Theoretical Approaches to Queueing Systems and Their Simulation in Multichannel Environment

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Abstract: - This paper is concerned with queueing systems showing how to derive their characteristics if the requirement arrivals correspond to a Poisson process and the service times have the exponential distribution. However, the requirements of stationarity, regularity, and independence of increases needed to model these processes by Markov chains and to define the transition probabilities may not be satisfied, or no information may be available on such parameters. Using randomly generated data, we propose a strategy of processing the requirements in multichannel systems and a way of evaluating the probabilities necessary to express the characteristics of the systems comparing these results with the theoretical ones.

Key-Words: - queueing system, Poisson process, Markovian chain, system transition, distribution, simulation

1 Introduction

The Danish mathematician A. K. Erlang formulated the fundamentals of the queueing theory about hundred years ago, the further development of the theory is mainly associated with the Russian mathematician A. N. Kolmogorov, but its current classification was proposed by the English mathematician D. G. Kendall. All details may be found, e.g., in [1], [2], [3], [4], [6], [7].

Generally, at random moments, customers (requirements) enter the system and require servicing. Service options may be limited, e.g., the number of service channels (or service lines). If at least one serving line is empty, the demand entering the system is immediately processed. However, the service time is also random in nature because the performance requirements may vary. If all service lines are busy, then the requirements (customers) must wait for their turn in a queue for the processing of previous requirements, or be rejected (e.g. a telephone call).

Service lines are frequently arranged in parallel, e.g., at the hairdresser's where customers waiting for a haircut are served by several stylists, or at a gas station, where motorists call at several stands of fuel. However, there is a serial configuration of the queue system.

1.1 Classification of Queueing Systems

The queue is usually understood in the usual FIFO sense – *first in*, *first out*), but a LIFO operation (*last in*, *first out*) is also possible, which is also referred to as a LCFS (*last come*, *first served*) strategy.

Besides the FIFO and LIFO service, we can also meet random selection of requirements from the queue to the service system (SIRO - selection in random order) and service managed by priority requirements (PRI - Priority).

The *queue length* may be *limited* by rejecting additional requirements if a certain (predefined) number of requirements is achieved, such as the number of reservations for the book in a library that is currently checked out or (virtually) *unlimited*.

The requirements in the queue may have *limited* or *unlimited patience*. In the case of unlimited patience, requirements wait for their turn while in, a system with limited patience entering the queue significantly depends on the queue length. Instead of the queue length the concept of system *capacity* may also be used, which means the maximum number of requirements that may be present in the system.

In 1951, Kendall proposed a classification based on three main aspects in the form A/B/C, where

Here, we restrict our considerations on systems of the form A/B/C/D/E/F, where

A characterises the probability distribution of random variable period (interval) between the requirement arrivals to the system,

B the probability distribution of random variable *service time of a requirement*,

C is the number of parallel service channels, in the case of "unrestricted" (i.e. very large) number of channels is usual to express the parameter C by ∞.

As already mentioned, the system can be characterised by a larger number of features, so Kendall classification was further extended to the form

A/B/C/D/E/F,

where the meanings of the symbols D, E and F are as follows:

D integer indicating the maximum number of requirements in the system (i.e. the capacity of the system), unless explicitly restricted, expressed by ∞,

E integer expressing the maximum number of requirements in the input stream (or in a resource requirements), if it is unlimited, ∞ is used,

F queue type (FIFO/LIFO/SIRO/PRI).

Let us first assume that parameters A and B equal to M, i.e. intervals between the arrivals of requirements and requirement-service-time are mutually stochastically independent and have exponential distribution, this means that the input stream represents a Poisson (Markov) process, that satisfies the following properties:

(1) *Stationarity* (*homogeneity over time*) - the number of events in equally long time intervals is constant.

(2) *Regularity* - the probability of more than one event at a sufficiently small interval of length Δt is negligibly small. This means that in, the interval $(t, t+\Delta t)$, there is either exactly one event with probability $\lambda\Delta t$ or no event with probability $1-\lambda\Delta t$. In other words, in a Poisson process, the only system transition to the next "higher" state is possible or the system remains in the same condition.

