On Maintaining Optimal Temperatures In Greenhouses

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Abstract: We study the problem of the optimal temperature regulation in industrial greenhouses. We consider a model based on the one-dimensional heat equation having a non-constant coefficient on a bounded interval with quadratic cost functional, prove the existence and uniqueness of a control function from a prescribed set, and study the structure of the set of accessible temperature functions.

Key–Words: Greenhouse, Climate control, Heat equation, Extremal problem.

1 Introduction

When growing plants in industrial greenhouses, some temperature conditions are needed at some fixed height corresponding to the growth point of the plants. These conditions should be maintained according to a circadian schedule with small deviations admitted. One can make the temperature to rise by heating the floor of the greenhouse and to fall by opening ventilator windows at the ceil. A greenhouse can be treated as an elongated parallelepiped. Consider its cross-sections that are perpendicular to its longer side. Now we can propose a mathematical model to solve the task.

2 Mathematical model based on the heat equation

The model is based on the heat equation. The first results concerning this model are published in [1], [2], [3]. Some methods of proof of the main results are contained in [4] and [5]. Similar extremum problems for integral functionals were considered by different authors (see [6], [7], [8], [10]). The review of early results in this problematic is contained in [9], bibliography of later works is contained in [11]. The problem of minimization of functional with final observation and the problem of optimal time of control were considered in [6], [7], [8], [9], [11]. See also [12], [13].

3 Boundary Value Problem

Let us consider the mixed problem for the heat equation

$$u_t = (a(x)u_x)_x, \quad 0 < x < l, \quad t > 0, \tag{1}$$

with a sufficiently smooth coefficient $a(x)$ satisfying the condition

$$0 < a_0 \le a(x), \quad x \in [0, 1], \tag{2}$$

with the boundary conditions

$$u(0, t) = \phi(t), \quad u_x(l, t) = \psi(t), \quad t > 0, \tag{3}$$

and the initial condition

$$u(x, 0) = 0, \quad 0 < x < l, \tag{4}$$

with $\phi \in W_2^1(0, T)$, $\psi \in W_2^1(0, T)$ for any $T > 0$. In this article we mean that $\psi(t)$ is a fixed function and $\phi(t)$ is a control function to be found.

Put $Q_T = (0, l) \times (0, T)$. Just as in [14], p. 15, by $V_2^{1,0}(Q_T)$ we denote the Banach space of functions

$u \in W_2^{1,0}(Q_T)$ having the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0<t<T} \left(\int_0^l u(x,t)^2 \, dx \right)^{1/2} + \left(\int_{Q_T} u_x(x,t)^2 \, dx \, dt \right)^{1/2} \quad (5)$$

and such that $t \mapsto u(\cdot, t)$ is a continuous mapping $[0,T] \to L_2(0,l)$.

By $\widetilde{W}_2^1(Q_T)$ denote the space of all $\eta \in W_2^1(Q_T)$ such that $\eta(x,T) = 0,\ \eta(0,t) = 0$.

We will consider the energy class of weak solutions to problem (1)–(4), i. e. the set of functions $u \in V_2^{1,0}(Q_T)$ satisfying the boundary condition $u(0,t) = \phi(t)$ and the integral identity

$$\int_{Q_T} (a(x) u_x \eta_x - u \eta_t) \, dx \, dt = \int_0^T \psi(t) \, \eta(l,t) \, dt \quad (6)$$

for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$.

Lemma 1 *There exists a unique weak solution to problem (1)–(4) belonging to $V_2^{1,0}(Q_T)$.*

Proof: To prove the uniqueness of such solution, we can use the same considerations as in [14]. To prove the existence we represent the solution to problem (1)–(4) as the sum of functions

$$u(x,t) = v(x,t) + \phi(t) + x\psi(t). \quad (7)$$

Then, by (1)–(4), (7) we have the following problem on the function v :

$$\begin{aligned} v_t &= (a(x) v_x)_x + g(x,t), && (8) \\ & \quad 0 < x < l, \quad t > 0, \\ v(0,t) &= 0, \quad v_x(l,t) = 0, \quad t > 0, && (9) \\ v(x,0) &= h(x), \quad 0 < x < l, && (10) \end{aligned}$$

where

$$\begin{aligned} h(x) &= -\phi(0) - x\psi(0), \\ g(x,t) &= a'(x)\psi(t) - \phi'(t) - x\psi'(t). \end{aligned}$$

Denote by $\{y_n(x)\}_{n=1}^{\infty}$ the sequence of normalized in $L_2(0,l)$ eigenfunctions of the self-adjoint boundary value problem

$$\begin{gathered} (a(x)y')' + \lambda y = 0, \\ y(0) = 0, \quad y'(l) = 0, \end{gathered}$$

by $\{\lambda_n\}_{n=1}^{\infty}$ we denote the corresponding sequence of positive eigenvalues.

