Existence of Bounded Oscillatory Solutions to Second Order Forced Neutral Dynamic Equations with Time Delay on Time Scales

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Abstract: In this paper, we study the existence of oscillatory solutions to a class of second order forced neutral dynamic equations with time delay on time scales and give several sufficient conditions for oscillation of bounded solutions.

Key–Words: time scales; time delay; neutral; dynamic equations; oscillatory solution

1 Introduction

The theory of oscillation of neutral differential equations is rich and has wide applications, and there are much research and abundant achievements on the theory of oscillation of differential equations. In the past ten years, oscillation and nonoscillation of dynamic equations on time scales have aroused wide attention of scholars at home and abroad, and the theory of time scales is also gradually improved[1−2]. However, the research results of oscillation of dynamic equations with time delay on time scales is relatively little.

Agarwal et al.[3] studied the following second order neutral differential equation

$$x^{\Delta\Delta}(t)+p(t)x(\tau(t))=0, t\in\mathbb{T},$$

and gave several sufficient conditions for oscillation of the above equation.

Erbe et al.[4] considered second order nonlinear dynamic equation with time delay on time scales

$$[r(t)x^{\Delta}(t)]^{\Delta}+p(t)f(x(\tau(t)))=0, t\in\mathbb{T},$$

and obtained some new sufficient conditions for ensuring that every solution oscillates or converges to zero by $Riccati$ transformation techniques.

Sahiner[5] studied the oscillation of solution to the following second order neutral dynamic equation with time delay on time scales

$$[r(t)((x(t)+p(t)(x(t-\tau))^{\Delta})^{\gamma}]^{\Delta}+f(t,x(t-\delta))=0, t\in\mathbb{T}.$$

Saker[6] studied the following second order nonlinear neutral dynamic equation with time delay on time scales

$$[r(t)((x(t)+p(t)(x(\tau(t))^{\Delta})^{\gamma}]^{\Delta}+f(t,x(\delta(t)))=0, t\in\mathbb{T},$$

and gained several sufficient conditions for oscillation of the above equation.

Based on the above results, we study the existence of oscillatory solution to more general neutral equations by adding forced term, and give several sufficient conditions for oscillation of bounded solutions.

In this paper, we consider the existence of bounded oscillatory solutions to the forced second order neutral dynamic equation with time delay on time scales as follows

$$[r(t)z^{\Delta}(t)]^{\Delta}+\sum_{i=1}^{m}f_i(t,x(\delta_i(t)))=h(t),\ t\in[t_0,\infty)_{\mathbb{T}}, \tag{1}$$

where $z(t)=x(t)+p(t)x(\tau(t))$, and $\mathbb{T}$ is a unbounded time scale.

2 Preliminaries and Lemmas

A time scale is a nonempty closed subset in the set of real numbers $\mathbf{R}$. Assuming $\mathbb{T}$ is a time scale, when $t<\sup\mathbb{T}$, we define forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$; when $t>\inf\mathbb{T}$, we define back jump operator $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$. If $\sigma(t)>t$, we say that $t\in\mathbb{T}$ is right-scattered; while if $\sigma(t)=t$, we say that $t\in\mathbb{T}$ is right-dense. If $\rho(t)<t$, we say that $t\in\mathbb{T}$ is left-scattered; while if $\rho(t)=t$, we say that $t\in\mathbb{T}$ is left-dense. Defining the set $\mathbb{T}^{\kappa}$ as following: If $\mathbb{T}$ has a maximum left-scattered point $\mathbf{M}$, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{M\}$. Otherwise $\mathbb{T}^{\kappa}=\mathbb{T}$. A function $u:\mathbb{T}\to R$ is right-dense continuous or rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$, denoted

by $u \in C_{rd}(\mathbb{T})$. Assuming $f : \mathbb{T} \to R$, $t \in \mathbb{T}^{\kappa}$. If there is a constant α, for any $\varepsilon > 0$, and there is a neighborhood $U_{\mathbb{T}}(= U \bigcap \mathbb{T})$ of t, such that

$$|f(\sigma(t)) - f(s) - \alpha[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, s \in U_{\mathbb{T}},$$

then we say that f is Δ differentiable in t, and the derivative is α, which is defined as $f^{\Delta}(t)$. Similarly, we can define n order Δ derivative $f^{\Delta^n} = (f^{\Delta^{n-1}})^{\Delta}$. Other concept and calculation on time scales can be seen in [1-2,7].

We say that a nontrivial solution to Eq. (1) is an oscillatory solution if it is finally not positive or negative, otherwise we say it a nonoscillatory solution. If all solutions to Eq. (1) are oscillatory solutions, then we say that the equation is oscillatory.