(3) *Independence of increases* - the number of events that occur in one time interval does not depend on the number of events in other intervals.

2 The M/M/1/1/∞/FIFO System

Consider first the situation at the input *separately* from the service process and introduce the random variable *number of requests that come into the system during the interval* $\langle t_0, t_0+\Delta t\rangle$, where $\Delta t \in (0,\infty)$. Due to the stacionarity of the Poisson process, number of requests does not depend on the choice of initial time t_0 and the importance it has only considered the length of the interval Δt.

Let $p_k(t)$ denotes the probability that at time t just k the requirements are in the system. The regularity of the Poisson process implies that the probability that at time $t+\Delta t$ k requirements will be in the system is equal to the probability that at time t $k-1$ requirements were in the system and during Δt one requirement came with probability $\lambda\Delta t$ or at time t k requirements were in the system and during Δt with probability $1-\lambda\Delta t$ any new requirement did not come. From the rules for calculating probabilities of conjunction and disjunction of independent events then we get the equation:

$$p_k(t+\Delta t) = p_{k-1}(t).\,\lambda\Delta t + p_k(t).(1-\lambda\Delta t),\ k = 1,2,\ \ldots \quad (1)$$

The probability that at time $t+\Delta t$ no requirement is in the system is given by the probability that there was no requirement, nor during time Δt entered, is

$$p_0(t+\Delta t) = p_0(t).(1-\lambda\Delta t) \quad (2)$$

After the easy adjustment of equations (1) and (2) we get equations (3) and (4).

$$\frac{p_k(t+\Delta t)-p_k(t)}{\Delta t} = \lambda\, p_{k-1}(t) - \lambda\, p_k(t), \quad k = 1,2,\ldots \quad (3)$$

$$\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t} = -\lambda\, p_0(t) \quad (4)$$

Make now limit transition in equations (3) and (4) for $\Delta t \to 0$. We get:

$$\lim_{\Delta t\to 0}\frac{p_k(t+\Delta t)-p_k(t)}{\Delta t} = \lim_{\Delta t\to 0}\lambda\, p_{k-1}(t) - \lambda\, p_k(t),\ k = 1,2,\ldots$$

$$\lim_{\Delta t\to 0}\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t} = \lim_{\Delta t\to 0}(-\lambda\, p_0(t))$$

The expressions on the left side of the previous two equations are derivatives of the functions $p_k(t)$ and $p_0(t)$ at point t, i.e. $p_k'(t)$ and $p_0'(t)$, while their right sides the limit transition has no effect. Hence we obtain recurrence equations (5) (6)

$$p_k'(t) = \lambda\, p_{k-1}(t) - \lambda\, p_k(t), \quad k = 1,2,\ldots \quad (5)$$

$$p_0'(t) = -\lambda\, p_0(t) \quad (6)$$

These recurrence equations are a set of infinitely many ordinary differential equations of the first order. To solve them we need to know the initial conditions. However, it is clear that at time 0 no requirements are in the system, and therefore

$$p_k(0) = 0, \quad k = 1,2,\ldots \tag{7}$$

$$p_0(0) = 1 \tag{8}$$

From the theory of ordinary differential equations is known that the solution to the system of equations (5) and (6) with initial conditions (7) and (8) is a system functions

$$p_k(t) = e^{-\lambda t}\,\frac{(\lambda t)^k}{k!}, \quad k = 0,1,2,\ldots \tag{9}$$

Specially for k=0 we get

$$p_0(t) = e^{-\lambda t} \tag{10}$$

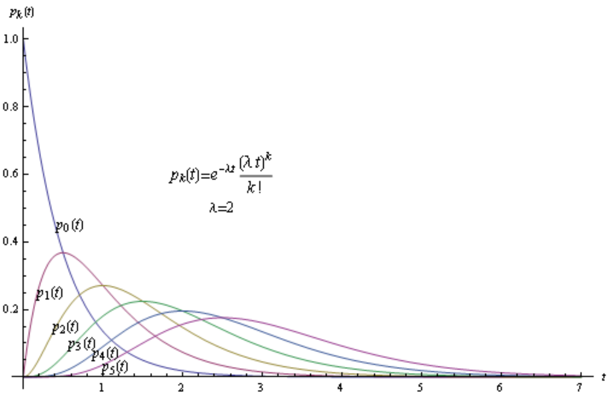

Fig. 1: Graphs of functions $p_k(t)$ for k=0,1, … , 5 and λ=2.