Finding the solution to problem (8)–(10) by the Fourier method, we obtain the following formal representation:

$$\begin{aligned} u(x,t) &= \phi(t) + x\psi(t) \\ &- \sum_{n=0}^{\infty} e^{-\lambda_n t} y_n(x) \\ &\times \left(\int_0^l (\phi(0) + z\psi(0)) y_n(z) dz \right. \\ &+ \int_0^t e^{\lambda_n \tau} \int_0^l (\phi'(\tau) + z\psi'(\tau) \\ &- \left. a'(z)\psi(\tau)) y_n(z) dz d\tau \right). \quad (11) \end{aligned}$$

Series (11) can be proved to converge in the space $V_2^{1,0}(Q_T)$ and to satisfy the integral identity (6). So, it gives a weak solution to problem (1)–(4). Lemma 1 is proved. □

4 Existence and uniqueness of solution to extremum problem

Now we raise the problem to maintain the temperature $z(t)$ at some given height c during the whole time interval $0 \le t \le T$. Consider the problem

$$\begin{aligned} u_t &= (a(x) u_x)_x, && (12) \\ & \quad 0 < x < l, \quad t > 0, \\ u(0,t) &= \phi(t), \quad u_x(l,t) = \psi(t), && (13) \\ & \quad t > 0, \\ u(x,0) &= 0, \quad 0 < x < l, && (14) \end{aligned}$$

with $\phi(t) \in W_2^1(0,T),\ \psi(t) \in W_2^1(0,T)$ for any $T > 0$.

Hereafter denote by u_ϕ its unique solution existing according to Lemma 1. In this notation ψ and l are not mentioned because they are fixed throughout the paper.

Suppose $T > 0,\ z \in L_2(0,T)$. By Φ_M with $M > 0$ denote the set of functions

$$\Phi_M = \left\{ \phi \in W_2^1(0,T) : \ \|\phi\|_{W_2^1(0,T)} \le M \right\}.$$

For some $c \in (0,l]$ define the functional

$$J[\phi] = \int_0^T (u_\phi(c,t) - z(t))^2 dt.$$

Consider the minimization problem for this functional and put

$$m = \inf_{\phi \in \Phi_M} J[\phi].$$

Physically, at one endpoint of an infinitely thin rod, the temperature $\phi(t)$ (the control function) is maintained during the time T, and the heat flow $\psi(t)$ is given at another endpoint. The problem consists in finding the control function $\phi_0(t)$ making the temperature at some point c maximally close to the given one $z(t)$. The quality of the control is estimated by the functional $J[\phi]$.

We investigate the existence and uniqueness of the control function $\phi_0(t) \in \Phi_M$ (the minimizer) giving the minimum of the functional $J[\phi]$. The proofs of results on the existence and uniqueness are based on the following Lemma concerning the best approximation in Hilbert spaces.

Lemma 2 *Let A be a convex closed set in a Hilbert space H. Then for any $x \in H$ there exists a unique element $y \in A$ such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Proof: Denote $d = \inf_{z \in A} \|x - z\|$. By the parallelogram property for any $x \in H$ and $y_1, y_2 \in A$ we have

$$2\left(\|x-y_1\|^2 + \|x-y_2\|^2\right) = \|y_1 - y_2\|^2 + 4\Big\|x - \frac{y_1+y_2}{2}\Big\|^2.$$

By the convexity of the set A we have $\frac{1}{2}(y_1+y_2) \in A$, whence

$$\Big\|x - \frac{y_1+y_2}{2}\Big\| \geq d.$$

Therefore,

$$\|y_1 - y_2\|^2 \leq 2\left(\|x-y_1\|^2 + \|x-y_2\|^2\right) - 4d^2,$$

which invokes uniqueness of the minimizer.