In this paper, we give the following assumptions:
(H_1) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$,
$p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$,
$h \in C_{rd}([t_0, \infty)_{\mathbb{T}}, R), h(t)$ is bounded,

$$\int_{t_0}^{+\infty} \mid h(t) \mid \Delta t < +\infty,$$

and $\exists d > 0, s.t.\ r(t) \leq d, \forall t \in [t_0, \infty)_{\mathbb{T}}$;

(H_2) $\tau, \delta_i \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$, $\tau(t) \leq t, \delta_i(t) \leq t, \forall t \in [t_0, \infty)_{\mathbb{T}}$, and $\lim\limits_{t \to +\infty} \tau(t) = \lim\limits_{t \to +\infty} \delta_i(t) = +\infty, i = 1, 2, \cdots, m$;

(H_3) $f_i \in C([t_0, \infty)_{\mathbb{T}} \times R, R)$, $f_i(t, x)$ does not increase about x, $\dfrac{f_i(t, x)}{x} > 0 (x \neq 0), \forall t \in [t_0, \infty)_{\mathbb{T}}$, and $\lim\limits_{t \to +\infty} \mid f_i(t, \lambda) \mid = +\infty, i = 1, 2, \cdots, m$, λ is an arbitrary constant which has nothing to do with t.

3 Main Results

Theorem 1 Assuming that $(H_1) \sim (H_3)$ hold, and

$$\sum_{i=1}^{m} \int_{t_0}^{+\infty} \mid f_i(t, c) \mid \Delta t = +\infty,$$

where c is an arbitrary constant which has nothing to do with t. Then all bounded solutions to Eq. (1) oscillate.

Proof: Supposing that $x(t)$ is a final bounded positive solution of Eq. (1)(final bounded negative solution can be proved in the same way), then there exists sufficient large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [t_1, \infty)_{\mathbb{T}}$, $x(t) > 0, x(\delta_i(t)) > 0 (i = 1, 2, \cdots, m), x(\tau(t)) > 0, z(t) > 0$. Because $x(t)$ is bounded, there exists $M > 0$, $x(t) \leq M$ for arbitrary $t \in [t_1, \infty)_{\mathbb{T}}$. And because $\lim\limits_{t \to +\infty} \delta_i(t) = +\infty, i = 1, 2, \cdots, m$, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$, for arbitrary $t \in [t_2, \infty)_{\mathbb{T}}, 0 < x(\delta_i(t)) \leq M, i = 1, 2, \cdots, m.$

By Eq. (1) and (H_3), when $t \in [t_2, \infty)_{\mathbb{T}}$,

$$\begin{aligned} [r(t)z^{\Delta}(t)]^{\Delta} &= -\sum_{i=1}^{m} f_i(t, x(\delta_i(t))) + h(t) \\ &\leq -\sum_{i=1}^{m} f_i(t, M) + \mid h(t) \mid . \quad (2) \end{aligned}$$

By $(H_1), (H_3)$, when $t \to +\infty$, the right side tends to $-\infty$, so $[r(t)z(t)^{\Delta}]^{\Delta}$ is finally negative.

Integrating both sides of Eq. (2) from t_2 to t

$$r(t)z^{\Delta}(t) \leq r(t_2)z^{\Delta}(t_2) - \sum_{i=1}^{m} \int_{t_2}^{t} f_i(s, M)\Delta s + \int_{t_2}^{t} \mid h(s) \mid \Delta s.$$

According to $(H_1), (H_3)$, when $t \to +\infty$, the right side tends to $-\infty$, so $r(t)z^{\Delta}(t)$ is finally negative. Also $[r(t)z^{\Delta}(t)]^{\Delta}$ is finally negative. Therefore, there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}, M_1 < 0$, such that $r(t_3)z^{\Delta}(t_3) = M_1$ holds and when $t \in [t_3, \infty)_{\mathbb{T}}$,

$$r(t)z^{\Delta}(t) \leq r(t_3)z^{\Delta}(t_3) = M_1.$$

So $z^{\Delta}(t) \leq \dfrac{M_1}{r(t)} \leq \dfrac{M_1}{d}$, integrating from t_3 to t,

$$z(t) \leq z(t_3) + \frac{M_1}{d}(t - t_3) \to -\infty (t \to +\infty),$$

and this contradicts that $z(t)$ is finally positive. Thus all bounded solutions to Eq. (1) oscillate. The proof is completed. ☐

Theorem 2 Assuming that $(H_1) \sim (H_3)$ hold, $0 \leq p(t) \leq 1$, and there exist

$$\lambda > 0, Q_i \in C_{rd}([t_0, \infty)_{\mathbb{T}}, R^+), F_i \in C_{rd}(R, R),$$

s.t.

$$\mid f_i(t, x) \mid \geq Q_i(t) \mid F_i(x) \mid, \frac{F_i(x)}{x} \geq \lambda,$$

$i = 1, 2, \cdots, m$ and

$$\sum_{i=1}^{m} \int_{t_0}^{+\infty} Q_i(t)[1 - p(\delta_i(t))]\Delta t = +\infty.$$

Then all bounded solutions to Eq. (1) oscillate.

