_4) = \left(j_1(p_1), j_2(p_2), j_3(p_3), j_4(p_4) \right)$$
$$j^2(p_1,p_2,p_3,p_4) = \left(j_1^2(p_1), j_2^2(p_2), j_3^2(p_3), j_4^2(p_4) \right)$$

…………………………………………………….

In particular,

$$j(p,p,p,p) = \left(j_1(p), j_2(p), j_3(p), j_4(p) \right)$$

The set M turns out into a complete uniform space with a saturated family of pseudometrics (cf. [13])

$$\rho_{(p_1,p_2,p_1,p_2)}\left((I_1,I_2,I_3,I_4),(\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4)\right)= \\ =\rho^{(p_1)}(I_1,\overline{I}_1)+\rho^{(p_2)}(I_2,\overline{I}_2)+\rho^{(p_1)}(I_3,\overline{I}_3)+\rho^{(p_2)}(I_4,\overline{I}_4).$$

Now we formulate the main problem: to find a T_0-periodic solution of:

$$\begin{aligned} I_1(t)&=A_{10}(t)+A_{11}I_3(t-T_1)+A_{12}I_4(t-T_2)\\ I_2(t)&=A_{20}(t)+A_{21}I_3(t-T_1)+A_{22}I_4(t-T_2)\\ I_3(t)&=B_{11}I_1(t-T_1)+B_{12}I_2(t-T_2)\\ I_4(t)&=B_{21}I_1(t-T_1)+B_{22}I_2(t-T_2)\end{aligned}, \quad (19)$$

where

$$I_1(t)=I_{10}(t),t\in[-T_1,0],\quad I_2(t)=I_{20}(t),t\in[-T_2,0],\\ I_3(t)=I_{30}(t),t\in[-T_1,0],\quad I_4(t)=I_{40}(t),t\in[-T_2,0]$$

We define an operator with components $B=(B_1, B_2, B_3, B_4)$ by the formulas:

$$B_1(I_1,I_2,I_3,I_4)(t):=\begin{cases}A_{10}(t)+A_{11}I_3(t-T_1)+A_{12}I_4(t-T_2)\\ I_3(t)=I_{30}(t),t\in[-T_1,0]\\ I_4(t)=I_{40}(t),t\in[-T_2,0]\end{cases}\\ t\in[0,\infty)$$

$$B_2(I_1,I_2,I_3,I_4)(t):=\begin{cases}A_{20}(t)+A_{21}I_3(t-T_1)+A_{22}I_4(t-T_2)\\ I_3(t)=I_{30}(t),t\in[-T_1,0]\\ I_4(t)=I_{40}(t),t\in[-T_2,0]\end{cases}\\ t\in[0,\infty)$$

$$B_3(I_1,I_2,I_3,I_4)(t):=\begin{cases}B_{11}I_1(t-T_1)+B_{12}I_2(t-T_2)\\ I_1(t)=I_{10}(t),t\in[-T_1,0]\\ I_2(t)=V_{20}(t),t\in[-T_2,0]\end{cases}\\ t\in[0,\infty)$$

$$B_4(I_1,I_2,I_3,I_4)(t):=\begin{cases}B_{21}I_1(t-T_1)+B_{22}I_2(t-T_2)\\ I_1(t)=I_{10}(t),t\in[-T_1,0]\\ I_2(t)=I_{20}(t),t\in[-T_2,0]\end{cases}.\\ t\in[0,\infty)$$

6 Existence–uniqueness of periodic solution

The main result is:

Theorem 1. Let the following conditions be fulfilled:

$$I_{10}(.),I_{30}(.)\in C^1_{T_0}[-T_1,0],\quad I_{20}(.),I_{40}(.)\in C^1_{T_0}[-T_2,0]\\ U_S(.)\in C_{T_0}[0,\infty),\quad \breve{U}_S=\max\left\{\left|U_S(t)\right|:t\in[0,T_0]\right\}$$

$$\text{Assumptions (D) and (L) are valid;}\\ T_1=m_1T_0, T_2=m_2T_0 \text{ for positive integers } m_1,m_2 \quad (21)$$

$$\frac{\left|q_2+R_{NE}\gamma_2\right|}{\left|\Delta_{12}\right|}\breve{U}_S=\frac{1}{\left|\Delta_{12}\right|}\left|\begin{matrix}\lambda_2L_{12}+\left(L_{22}\lambda_2+R_{NE}\right)\times\\ \times\dfrac{1-\lambda_2^2\left(L_{11}C_{11}+L_{12}C_{12}\right)}{\lambda_2^2\left(L_{12}C_{11}+L_{22}C_{12}\right)}\end{matrix}\right|\breve{U}_S< \\ <\min\left\{I_{10},I_{20}\right\} \quad (22)$$