To prove the existence of the minimizer, consider a sequence of elements $y_k \in A$ such that

$$\lim_{k\to\infty} \|x - y_k\| = d.$$

The sequence $\{y_k\}$ is a fundamental one since

$$\|y_k - y_l\|^2 \leq 2\left(\|x-y_k\|^2 + \|x-y_l\|^2\right) - 4d^2 \to 0, \quad k, l \to \infty.$$

Suppose $\lim_{k\to\infty} y_k = y \in H$. Then $\|x - y\| = d$ and $y \in A$ due to the closeness of the set A. Lemma 2 is proved. □

Theorem 3 *There exists a unique function $\phi_0(t) \in \Phi_M$ such that $m = J[\phi_0]$.*

Proof: Denote

$$B_M = \{y = u_\phi(c,\cdot) : \ \phi \in \Phi_M\} \subset L_2(0,T).$$

Let us prove that the set B_M is a convex closed subset in $L_2(0,T)$. Suppose $y_1, y_2 \in B_M$ with $y_j = u_{\phi_j}(c,\cdot)$. Then $\|\phi_j\|_{W_2^1(0,T)} \leq M$, $j \in \{1,2\}$, and for any $\alpha \in (0,1)$ we have

$$\|\alpha\phi_1 + (1-\alpha)\phi_2\|_{W_2^1(0,T)} \leq \alpha\|\phi_1\|_{W_2^1(0,T)} + (1-\alpha)\|\phi_2\|_{W_2^1(0,T)} \leq M,$$

whence $\alpha y_1 + (1-\alpha)y_2 \in B_M$ and the set B_M is convex.

Now we prove that B_M is a closed subset in $L_2(0,T)$. Let $\{y_k(t)\}_{k=1}^\infty \subset B_M$ be a fundamental sequence in $L_2(0,T)$ having the limit $y \in L_2(0,T)$. The corresponding sequence $\{\phi_k\} \subset \Phi_M$ is a weakly precompact set in $W_2^1(0,T)$. Hence, some subsequence ϕ_{k_j} tends weakly, as $j \to \infty$, to a function $\phi \in W_2^1(0,T)$. By the properties of weakly convergent sequences in Hilbert spaces ([15], Chapter 1, Section 1, Theorem 1.1) we obtain

$$\|\phi\|_{W_2^1(0,T)} \leq \limsup_{j\to\infty} \|\phi_{k_j}\|_{W_2^1(0,T)} \leq M, \tag{15}$$

whence $\phi \in \Phi_M$.

Next, by the Banach-Saks Theorem ([16], Chapter 2, Section 3) there exists a subsequence k_{j_n} such that

$$\lim_{n\to\infty} \|\tilde{\phi}_n - \phi\|_{W_2^1(0,T)} = 0, \tag{16}$$

where

$$\tilde{\phi}_n = \frac{1}{n}\sum_{l=1}^n \phi_{k_{j_l}}. \tag{17}$$

Therefore,

$$\|\tilde{\phi}_n\|_{W_2^1(0,T)} \leq \frac{1}{n}\sum_{l=1}^n \|\phi_{k_{j_l}}\|_{W_2^1(0,T)} \leq M$$

and by (15) we obtain

$$\tilde{y}_n = \frac{1}{n}\sum_{l=1}^n y_{k_{j_l}} \in B_M.$$

By standard technique (see [14], [15]) we can obtain the following estimate for the solution to problem (1)–(4):

$$\|u_\phi\|_{V_2^{1,0}(Q_T)} \leq C_1(\|\phi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}),$$

where the constant C_1 is independent of ϕ and ψ. Therefore, for the corresponding sequence of solutions

$$\tilde{u}_n = \frac{1}{n}\sum_{l=1}^n u_{k_{j_l}},$$

we obtain the inequalities

$$\|\tilde{u}_m - \tilde{u}_n\|_{V_2^{1,0}(Q_T)} \le C_1\|\tilde{\phi}_m - \tilde{\phi}_n\|_{W_2^1(0,T)} \to 0, \quad m, n \to \infty, \tag{18}$$

whenever $\tilde{u}_n = u_{\tilde{\phi}_n}$. This means that $\tilde{u}_n(0,t) = \tilde{\phi}_n(t)$ and the integral identity