From equation (9) we can see that in the M/M/1 system, the random variable number of requests that come into the system during a time interval of length t has a Poisson distribution with parameter λt.

The mean of this random variable is λt and specially for t=1 the mean value of the random variable number of requests that come into the system per unit of time is equal to λ. We say that λ is the *mean intensity of the input* or shortly the *input intensity* and it expresses the average number of requests that came into the system per unit time.

We show further that the random variable interval between arrivals of requests has an exponential distribution. Denote the variable T. Then the probability that after a request no further requirement for the entire time interval t entered into the system is equal to $p_0(t)$, and therefore, according to equation (10)

$$P(T > t) = p_0(t) = e^{-\lambda t} \tag{11}$$

From here we obtain the *distribution function* $F(t)$ of the *exponential distribution* with parameter λ.

$$F(t) = P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t} \tag{12}$$

The *mean value* of the random variable T representing the average time between two consecutive requests is

$$E(T)=1/\lambda \tag{13}$$

Analogously, we can now examine the service process. We assume that the random variable *service time of one requirement* (shortly *service time*) has an exponential distribution. Denote the distribution of this parameter μ, generally $\mu \neq \lambda$. Mean value of the random variable service time T_O is

$$E(T_O)=1/\mu \tag{14}$$

and parameter μ indicates the mean number of requests served per time unit of work time channel, briefly *mean service intensity*, *service intensity* shortly.

To derive the characteristics of the system is more convenient to describe the system activity by a *graph of system transitions*. The nodes of the graph represent states and directed edges transitions from one state to another, and evaluation of these edges is described by the probability of transition from one state to another. State S_n for fixed $t\in\langle 0,\infty)$, thus more exactly $S_n(t)$ is a random variable and indicates that at time t n requests are in the system. If exactly n requirements, $n \ge 1$, are in the system M/M/1 / ∞ / ∞ / FIFO, then one is operating in a single line system (service channel) operated and the remaining n–1 are waiting in the queue. Transitions between states which differ in a number of requirements in a system can be understood as a process of birth and death, where the request birth represents request entry into the system and death corresponds to request leaving from the system after finishing its operation. For given input assumptions the Poisson stream of requests with a parameter λ and an exponential distribution of service time with parameter μ it is possible the queueing system behaviour describe by the *Markov processes*.

Due to the regularity, have sense only transition probabilities $P(S_i \to S_j)$, where either $i=j$ or i and j differ by 1 For example transition probability $P(S_0 \to S_0)$ corresponds to the probability of the event that during the time interval of length Δt no requirement enters the system, transition probability $P(S_k \to S_{k-1})$, $k\ge 1$, is the probability of the event that during the time interval of length Δt no requirement enters the system and at the same time one request will be served and leaves the system, transition probability $P(S_k \to S_k)$, $k\ge 1$, is equal to the probability of the event that during the time interval of length Δt no requirement enters the system and also no requirement enters the system leaves the system or during this interval one requirement enters and one requirement will be served and one requirement leaves the system.

From the regularity property and the method of calculating the total probability resulting from the partial probabilities of conjunction and disjunction of independent events we get transition probabilities, in the neglect of the powers of the interval length Δt, as follows:

$$P(S_0 \to S_0) = 1 - \lambda\,\Delta t \tag{15}$$

$$P(S_0 \to S_1) = \lambda\,\Delta t \tag{16}$$

$$P(S_k \to S_{k-1}) = (1-\lambda\,\Delta t)\,\mu\Delta t = \mu\Delta t - \lambda\,\mu\Delta t^2 \cup \cup \mu\Delta t \tag{17}$$

$$P(S_k \to S_k) = (1-\lambda\,\Delta t)\,(1-\mu\Delta t) + \lambda\,\Delta t\;\mu\Delta t = =1-\mu\Delta t - \lambda\,\Delta t + 2\lambda\,\mu\Delta t^2 \cup 1-(\lambda+\mu)\,\Delta t \tag{18}$$

$$P(S_k \to S_{k+1}) = \lambda\,\Delta t\,(1-\mu\Delta t) = \lambda\,\Delta t - \lambda\,\mu\Delta t^2 \cup \cup \lambda\,\Delta t \tag{19}$$