$$\frac{\left|q_1+R_{NE}\gamma_1\right|}{\left|\Delta_{12}\right|}\breve{U}_S=\frac{1}{\left|\Delta_{12}\right|}\left|\begin{matrix}\lambda_1L_{12}+\left(L_{22}\lambda_1+R_{NE}\right)\times\\ \times\dfrac{1-\lambda_1^2\left(L_{11}C_{11}+L_{12}C_{12}\right)}{\lambda_1^2\left(L_{12}C_{11}+L_{22}C_{12}\right)}\end{matrix}\right|\breve{U}_S< \\ <\min\left\{I_{10},I_{20}\right\} \quad (23)$$

Then there exists a unique T_0-periodic solution of (19).

Proof: The set $M_1\times M_2\times M_3\times M_4$ is a uniform space with the above saturated family of pseudo-metrics. In view of $\lambda_1>\lambda_2\Rightarrow\Lambda/\lambda_2>\Lambda/\lambda_1\Rightarrow T_2>T_1$ we show that B maps $M_1\times M_2\times M_3\times M_4$ into itself. It is easy to verify that all components of the operator B are periodic functions.

For $t\in[kT_0,(k+1)T_0]$ and sufficiently large $\mu>0$, having in mind (12), (13) we obtain:

$$\begin{aligned}&\left|B_1^{(k)}(I_1,I_2,I_3,I_4)(t)\right|\le\\ &\le\left|A_{10}(t)\right|+\left|A_{11}\right|\left|I_3(t-T_1)\right|+\left|A_{12}\right|\left|I_4(t-T_2)\right|\le\\ &\le\left|A_{10}(t)\right|+\left|A_{11}\right|I_{30}e^{\mu(t-T_1-kT_0)}+\left|A_{12}\right|I_{40}e^{\mu(t-T_2-kT_0)}\le\\ &\le e^{\mu(t-kT_0)}\left(\frac{\left|q_2+R_{NE}\gamma_2\right|}{\left|\Delta_{12}\right|}\breve{U}_S+\left|A_{11}\right|I_{30}e^{-\mu T_1}+\left|A_{12}\right|I_{40}e^{-\mu T_2}\right)\le\\ &\le e^{\mu(t-kT_0)}I_{10}\end{aligned}$$

$$\begin{aligned}&\left|B_2^{(k)}(I_1,I_2,I_3,I_4)(t)\right| \le\\ &\le\left|A_{20}(t)\right|+\left|A_{21}\right|\left|I_3(t-T_1)\right|+\left|A_{22}\right|\left|I_4(t-T_2)\right|\le\\ &\le\frac{\left|q_1+R_{NE}\gamma_1\right|\breve{U}_S}{\left|\Delta_{12}\right|}+\left|A_{21}\right|I_{30}e^{\mu(t-T_1-kT_0)}+\left|A_{22}\right|I_{40}e^{\mu(t-T_2-kT_0)}\le\\ &\le e^{\mu(t-kT_0)}\left(\frac{\left|q_1+R_{NE}\gamma_1\right|\breve{U}_S}{\left|\Delta_{12}\right|}+\left|A_{21}\right|I_{30}e^{-\mu T_1}+\left|A_{22}\right|I_{40}e^{-\mu T_2}\right)\le\\ &\le e^{\mu(t-kT_0)}I_{20}\end{aligned}$$

$$\begin{aligned}&\left|B_3^{(k)}(I_1,I_2,I_3,I_4)(t)\right| \le\\ &\le\left|B_{11}\right|\left|I_1(t-T_1)\right|+\left|B_{12}\right|\left|I_2(t-T_2)\right|\le\\ &\le\left|B_{11}\right|I_{10}e^{\mu(t-T_1-kT_0)}+\left|B_{12}\right|I_{20}e^{\mu(t-T_2-kT_0)}\le\\ &\le e^{\mu(t-kT_0)}\left(\left|B_{11}\right|I_{10}e^{-\mu T_1}+\left|B_{12}\right|I_{20}e^{-\mu T_2}\right)\le\\ &\le e^{\mu(t-kT_0)}I_{30}\end{aligned}$$

$$\begin{aligned}&\left|B_4^{(k)}(I_1,I_2,I_3,I_4)(t)\right| \le\\ &\le\left|B_{21}\right|\left|I_1(t-T_1)\right|+\left|B_{22}\right|\left|I_2(t-T_2)\right|\le\\ &\le\left|B_{21}\right|I_{10}e^{\mu(t-T_1-kT_0)}+\left|B_{22}\right|I_{20}e^{\mu(t-T_2-kT_0)}\le\\ &\le e^{\mu(t-kT_0)}\left(\left|B_{21}\right|I_{10}e^{-\mu T_1}+\left|B_{22}\right|I_{20}e^{-\mu T_2}\right)\le\\ &\le e^{\mu(t-kT_0)}I_{40}\end{aligned}$$