$$\int_{Q_T} (a(x)(\tilde{u}_n)_x\eta_x - (\tilde{u}_n)\eta_t)\,dx\,dt = \int_0^T \psi(t)\,\eta(l,t)\,dt \tag{19}$$

holds for any function $\eta(x,t) \in \widetilde{W}_2^1(Q_T)$. Taking into account relations (16), (18), and (19), we see that there exists the limit function $u \in V_2^{1,0}(Q_T)$, which is a weak solution to problem (1)–(4) with the boundary function ϕ, and

$$\|u - \tilde{u}_n\|_{V_2^{1,0}(Q_T)} \le C_1\|\phi - \tilde{\phi}_n\|_{W_2^1(0,T)}.$$

So, by the embedding estimate (see [15], Chapter 1, Section 6, Formula 6.15) we obtain

$$\|u(c,\cdot) - \tilde{u}_n(c,\cdot)\|_{L_2(0,T)} \le C_2\|u - \tilde{u}_n\|_{V_2^{1,0}(Q_T)} \le C_1C_2\|\phi - \tilde{\phi}_n\|_{W_2^1(0,T)},$$

whence $y = u(c,\cdot) \in B_M$ and B_M is a closed subset in $L_2(0,T)$.

Therefore, by Lemma 2, there exists a unique function $y = u(c,\cdot)$, where $u \in V_2^{1,0}(Q_T)$ is a solution to problem (1)–(4) with some $\phi_0 \in \Phi_M$ such that

$$\inf_{\phi\in\Phi_M} J[\phi] = J[\phi_0].$$

Let us prove that such $\phi_0 \in \Phi_M$ is unique. If not, consider a pair of such functions ϕ_1, ϕ_2 and the corresponding pair of solutions u_1, u_2. The function $\tilde{u} = u_1 - u_2$ is a solution to the problem

$$\tilde{u}_t = (a(x)\tilde{u}_x)_x, \quad 0 < t < T, \quad 0 < x < l, \tag{20}$$

$$\tilde{u}(0,t) = \tilde{\phi}(t), \quad 0 < t < T, \quad \tilde{\phi}(t) = \phi_1(t) - \phi_2(t), \tag{21}$$

$$\tilde{u}(c,t) = 0, \quad 0 < t < T, \tag{22}$$

$$\tilde{u}_x(l,t) = 0, \quad 0 < t < T, \tag{23}$$

$$\tilde{u}(x,0) = 0, \quad 0 < x < l. \tag{24}$$

Taking into account the integral identity (6) with the function $\eta(x,t)$ equal to 0 on $[0,c]\times[0,T]$, we obtain that the function $\tilde{u}$ on the rectangle $Q_T^{(c)} = (c,l)\times(0,T)$ equals the solution to the problem

$$\hat{u}_t = (a(x)\hat{u}_x)_x, \quad 0 < t < T, \quad c < x < l, \tag{25}$$

$$\hat{u}(c,t) = 0, \quad 0 < t < T, \tag{26}$$

$$\hat{u}_x(l,t) = 0, \quad 0 < t < T, \tag{27}$$

$$\hat{u}(x,0) = 0, \quad c < x < l. \tag{28}$$

But the solution to problem (25)–(28) vanishes on $[c,l]\times[0,T]$, whence we have

$$\tilde{u}(x,t) = 0, \quad c < x < l, \quad 0 < t < T. \tag{29}$$

Now we will prove that

$$\tilde{u}(x,t) = 0, \quad 0 < x < l, \quad 0 < t < T. \tag{30}$$

Note that by Theorem 2 from [18], Section 11, the weak solution $\tilde{u}$ is a classical solution to equation (20) in Q_T. Now we use Theorem 5 from [17], Section 3. It establishes the following.

Consider a function $u(x,t) \in C^{2,1}(\Omega)$, $\Omega \subset R^2$, such that $u_t = (a(x)u_x)_x$ on Ω. Suppose G_0 is a connected component of the set $\Omega \cap \{t = t_0\}$, and $\widetilde{G}$ is a connected open subset of G_0. If $u|_{\widetilde{G}} = 0$, then $u|_{G_0} = 0$.