Equations (17), (18) and (19) are satisfied for k = 1,2, …

Graph of M/M/1/∞/∞/FIFO system transitions is shown in Fig. 2. For simplicity, nodes are indicated only by numbers instead of symbols S_i. Instead of the general denotations of transition probabilities, we write the specific expressions determined by equations (15) - (19).

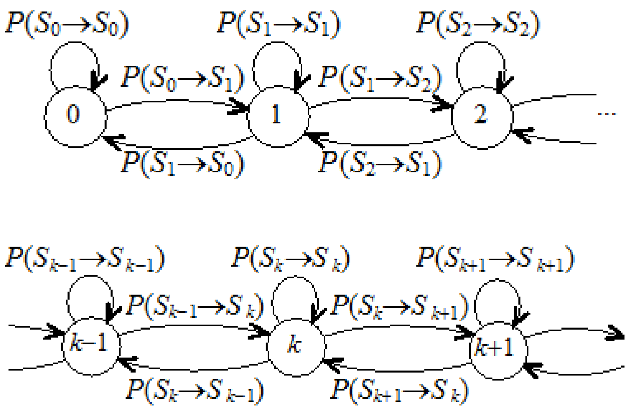

Fig. 2: Graph of M/M/1 / ∞ / ∞ / FIFO system transitions.

Using the transition probabilities between states we can determine the probabilities $p_k(t)$ indicating that at time t exactly k are in the system, however not separately for entries and services, but together

$$p_0(t+\Delta t) = P(S_0 \to S_0) + P(S_1 \to S_0) = =p_0(t).(1-\lambda\,\Delta t) + p_1(t).\mu\Delta t \tag{20}$$

$$p_k(t+\Delta t) = P(S_{k-1} \to S_k) + P(S_k \to S_k) + P(S_{k+1} \to S_k) = p_{k-1}(t).\,\lambda\,\Delta t + p_k(t).[1-(\lambda+\mu)\,\Delta t] + p_{k+1}(t).\,\mu\Delta t,\quad k = 1,2,\ \dots \tag{21}$$

After the easy modification of the equations (20) and (21) we obtain equations (22) and (23)

$$\frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda\, p_0(t) + \mu\, p_1(t) \tag{22}$$

$$\frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lambda\, p_{k-1}(t) - (\lambda+\mu)\, p_k(t) + \mu\, p_{k+1}(t), \quad k = 1,2,... \tag{23}$$

Make now the limit transition for $\Delta t \to 0$ in the equations (22) and (23). We get:

$$\lim_{\Delta t\to 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = \lim_{\Delta t\to 0} [-\lambda\, p_0(t) + \mu\, p_1(t)]$$

$$\lim_{\Delta t\to 0} \frac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = \lim_{\Delta t\to 0} [\lambda\, p_{k-1}(t) - -(\lambda+\mu)\, p_k(t) + \mu\, p_{k+1}(t)], \quad k = 1,2,...$$

The expressions on the left side of the previous two equations are derivatives of the functions $p_0(t)$ and $p_k(t)$ at point t, i.e. $p_0'(t)$ and $p_k'(t)$, while on their right sides the limit transition does not have any effect. Hence we get recurrence equations (24) (25) as follows:

These recurrence equations are a set of infinitely many ordinary differential equations of the first order. To address them we need to know the initial conditions, which are given by the state of a system at time t_0=0. If there are k_0 requirements in a system at time t_0=0, then the initial conditions are given by (26) and (27)

$$p_{k_0}(0) = 1 \tag{26}$$

$$p_k(0) = 0, \quad k \geq 1,\, k \neq k_0 \tag{27}$$

Hereafter, we assume that $\lambda < \mu$, i.e. $\lambda/\mu < 1$. Denote the ratio λ/μ by ψ symbol. We call it the *intensity of the system load*. Condition (28)

$$\psi = \frac{\lambda}{\mu} < 1 \tag{28}$$

is a necessary and sufficient condition for not queue growing beyond all bounds. This condition also ensures that after a sufficiently long time since the opening of a queueing system its situation stabilizes, i.e. there are limits

$$\lim_{t\to\infty} p_k(t) = p_k, \quad k = 0,1,\dots, \tag{29}$$

and then after a sufficiently long time since the opening of a queueing system probabilities $p_k(t)$ can be considered as constant, i.e.

$$p_k(t) = p_k = \text{const} \tag{30}$$

Since the derivatives of constants are zero, we get from this fact and from equations (24) and (25) infinite set of linear algebraic equations determined by (31) and (32).

$$0 = -\lambda\, p_0 + \mu\, p_1 \qquad (31)$$

$$0 = \lambda\, p_{k-1} - (\lambda+\mu)\, p_k + \mu\, p_{k+1}, \quad k = 1,2,\ldots \qquad (32)$$

It is clear that (33) is satisfied

$$\sum_{k=0}^{\infty} p_k = 1 \qquad (33)$$

We express p_1 from equation (31) and get

$$p_1 = \frac{\lambda}{\mu}\, p_0 = \psi\, p_0 \qquad (34)$$

and from (32) we express p_k for $k \geq 2$. For k=1 we get from (32)

$$p_2 = \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu) p_1] = \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu)\psi p_0] =$$
$$= \frac{1}{\mu}[-\lambda p_0 + (\lambda+\mu)\frac{\lambda}{\mu} p_0] =$$
$$= \frac{\lambda}{\mu}[-p_0 + \frac{\lambda}{\mu} p_0 + p_0] = \left(\frac{\lambda}{\mu}\right)^2 p_0 = \psi^2\, p_0 \qquad (35)$$

and generally for k=1,2, … equation (36) is satisfied

$$p_k = \psi^k\, p_0 \qquad (36)$$

Now p_0 remains to be determined. To do this, we use equations (33) and (36).

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} (\psi^k\, p_0) = p_0 \sum_{k=0}^{\infty} \psi^k = 1 \qquad (37)$$

Since the sum in (37) is a geometric series with quotient ψ, first element of ψ^0=1 and the sum $\frac{1}{1-\psi}$, we get from (37) $p_0 \frac{1}{1-\psi} = 1$, and thus

$$p_0 = 1 - \psi \qquad (38)$$

Using (38) equation (36) can be expressed as

$$p_k = \psi^k (1-\psi), \quad k = 1,2,\ldots \qquad (39)$$

These equations allow derive other important characteristics of the M/M/1/∞/∞ /FIFO system, which include:

1. *Mean number of requirements in the system*:

$$E(N_s) = \overline{n_s} = \sum_{k=0}^{\infty} k\, p_k = \sum_{k=1}^{\infty} [k\,\psi^k (1-\psi)] =$$
$$= (1-\psi)\sum_{k=1}^{\infty} k\,\psi^k = (1-\psi)\psi \sum_{k=1}^{\infty} k\,\psi^{k-1} =$$
$$= (1-\psi)\psi \frac{d}{d\psi}\int \sum_{k=1}^{\infty} k\,\psi^{k-1} d\psi =$$
$$= (1-\psi)\psi \frac{d}{d\psi}\sum_{k=1}^{\infty}\psi^k = (1-\psi)\psi \frac{d}{d\psi}\left(\frac{\psi}{1-\psi}\right) =$$
$$= (1-\psi)\psi \frac{(1-\psi)+\psi}{(1-\psi)^2} = (1-\psi)\psi \frac{1}{(1-\psi)^2} =$$
$$= \frac{\psi}{1-\psi} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} \qquad (40)$$