It remains to show that B is contractive operator. We notice that $T_2>T_1\Rightarrow-\mu T_2<-\mu T_1\Rightarrow e^{-\mu T_2}<e^{-\mu T_1}$.

Then

$$\begin{aligned}&\max\left\{\begin{matrix}\left|A_{11}\right|e^{-\mu T_1}+\left|A_{12}\right|e^{-\mu T_2};\ \left|A_{21}\right|e^{-\mu T_1}+\left|A_{22}\right|e^{-\mu T_2};\\ \left|B_{11}\right|e^{-\mu T_1}+\left|B_{12}\right|e^{-\mu T_2};\ \left|B_{21}\right|e^{-\mu T_1}+\left|B_{22}\right|e^{-\mu T_2}\end{matrix}\right\}\le\\ &\le e^{-\mu T_1}\max\left\{\begin{matrix}\left|A_{11}\right|+\left|A_{12}\right|;\ \left|A_{21}\right|+\left|A_{22}\right|;\\ \left|B_{11}\right|+\left|B_{12}\right|;\ \left|B_{21}\right|+\left|B_{22}\right|\end{matrix}\right\}=K(\mu)\end{aligned}$$

and for $t\in[kT_0,(k+1)T_0]$ we have

$$\begin{aligned}&\left|B_1^{(k)}(I_3,I_4)(t)-B_1^{(k)}(\bar I_3,\bar I_4)(t)\right|\le\\ &\le\left|A_{11}\right|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|+\left|A_{12}\right|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|\le\\ &\le\left|A_{11}\right|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+\\ &\quad+\left|A_{12}\right|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\le\\ &\le e^{\mu(t-kT_0)}\left[\left|A_{11}\right|e^{-\mu T_1}\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\left|A_{12}\right|e^{-\mu T_2}\rho^{j_2(k)}\left(I_4,\bar I_4\right)\right]\end{aligned}$$

that implies

$$\begin{aligned}&\rho^{(k)}\left(B_1^{(k)}(I_3,I_4),B_1^{(k)}(\bar I_3,\bar I_4)\right)\le\\ &\le\left|A_{11}\right|e^{-\mu T_1}\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\left|A_{12}\right|e^{-\mu T_2}\rho^{j_2(k)}\left(I_4,\bar I_4\right)\le\\ &\le\left(\left|A_{11}\right|e^{-\mu T_1}+\left|A_{12}\right|e^{-\mu T_2}\right)\left(\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\right)\le\\ &\le\left(\left|A_{11}\right|e^{-\mu T_1}+\left|A_{12}\right|e^{-\mu T_2}\right)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\bar I_1\right)+\rho^{j_2(k)}\left(I_2,\bar I_2\right)+\\ +\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\end{pmatrix}\le\\ &\le K(\mu)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\bar I_1\right)+\rho^{j_2(k)}\left(I_2,\bar I_2\right)+\\ +\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}&\left|B_2^{(k)}(I_3,I_4)(t)-B_2^{(k)}(\bar I_3,\bar I_4)(t)\right|\le\\ &\le\left|A_{21}\right|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|+\left|A_{22}\right|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|\le\\ &\le\left|A_{21}\right|\left|I_3(t-T_1)-\bar I_3(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+\\ &\quad+\left|A_{22}\right|\left|I_4(t-T_2)-\bar I_4(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\\ &\le e^{\mu(t-kT_0)}\left[\left|A_{21}\right|e^{-\mu T_1}\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\left|A_{22}\right|e^{-\mu T_2}\rho^{j_2(k)}\left(I_4,\bar I_4\right)\right]\end{aligned}$$