Proof: Supposing that $x(t)$ is a final bounded positive solution to Eq. (1) (final bounded negative solution can be proved in the same way), then there exists sufficient large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [t_1, \infty)_{\mathbb{T}}$, $x(t) > 0, x(\delta_i(t)) > 0, i = 1, 2, \cdots, m, x(\tau(t)) > 0, z(t) > 0, z(t) \geq x(t)$. Because $x(t)$ is bounded, there exists $M > 0$, $x(t) \leq M$ for arbitrary $t \in [t_1, \infty)_{\mathbb{T}}$. And because $\lim\limits_{t \to +\infty} \delta_i(t) = +\infty, i = 1, 2, \cdots, m$, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$, for arbitrary $t \in [t_2, \infty)_{\mathbb{T}}, 0 < x(\delta_i(t)) \leq M, i = 1, 2, \cdots, m$.

From Eq. (1) and (H_3), when $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$[r(t)z(t)^{\Delta}]^{\Delta} = -\sum_{i=1}^{m} f_i(t, x(\delta_i(t))) + h(t)$$

$$\leq -\sum_{i=1}^{m} f_i(t, M) + \mid h(t) \mid .$$

From $(H_1), (H_3)$, when $t \to +\infty$, the right side tends to $-\infty$, so $[r(t)z^{\Delta}(t)]^{\Delta}$ is finally negative and $r(t)z^{\Delta}(t)$ is finally positive or negative.

If $r(t)z^{\Delta}(t)$ is finally negative, by $[r(t)z^{\Delta}(t)]^{\Delta}$ being finally negative, then there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}, M_1 < 0$, such that $r(t_3)z^{\Delta}(t_3) = M_1$ holds and when $t \in [t_3, \infty)_{\mathbb{T}}$,

$$r(t)z^{\Delta}(t) \leq r(t_3)z^{\Delta}(t_3) = M_1.$$

So $z^{\Delta}(t) \leq \dfrac{M_1}{r(t)} \leq \dfrac{M_1}{d}$, integrating from t_3 to t,

$$z(t) \leq z(t_3) + \frac{M_1}{d}(t - t_3) \to -\infty (t \to +\infty),$$

and this contradicts that $z(t)$ is finally positive.

If $r(t)z^{\Delta}(t)$ is finally positive, then $z^{\Delta}(t)$ is also finally positive. And because $z(t)$ is finally positive, $\lim\limits_{t \to +\infty} \delta_i(t) = +\infty, i = 1, 2, \cdots, m$, then there exist $T_1 \in [t_1, \infty)_{\mathbb{T}}, M_2 > 0$, when $t \in [T_1, \infty)_{\mathbb{T}}$, $z(\delta_i(t)) \geq M_2, i = 1, 2, \cdots, m$. From Eq. (1), one obtains

$$[r(t)z^{\Delta}(t)]^{\Delta} = -\sum_{i=1}^{m} f_i(t, x(\delta_i(t))) + h(t)$$
$$\leq -\sum_{i=1}^{m} Q_i(t)F_i(x(\delta_i(t))) + h(t)$$
$$\leq -\lambda \sum_{i=1}^{m} Q_i(t)x(\delta_i(t)) + h(t)$$
$$= -\lambda \sum_{i=1}^{m} Q_i(t)[z(\delta_i(t)) - p(\delta_i(t))x(\tau(\delta_i(t)))] + h(t).$$

For $z(t) \geq x(t), z^{\Delta}(t)$ being finally positive, there exists $T_2 \in [t_1, \infty)_{\mathbb{T}}$, when $t \in [T_2, \infty)_{\mathbb{T}}$, $0 < x(\tau(\delta_i(t))) \leq z(\tau(\delta_i(t))) \leq z(\delta_i(t)), i = 1, 2, \cdots, m$. Letting $T_3 = max\{T_1, T_2\}$, then when $t \in [T_3, \infty)_{\mathbb{T}}$,

$$[r(t)z^{\Delta}(t)]^{\Delta}$$
$$\leq -\lambda \sum_{i=1}^{m} Q_i(t)[1 - p(\delta_i(t))]z(\delta_i(t)) + h(t)$$
$$\leq -\lambda M_2 \sum_{i=1}^{m} Q_i(t)[1 - p(\delta_i(t))] + \mid h(t) \mid,$$

integrating from T_3 to t, we have

$$r(t)z^{\Delta}(t)$$
$$\leq r(T_3)z^{\Delta}(T_3)$$
$$-\lambda M_2 \sum_{i=1}^{m} \int_{T_3}^{t} Q_i(s)[1 - p(\delta_i(s))]\Delta s$$
$$+\int_{T_3}^{t} \mid h(s) \mid \Delta s$$
$$\to -\infty \quad (t \to +\infty),$$

and this contradicts that $z^{\Delta}(t)$ is finally positive. Therefore all bounded solutions to Eq. (1) oscillate. The proof is completed. □

4 Conclusion

In this paper, we study the existence of bounded oscillatory solutions to a class of second order neutral dynamic equations by adding forced term on time scales, and give several sufficient conditions for oscillation of bounded solutions.

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