Applying this theorem to the solution $\tilde{u}$ of problem (20)–(24) for any $t_0 \in (0,T)$ with $G_0 = (0,l)\times\{t_0\}$ and $\widetilde{G} = (c,l)\times\{t_0\}$, we obtain that (30) follows from (29). Therefore, $\tilde{u}(x,t) = 0$ for any $x \in (0,l)$ and $t \in (0,T)$. This means that $\tilde{\phi}(t) = \tilde{u}(0,t) = 0$. The proof of Theorem 3 is complete. □

By similar considerations we can obtain the existence and uniqueness theorems for other practically important classes of control functions.

By Φ_M^0 denote the class of control functions

$$\Phi_M^0 = \{\phi \in W_2^1(0,T),\ \|\phi\|_{W_2^1(0,T)} \le M,\ \phi(0) = 0\}.$$

Theorem 4 *There exists a unique function $\phi_0(t) \in \Phi_M^0$ such that*

$$\inf_{\phi\in\Phi_M^0} J[\phi] = J[\phi_0].$$

By $\bar{\Phi}_M^0$ denote the class of control functions

$$\bar{\Phi}_M^0 = \{\phi \in W_2^1(0,T),\ \|\phi\|_{W_2^1(0,T)} \le M,\ \phi_1 < \phi(t) < \phi_2\}$$

with some constants ϕ_1 and ϕ_2.

Theorem 5 *There exists a unique function $\phi_0(t) \in \bar{\Phi}_M^0$ such that*

$$\inf_{\phi\in\bar{\Phi}_M^0} J[\phi] = J[\phi_0].$$

5 On exact controllability

Besides the question of existence and uniqueness of the solution to the extremum problem, another important question concerns the exact controllability, which means the ability to obtain, at some point $x = c$, the restriction $u(c,t)$ equal almost everywhere on $[0,T]$ to a given function $z(t)$. Respectively, by the exact control we mean the function $\phi_0(t) \in \Phi_M^0$ making the functional $J[\phi]$ to vanish:

$$J[\phi_0] = \int_0^T \left(u_{\phi_0}(c,t) - z(t)\right)^2 dt = 0.$$

The next theorem shows that the set of functions $z(t) \in L_2(0,T)$ admitting exact controllability is sufficiently "small".

Theorem 6 *The set of all functions $z \in L_2(0,T)$ admitting exact control, i. e. such that $J[\phi] = 0$ for some $\phi(t) \in \Phi_M$, is a first category subset in $L_2(0,T)$.*

Proof: Consider equation (1) for a function $u_1(x,t) \in V_2^{1,0}(Q_T)$ with the boundary conditions

$$\begin{aligned} u_1(0,t) &= \phi_1(t), \\ (u_1)_x(l,t) &= \psi(t), \end{aligned}$$

and the same equation for a function $u_2(x,t) \in V_2^{1,0}(Q_T)$ with the boundary conditions

$$\begin{aligned} u_2(0,t) &= \phi_2(t), \\ (u_2)_x(l,t) &= \psi(t). \end{aligned}$$

Denote $\tilde{u} = u_1 - u_2$. The function $\tilde{u} = u_1 - u_2$ is a solution of equation (1) with the boundary conditions

$$\tilde{u}(0,t) = \tilde{\phi}(t) = \phi_1(t) - \phi_2(t), \quad (31)$$

$$\tilde{u}_x(l,t) = 0, \quad (32)$$

and the initial condition

$$\tilde{u}(x,0) = 0. \quad (33)$$

Now, in the rectangle $Q_T^{(2l)} = (0,2l) \times (0,T)$ consider the problem

$$\bar{u}_t = (\bar{a}(x)\bar{u}_x)_x, \quad (34)$$

$$0 < x < 2l, \quad 0 < t < T,$$

$$\bar{u}(0,t) = \tilde{\phi}(t), \quad (35)$$

$$\bar{u}(2l,t) = \tilde{\phi}(t), \quad (36)$$

$$\bar{u}(x,0) = 0, \quad (37)$$

where $\bar{a}(x) = a(2l - x)$, $x \in (l,2l)$. The weak solution of problem (34) – (37) is a function $\bar{u}(x,t) \in V_2^{1,0}(Q_T^{(2l)})$ satisfying the boundary condition $u(0,t) = u(2l,t) = \tilde{\phi}(t)$ and the integral identity