which implies

$$\begin{aligned}&\rho^{(k)}\left(B_2^{(k)}(I_3,I_4),B_2^{(k)}(\bar I_3,\bar I_4)\right)\le\\ &\le\left|A_{21}\right|e^{-\mu T_1}\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\left|A_{22}\right|e^{-\mu T_2}\rho^{j_2(k)}\left(I_4,\bar I_4\right)\le\\ &\le\left(\left|A_{21}\right|e^{-\mu T_1}+\left|A_{22}\right|e^{-\mu T_2}\right)\left(\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\right)\le\\ &\le\left(\left|A_{21}\right|e^{-\mu T_1}+\left|A_{22}\right|e^{-\mu T_2}\right)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\bar I_1\right)+\rho^{j_2(k)}\left(I_2,\bar I_2\right)+\\ +\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\end{pmatrix}\le\\ &\le K(\mu)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\bar I_1\right)+\rho^{j_2(k)}\left(I_2,\bar I_2\right)+\\ +\rho^{j_1(k)}\left(I_3,\bar I_3\right)+\rho^{j_2(k)}\left(I_4,\bar I_4\right)\end{pmatrix}\end{aligned}$$

Further on we have

$$\begin{aligned}&\left|B_3^{(k)}(I_1,I_2)(t)-B_3^{(k)}(\bar I_1,\bar I_2)(t)\right|\le\\ &\le\left|B_{11}\right|\left|I_1(t-T_1)-\bar I_1(t-T_1)\right|+\left|B_{12}\right|\left|I_2(t-T_2)-\bar I_2(t-T_2)\right|\le\\ &\le\left|B_{11}\right|\left|I_1(t-T_1)-\bar I_1(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+\\ &\quad+\left|B_{12}\right|\left|I_2(t-T_2)-\bar I_2(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\le\\ &\le e^{\mu(t-kT_0)}\left[\left|B_{11}\right|e^{-\mu T_1}\rho^{j_1(k)}\left(I_1,\bar I_1\right)+\left|B_{12}\right|e^{-\mu T_2}\rho^{j_2(k)}\left(I_2,\bar I_2\right)\right]\end{aligned}$$

which implies

$$
\begin{aligned}
&\rho^{(k)}\left(B_3^{(k)}(I_1,I_2),B_3^{(k)}(\overline{I}_1,\overline{I}_2)\right)\le\\
&\quad\le|B_{11}|e^{-\mu T_1}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+|B_{12}|e^{-\mu T_2}\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)\le\\
&\quad\le\left(|B_{11}|e^{-\mu T_1}+|B_{12}|e^{-\mu T_2}\right)\left(\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)\right)\le\\
&\quad\le\left(|B_{11}|e^{-\mu T_1}+|B_{12}|e^{-\mu T_2}\right)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)+\\+\rho^{j_2(k)}\left(I_3,\overline{I}_3\right)+\rho^{j_1(k)}\left(I_4,\overline{I}_4\right)\end{pmatrix}\le\\
&\quad\le K(\mu)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)+\\+\rho^{j_2(k)}\left(I_3,\overline{I}_3\right)+\rho^{j_1(k)}\left(I_4,\overline{I}_4\right)\end{pmatrix}
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\left|B_4^{(k)}(I_1,I_2)(t)-B_4^{(k)}(\overline{I}_3,\overline{I}_4)(t)\right|\le\\
&\quad\le|B_{21}|\left|I_1(t-T_1)-\overline{I}_1(t-T_1)\right|+|B_{22}|\left|I_{22}(t-T_2)-\overline{I}_2(t-T_2)\right|\le\\
&\quad\le|B_{21}|\left|I_1(t-T_1)-\overline{I}_1(t-T_1)\right|e^{-\mu(t-kT_0-T_1)}e^{\mu(t-kT_0-T_1)}+\\
&\qquad+|B_{22}|\left|I_2(t-T_2)-\overline{I}_2(t-T_2)\right|e^{-\mu(t-kT_0-T_2)}e^{\mu(t-kT_0-T_2)}\le\\
&\quad\le e^{\mu(t-kT_0)}\left[|B_{21}|e^{-\mu T_1}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+|B_{22}|e^{-\mu T_2}\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)\right]
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\rho^{(k)}\left(B_4^{(k)}(I_1,I_2),B_4^{(k)}(\overline{I}_1,\overline{I}_2)\right)\le\\
&\quad\le|B_{21}|e^{-\mu T_1}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+|B_{22}|e^{-\mu T_2}\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)\le\\
&\quad\le\left(|B_{21}|e^{-\mu T_1}+|B_{22}|e^{-\mu T_2}\right)\left(\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)\right)\le\\
&\quad\le\left(|B_{21}|e^{-\mu T_1}+|B_{22}|e^{-\mu T_2}\right)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)+\\+\rho^{j_1(k)}\left(I_3,\overline{I}_3\right)+\rho^{j_2(k)}\left(I_4,\overline{I}_4\right)\end{pmatrix}\le\\
&\quad\le K(\mu)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)+\\+\rho^{j_1(k)}\left(I_3,\overline{I}_3\right)+\rho^{j_2(k)}\left(I_4,\overline{I}_4\right)\end{pmatrix}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\rho^{(k)}\begin{pmatrix}\left(B_1^{(k)}(I_3,I_4),B_2^{(k)}(I_3,I_4),B_3^{(k)}(I_1,I_2),B_4^{(k)}(I_1,I_2)\right),\\\left(B_1^{(k)}(\overline{I}_3,\overline{I}_4),B_2^{(k)}(\overline{I}_3,\overline{I}_4),B_3^{(k)}(\overline{I}_1,\overline{I}_2),B_4^{(k)}(\overline{I}_1,\overline{I}_2)\right)\end{pmatrix}\le\\
&\quad\le 4K(\mu)\begin{pmatrix}\rho^{j_1(k)}\left(I_1,\overline{I}_1\right)+\rho^{j_2(k)}\left(I_2,\overline{I}_2\right)+\\+\rho^{j_1(k)}\left(I_3,\overline{I}_3\right)+\rho^{j_2(k)}\left(I_4,\overline{I}_4\right)\end{pmatrix}
\end{aligned}
$$

It is easy to see that $j^n(k,k,k,k)<Q(k)<\infty$ (n=1,2,…) is uniformly bounded by n; Q is a positive constant not depending on n. Indeed, every interval goes to the left from the initial point after a finite number n of iteration of j. This means that the operator B is contractive one in the sense of definition from [13] and has a unique fixed point $\left(I_1(t),I_2(t),I_3(t),I_4(t)\right)$, which is a solution of (19).

Finally, we note that the solution can be approximated by a sequence of successive approximations with advanced prescribed accuracy.

Theorem 1 is thus proved.

7 Results validation

Since our goal is to find $U_{NE}=u_2(0,t);U_{FE}=u_2(\Lambda,t)$ we have (cf. (12), (13)):

$$
\begin{aligned}
u_2(0,t)&=q_1I_1(0,t)+q_2I_2(0,t)-q_1I_3(0,t)-q_2I_4(0,t)=\\
&=q_1I_1(t)+q_2I_2(t)-q_1I_3(t-T_1)-q_2I_4(t-T_2)
\end{aligned}
$$

$$
\begin{aligned}
u_2(\Lambda,t)&=q_1I_1(\Lambda,t)+q_2I_2(\Lambda,t)-q_1I_3(\Lambda,t)-q_2I_4(0,t)=\\
&=q_1I_1(t-T_1)+q_2I_2(t-T_2)-q_1I_3(t)-q_2I_4(t)
\end{aligned}
$$

where $\left(I_1(t),I_2(t),I_3(t),I_4(t)\right)$ is the solution obtained in the above theorem.

We have to check the conditions of our Theorem 1 referring to the data from [9] :

$L_{11}=L_G=L_R=L_{22}=0.8529$ μH/m;

$L_m=0.3725$ μH/m;

$L_{12}=L_{21}=L_m$;

$C_{11}=C_G+C_m=C_R+C_m=C_{22}=46:762$ pF/m;

$C_{12}=C_{21}=-C_m=-18.036$ pF/m;

$L_{12}C_{11}+L_{22}C_{12}=L_m(C_G+C_m)-L_RC_m=0.3725\times$ $\times 46.762-0.8529\times 18.036=2.036\neq 0$;

$L_{12}C_{22}+L_{11}C_{12}=L_m(C_R+C_m)-L_GC_m=0.3725\times$ $\times 46.762-0.8529\times 18.036=2.036\neq 0$;

$\Delta_C=C_{11}C_{22}-C^2_{12}=(C_G+C_m)(C_R+C_m)-C^2_m=$ $=46.762^2-(-18.036)^2\approx 1861.3874>0$;

$\Delta_L=L_GL_R-L^2_m=L_{11}L_{22}-L^2_{12}=$ $=0.8529^2-0.3725^2\approx 0.5887>0$;

$$
\lambda_1=\sqrt{\begin{array}{l}\dfrac{L_{11}C_{11}+2L_{12}C_{12}+L_{22}C_{22}}{2\Delta_C\Delta_L}+\\+\dfrac{\sqrt{\left(L_{11}C_{11}+2L_{12}C_{12}\right)^2-4\Delta_C\Delta_L}}{2\Delta_C\Delta_L}\end{array}}\approx
$$

$$
\approx\sqrt{0.0321157}\approx 0.1792
$$

$$\lambda_2 = \sqrt{\frac{L_{11}C_{11} + 2L_{12}C_{12} + L_{22}C_{22}}{2\Delta_C\Delta_L} - \frac{\sqrt{\left(L_{11}C_{11} + 2L_{12}C_{12}\right)^2 - 4\Delta_C\Delta_L}}{2\Delta_C\Delta_L}} \approx ;$$
$$\approx \sqrt{0.0284} \approx 0.1686$$