$$\int_{Q_T^{(2l)}} (\bar{a}\bar{u}_x\eta_x - \bar{u}\eta_t)dxdt = 0 \quad (38)$$

for any function $\eta(x,t) \in W_2^1(Q_T)$ such that $\eta(x,T) = 0$, $\eta(0,t) = 0$, $\eta(2l,t) = 0$. It follows from equality (38) that

$$\bar{u}(x,t) = \tilde{u}(x,t), \quad 0 < x < l, \quad 0 < t < T. \quad (39)$$

By the maximum principle for weak solutions ([14], Ch. 3, Sec. 7, Th. 7.2), the solution $\bar{u}(x,t)$ satisfies the inequalities

$$\text{ess} \inf_{t \in [0,T]} \tilde{\phi}(t) \le \bar{u}(x,t) \le \text{ess} \sup_{t \in [0,T]} \tilde{\phi}(t). \quad (40)$$

From (40) therefore

$$\operatorname{ess\,sup}_{Q_T} |\tilde{u}| \le \text{ess} \sup_{0<t<T} |\phi_1(t) - \phi_2(t)|, \quad (41)$$

and, consequently

$$\text{ess} \sup_{0<t<T} |\tilde{u(c,t)}| \le \text{ess} \sup_{0<t<T} |\phi_1(t) - \phi_2(t)|. \quad (42)$$

Integrating inequality (42), we obtain

$$\int_0^T \tilde{u}^2(c,t)dt \le T(\sup_{0<t<T} |\phi_1(t) - \phi_2(t)|)^2. \quad (43)$$

Suppose the functions $\phi_1(t)$ and $\phi_2(t)$ are the exact control functions for given $z_1(t)$ and $z_2(t)$. This means that

$$\begin{aligned} J[\phi] &= \int_0^T (u_1(c,t) - z_1(t))^2 dt = 0, \\ J[\phi] &= \int_0^T (u_2(c,t) - z_2(t))^2 dt = 0. \end{aligned}$$

In this situation inequality (43) invokes the inequality

$$\int_0^T (z_1(t) - z_2(t))^2 dt \le T(\text{ess} \sup_{0<t<T} |\phi_1(t) - \phi_2(t)|)^2 \quad (44)$$

for arbitrary functions $z_1(t)$ and $z_2(t)$ admitting exact controllability.

Let $Z \subset L_2(0,T)$ be a set of exactly controllable functions. We have $Z = \cup_{M=1}^{\infty} Z_M$, where $Z_M \subset L_2(0,T)$ is the set of functions exactly controllable with $\phi(t) \in \Phi_M$. Consider an arbitrary sequence of control functions $\{\phi_k(t)\} \subset \Phi_M$, $M = 1, 2, \ldots$ and the corresponding sequence $\{z_k(t)\} \subset Z_M$. The set

Φ_M is a bounded set in $W_2^1(0,T)$. By the embedding theorem for $W_2^1(0,T)$, we have

$$|\phi_{k_l} - \phi_{k_j}| \to 0, \quad l,j \to \infty, \tag{45}$$

for some subsequence ϕ_{k_j}. Therefore, by (44), (45) we get for the sequence $\{z_{k_j}(t)\} \subset Z_M$ the relation

$$\int_0^T (z_{k_l}(t) - z_{k_j}(t))^2 dt \tag{46}$$
$$\le T(\text{ess}\sup_{0<t<T} |\phi_{k_l}(t) - \phi_{k_j}(t)|)^2 \to 0,$$
$$j,l \to \infty.$$

It follows from (46) that Z_M is a pre-compact set in $L_2(0,T)$. So, Z_M is nowhere dense in $L_2(0,T)$. Thus, since $Z = \cup_{M=1}^{\infty} Z_M$, we conclude that Z is a first category set in $L_2(0,T)$. Theorem 6 is proved. □

The following statements show that the exact controllability does not necessarily take place not only for functions $z \in L_2(0,T)$, but also for $z \in C([0,T])$. Consider the exact controllability question for problem (12)–(14) with $\psi(t) = 0$ (no heat flow through the right endpoint).

$$\bar{u}_t = (a(x)\bar{u}_x)_x, \tag{47}$$
$$0 < x < l, \quad 0 < t < T,$$
$$\bar{u}(0,t) = \phi(t), \quad \bar{u}_x(l,t) = 0, \tag{48}$$
$$0 < t < T,$$
$$\bar{u}(x,0) = 0, \quad 0 < x < l. \tag{49}$$

Theorem 7 *For any $M > 0$ there exists a function $z \in C([0,T])$ such that for any function $\phi(t) \in \Phi_M^0$, the solution to problem (47)–(49) satisfies the inequality $J[\phi] > 0$.*

Now consider the more general case with $\psi(t) \neq 0$, i. e. consider problem (12)–(14).