$L_{22} + \lambda_1\lambda_2\Delta_L C_{11} = 0.8529 + 0.1792 \times 0.1686 \times 0.5887 \times$
$\times 16.762 \approx 1.6846$;

$$C_{22}\Delta_L + L_{11}\sqrt{\Delta_L\Delta_C} = 46.762 \times 0.5887 + \\ + 0.8529\sqrt{0.5887 \times 1861.3874} \approx 55.7546 ;$$

$$\Delta_{12} = \frac{\sqrt{D}\left(\sqrt{\Delta_L/\Delta_C} + R_S R_{FE}\right) + \lambda_1 - \lambda_2}{L_{12}C_{11} + L_{22}C_{12}} \times \\ \times\left(\left(C_{22}\Delta_L + L_{11}\sqrt{\Delta_L\Delta_C}\right)R_{NE} + \frac{L_{22} + \lambda_1\lambda_2\Delta_L C_{11}}{\lambda_1\lambda_2}R_S\right) = \\ = 0.427 + 24.051 R_S R_{NE} + 3.5014 R_{NE} + 19.6 R_S ;$$

$$\Delta_{34} = \frac{\sqrt{D}\left(\sqrt{\Delta_L/\Delta_C} + R_\Lambda R_{FE}\right)}{L_{12}C_{11} + L_{22}C_{12}} - \frac{\lambda_1 - \lambda_2}{L_{12}C_{11} + L_{22}C_{12}} \times \\ \times\left(\left(C_{22}\Delta_L + L_{11}\sqrt{\Delta_L\Delta_C}\right)R_{NE} + \frac{L_{22} + \lambda_1\lambda_2\Delta_L C_{11}}{\lambda_1\lambda_2}R_\Lambda\right) = \\ = -(0.427 + 24.051 R_\Lambda R_{NE} + 3.5014 R_{NE} + 19.6 R_\Lambda);$$

$$\gamma_1 = \frac{1 - \lambda_1^2(L_{11}C_{11} + L_{12}C_{12})}{\lambda_1^2(L_{12}C_{11} + L_{22}C_{12})} = \\ = \frac{1 - 0.0321157(0.8529 \times 46.762 - 0.3725 \times 18.036)}{0.0321157(0.3725 \times 46.762 - 0.8529 \times 18.036)} \approx \\ \approx \frac{1 - 1.0646}{0.065386} \approx -0.987979;$$

$$\gamma_2 = \frac{1 - \lambda_2^2(L_{11}C_{11} + L_{12}C_{12})}{\lambda_2^2(L_{12}C_{11} + L_{22}C_{12})} = \\ = \frac{1 - 0.0284(0.8529 \times 46.762 - 0.3725 \times 18.036)}{0.0284(0.3725 \times 46.762 - 0.8529 \times 18.036)} \approx \\ \approx \frac{1 - 0.94}{0.0578} \approx 1.038$$

The inequalities from the main theorem are:

$$\frac{|\lambda_1 L_{12} + (L_{22}\lambda_1 + R_{NE})\gamma_1|}{\Delta_{12}}\breve{U}_S = \\ = \frac{|0.0667 - 0.98798(0.1528 + R_{NE})|}{0.427 + (24.051 R_S + 3.5014)R_{NE} + 19.6 R_S}\breve{U}_S \le \\ \le \min\{I_{10}, I_{20}\}$$

$$\frac{|\lambda_2 L_{12} + (L_{22}\lambda_2 + R_{NE})\gamma_2|}{\Delta_{12}}\breve{U}_S = \\ = \frac{|0.0628 + 1.038(0.1437 + R_{NE})|}{0.427 + (24.051 R_S + 3.5014)R_{NE} + 19.6 R_S}\breve{U}_S \le \\ \le \min\{I_{10}, I_{20}\}$$

In what follows we verify how the same data satisfy the conditions generated by the particular case under weak coupling assumptions. Indeed, the main system becomes

$$\left|\begin{array}{l} (C_G + C_m)\dfrac{\partial u_G(x,t)}{\partial t} + \dfrac{\partial i_G(x,t)}{\partial x} = 0 \\ -C_m\dfrac{\partial u_G(x,t)}{\partial t} + (C_R + C_m)\dfrac{\partial u_R(x,t)}{\partial t} + \dfrac{\partial i_R(x,t)}{\partial x} = 0 \\ L_G\dfrac{\partial i_G(x,t)}{\partial t} + \dfrac{\partial u_G(x,t)}{\partial x} = 0 \\ L_m\dfrac{\partial i_G(x,t)}{\partial t} + L_R\dfrac{\partial i_R(x,t)}{\partial t} + \dfrac{\partial u_R(x,t)}{\partial x} = 0 \end{array}\right.$$

recalling that

$$u_1(x,t) = u_G(x,t); \quad u_2(x,t) = u_R(x,t);$$
$$i_1(x,t) = i_G(x,t); \quad i_2(x,t) = i_R(x,t);$$
$$C_{11} = C_G + C_m; \quad C_{12} = C_{21} = -C_m; \quad C_{22} = C_R + C_m$$
$$L_{11} = L_G; \quad L_{12} = L_{21} = L_m; \quad L_{22} = L_R$$

or in a matrix form

$$\begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \\ 0 & 0 & L_{11} & 0 \\ 0 & 0 & L_{12} & L_{22} \end{bmatrix} + \begin{bmatrix} \partial u_1/\partial t \\ \partial u_2/\partial t \\ \partial i_1/\partial t \\ \partial i_2/\partial t \end{bmatrix} + \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \\ \partial i_1/\partial x \\ \partial i_2/\partial x \end{bmatrix} = 0.$$

Since

$$\Delta_C = C_{11}C_{22} = (C_G + C_m)(C_R + C_m) > 0$$
$$(L): \Delta_L = L_{11}L_{22} = L_G L_R > 0$$

it follows $|A| = \Delta_C\Delta_L \ne 0$

and therefore

$$A^{-1}=\begin{bmatrix} \frac{C_{22}}{\Delta_C} & 0 & 0 & 0 \\ \frac{-C_{12}}{\Delta_C} & C_{11}\ \Delta_C & 0 & 0 \\ 0 & 0 & \frac{L_{22}}{\Delta_L} & 0 \\ 0 & 0 & \frac{-L_{12}}{\Delta_L} & L_{11}\ \Delta_L \end{bmatrix}.$$

In view of

$$B=\begin{bmatrix} \frac{C_{22}}{\Delta_C} & 0 & 0 & 0 \\ \frac{-C_{12}}{\Delta_C} & C_{11}\ \Delta_C & 0 & 0 \\ 0 & 0 & \frac{L_{22}}{\Delta_L} & 0 \\ 0 & 0 & \frac{-L_{12}}{\Delta_L} & L_{11}\ \Delta_L \end{bmatrix}\times\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}=$$

$$=\begin{bmatrix} 0 & 0 & \frac{C_{22}}{\Delta_C} & 0 \\ 0 & 0 & \frac{-C_{12}}{\Delta_C} & C_{11}\ \Delta_C \\ \frac{L_{22}}{\Delta_L} & 0 & 0 & 0 \\ \frac{-L_{12}}{\Delta_L} & L_{11}\ \Delta_L & 0 & 0 \end{bmatrix}$$

we obtain

$$\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial i_1}{\partial t} \\ \frac{\partial i_2}{\partial t} \end{bmatrix}+\begin{bmatrix} 0 & 0 & \frac{C_{22}}{\Delta_C} & 0 \\ 0 & 0 & \frac{-C_{12}}{\Delta_C} & C_{11}\ \Delta_C \\ \frac{L_{22}}{\Delta_L} & 0 & 0 & 0 \\ \frac{-L_{12}}{\Delta_L} & L_{11}\ \Delta_L & 0 & 0 \end{bmatrix}\times\begin{bmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \\ \frac{\partial i_1}{\partial t} \\ \frac{\partial i_2}{\partial t} \end{bmatrix}=0.$$

We find the eigenvectors

$$H^{(k)}=(\xi_{1k},\xi_{2k},\xi_{3k},\xi_{4k})^T;(k=1,2,3,4)$$

of $(B^{-1}-\mu_k I)H^{(k)}=0;\ \mu_k=1/\lambda_k$, where

$$\begin{bmatrix} 0 & 0 & L_{11} & 0 \\ 0 & 0 & L_{12} & L_{22} \\ C_{11} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 \end{bmatrix}.$$