Theorem 8 *For any $M > 0$ and $M_1 > 0$ there exists a function $z \in C([0,T])$ such that for any function $\phi(t) \in \Phi_M^0$ and any $\psi(t) \in W_2^1(0,T)$ such that $\|\psi(t)\|_{W_2^1(0,T)} \le M_1$ the solution $u(x,t)$ to problem (12)–(14) satisfies the inequality*

$$J[\phi] = \int_0^T (u(c,t) - z(t))^2 \, dt > 0.$$

6 Conclusion

The results explained in the previous sections show that the we can obtain the existence and uniqueness of a control function in a prescribed class. Also we prove that the set of functions $z(t) \in L_2(0,T)$ admitting exact controllability is sufficiently "small". The results of this investigation were used for the development of control algorithms and software ([1], [3]). Afterwards this software was introduced to the climate control process in greenhouse complexes in Russia.

References:

[1] D. A. Lashin, Strategy of management to microclimate in greenhouses, *Gavrish.* Moscow, 2005, no. 1, pp. 33–35. (Russian)

[2] D. A. Lashin, On the optimal control of a temperature regime, *Differ. Equ.* 44, 2008, pp. 853.

[3] D. A. Lashin, On the existence of optimal control of temperature regimes, *J. of Math. Sci.* 158, 2009, pp. 219–227.

[4] I. V. Astashova, A. V. Filinovskiy, V. A. Kondratiev and L. A. Muravei, Some Problems in the Qualitative Theory of Differential Equations, *J. of Natural Geometry.* Jnan Bhawan, London. 23, 2003, pp. 1–126.

[5] *Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis,* edited by I. V. Astashova, UNITY–DANA, Moscow, 2012.

[6] A. G. Butkovsky, Optimal Control in the Systems with Distributed Parameters, *Avtomatika i Telemechanika*, 22, 1961, pp. 17–26.

[7] A. I. Egorov, *Optimal Control by Heat and Diffusion Processes,* Nauka, Moscow, 1978. (Russian)

[8] Yu. V. Egorov, Some Problems of Theory of Optimal Control, *Zhurnal. Vych. Mat. i Mat. Fiziki*, 3, 1963, pp. 887–904.

[9] A. G. Butkovsky, A. I. Egorov, K. A. Lurie, Optimal control of distributed systems, *SIAM J. Control,* 6, 1968, pp. 437–476.

[10] J. L. Lions, *Optimal control of systems governed by partial differential equations*, Berlin, Springer, 1971.

[11] A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and applications,* Nauchnaya Kniga, Novosibirsk, 1999.

[12] G. C. Goodwin, S. F. Graebe, M. E. Salgado, *Control system design,* Pearson, London–New-York, 2000.

[13] M. H. Farag, T. A. Talaat, E. M. Kamal, Existence and uniqueness solution of a class of quasilinear parabolic boundary control problems, *Cubo*, 15, 2013, pp. 111–119.

[14] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'seva, *Linear and quasi-linear equations of parabolic type,* Translations of Mathematical Monographs, 23, American Mathematical Society, Providence, RI, 1968.

[15] O. A. Ladyzhenskaya, *Boundary value problems of mathematical physics*, Fizmatlit, Moscow, 1973.

[16] F. Riesz, B. Szökefalvi-Nagy, *Functional Analysis*, Dover, New-York, 1990.

[17] A. M. Ilin, A. S. Kalashnikov, O. A. Oleinik, Linear equations of second order of parabolic type, *Russ. Math. Surways*, 17, 1962, issue 3, pp. 3–146.

[18] E. M. Landis, O. A. Oleinik, Generalized analiticity and some connected properties of solutions of elliptic and parabolic equations, *Russ. Math. Surways*, 29, 1974, issue 2, pp. 190–206.