In view of $L_{12}C_{11}+L_{22}C_{12}\neq 0$ and $L_{12}C_{22}+L_{11}C_{12}\neq 0$ we obtain the roots of

$$|B^{-1}-\mu I|=(L_{11}C_{11}-\mu^2)(L_{22}C_{22}-\mu^2)=0$$

From the matrix

$$\begin{pmatrix} -\mu & 0 & L_{11} & 0 \\ 0 & -\mu & L_{12} & L_{22} \\ C_{11} & 0 & -\mu & 0 \\ C_{12} & C_{22} & 0 & -\mu \end{pmatrix}\approx$$

$$\approx\begin{pmatrix} -\mu & 0 & L_{11} & 0 \\ 0 & -\mu & L_{12} & L_{22} \\ 0 & 0 & L_{11}C_{11}-\mu^2 & 0 \\ 0 & 0 & L_{11}C_{12}+L_{12}C_{22} & L_{22}C_{22}-\mu^2 \end{pmatrix}$$

we obtain for $\mu_1=-\mu_3=\sqrt{L_{11}C_{11}}$ the eigenvectors

$$\xi_{12}=\frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}},\quad \xi_{31}=\frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}$$

$$\xi_{41}=1,\quad \xi_{13}=-\frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}}$$

$$\xi_{43}=1,\quad \xi_{23}=-\frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}},$$

$$\xi_{33}=\frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}.$$

Similarly for $\mu_2=-\mu_4=\sqrt{L_{22}C_{22}}\Rightarrow$

$$\begin{pmatrix} -\sqrt{L_{22}C_{22}} & 0 & L_{11} & 0 \\ 0 & -\sqrt{L_{22}C_{22}} & L_{12} & L_{22} \\ 0 & 0 & L_{11}C_{11}-L_{22}C_{22} & 0 \\ 0 & 0 & L_{11}C_{12}-L_{12}C_{22} & 0 \end{pmatrix}$$

we have

$$\xi_{12}=0,\quad \xi_{22}=\sqrt{L_{22}/C_{22}},\quad \xi_{32}=0,\quad \xi_{42}=1,$$
$$\xi_{14}=0,\quad \xi_{24}=-\sqrt{L_{22}/C_{22}},\quad \xi_{34}=0,\quad \xi_{44}=1.$$

Consequently, the transformation matrix is :

$$\begin{pmatrix} \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}} & 0 & -\frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}} & 0 \\ \frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}} & \sqrt{\frac{L_{22}}{C_{22}}} & -\frac{L_{12}C_{11}+L_{22}C_{12}}{L_{11}C_{12}+L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}} & -\sqrt{\frac{L_{22}}{C_{22}}} \\ \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}} & 0 & \frac{L_{11}C_{11}-L_{22}C_{22}}{L_{11}C_{12}+L_{12}C_{22}} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

And then in view of (12) we have

$$u_1(x,t) = \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}}I_1(x,t) -$$

$$-\frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}}\sqrt{\frac{L_{11}}{C_{11}}}I_3(x,t);$$

$$u_2(x,t) = \frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{12} + L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}}I_1(x,t) + \sqrt{\frac{L_{22}}{C_{22}}}I_2(x,t) -$$

$$-\frac{L_{12}C_{11} + L_{22}C_{12}}{L_{11}C_{12} + L_{12}C_{22}}\sqrt{\frac{L_{12}}{C_{11}}}I_3(x,t) - \sqrt{\frac{L_{22}}{C_{22}}}I_4(x,t);$$

$$i_1(x,t) = \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}}I_1(x,t) + \frac{L_{11}C_{11} - L_{22}C_{22}}{L_{11}C_{12} + L_{12}C_{22}}I_3(x,t);$$

$$i_2(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t) + I_4(x,t).$$

If we take the specific parameters again from [9]

$L_{11} = L_G = L_R = L_{22} = 0.8529$ μH/m;

$C_{11} = C_G + C_m = C_R + C_m = C_{22} = 46.762$ pF/m

it is obvious that $L_{11}C_{11} - L_{22}C_{22} = 0$. This implies $u_1(x,t) = u_G(x,t) \equiv 0$. The contradiction obtained shows the advantages of our method.

8 Conclusion

In this paper we presented the full derivation of the equations describing a 3-conductor transmission line terminated by linear loads. In such way, we extended the general method from [12] by shrinking the mixed problem for the hyperbolic system expressing TEM propagation lengthwise the lines to a functional system on the boundary. In result, by applying the fixed point method we can obtain in an explicit form the solution to the system of functional equations by successive approximations beginning with simple initial approximation. Our method is applicable to nonlinear boundary conditions too. Besides, in this paper we prove existence-uniqueness of a more general periodic solution and demonstrated the benefits of our method on the samples related to examinations of cross-talks. It should be noted that the method elaborated here can be applied to nonlinear boundary conditions.

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