2. *Mean number of jobs in the queue* (*mean queue length*):

$$E(N_f) = \overline{n_f} = \sum_{k=1}^{\infty}(k-1)p_k = \sum_{k=1}^{\infty} k\, p_k - \sum_{k=1}^{\infty} p_k =$$
$$= \overline{n_s} - (1-p_0) = \overline{n_s} - [1-(1-\psi)] = \overline{n_s} - \psi =$$
$$= \frac{\psi}{1-\psi} - [1-(1-\psi)] == \frac{\psi}{1-\psi} - \psi = \frac{\psi^2}{1-\psi} = \psi\,\overline{n_s} \qquad (41)$$

3. *Mean time spent by a job in the system*:

$$E(T_s) = \overline{t_s} = \frac{\overline{n_s}}{\lambda} = \frac{\psi}{\lambda(1-\psi)} = \frac{\frac{\lambda}{\mu}}{\lambda\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu-\lambda} \qquad (42)$$

4. *Mean waiting time of a job in the queue*:

$$E(T_f) = \overline{t_f} = \frac{\overline{n_f}}{\lambda} = \frac{\psi^2}{\lambda(1-\psi)} = \frac{\psi}{\mu(1-\psi)} \qquad (43)$$

5. *Mean service time*:

$$E(T_O) = \frac{1}{\mu} \qquad (44)$$

6. *Factor of service channel idle time*

$$K_0 = p_0 = 1-\psi \qquad (45)$$

7. *Factor of service channel load*

$$K_1 = 1-p_0 = 1-(1-\psi) = \psi \qquad (46)$$

The equations (40)-(43) show that in the system M/M/1/∞ /∞/FIFO, $\lambda = \mu$, respectively $\psi = 1$ cannot by satisfied, because this would result in the growth of the parameters beyond all limits.

3 The M/M/n/n/∞/FIFO System

Using similar considerations and denotations as in the previous section we get

$$P(S_{k-1} \to S_k) = \lambda \Delta t, \; k = 1, \dots, n \tag{47}$$

$$P(S_k \to S_k) = (1 - \lambda \Delta t)(1 - k \mu \Delta t) \approx \approx 1 - (\lambda + k \mu)\Delta t, \; k = 0, \dots \tag{48}$$

$$P(S_{k+1} \to S_k) = (k+1)\mu \Delta t, \; k = 0, \dots, n-1 \tag{49}$$

Let $p_k(t)$ denote the probability that, at time t, just k requirements are in the system. Using the previous equations, we can calculate $p_0(t)$, $p_1(t)$, … , $p_k(t)$, …, $p_n(t)$.

$$p_0(t+\Delta t) = P(S_0 \to S_0) + P(S_1 \to S_0) = = p_0(t).(1 - \lambda \Delta t) + p_1(t).\mu \Delta t \tag{50}$$

$$p_1(t+\Delta t) = P(S_0 \to S_1) + P(S_1 \to S_1) + P(S_2 \to S_1) = = p_0(t).\, \lambda \Delta t + p_1(t).[1 - (\lambda + \mu)\Delta t] + p_2(t).\, 2\mu \Delta t \tag{51}$$

$$\dots$$

$$p_k(t+\Delta t) = P(S_{k-1} \to S_k) + P(S_k \to S_k) + P(S_{k+1} \to S_k) = p_{k-1}(t).\, \lambda \Delta t + p_k(t).[1 - (\lambda + k\mu)\Delta t] + + p_{k+1}(t).\,(k+1)\mu \Delta t, \quad k = 2, \dots, n-1 \tag{52}$$

However, if all channels are occupied and the queue is nonempty, the last equation changes to (7).

$$p_k(t+\Delta t) = p_{k-1}(t).\, \lambda \Delta t + p_k(t).[1 - (\lambda + n\mu)\Delta t] + + p_{k+1}(t).\, n\mu \Delta t, \quad k \geq n \tag{53}$$

After easy simplification of equations (50), (52) and (53), a limit transition for $\Delta t \to 0$ we get a set of first-order ordinary differential equations. Since the initial conditions may also be simply expressed, we can derive that

$$p_0 = \left[\sum_{k=0}^{n-1} \frac{\psi^k}{k!} + \frac{\psi^n}{n!} \frac{1}{1 - \frac{\psi}{n}} \right]^{-1} \tag{54}$$

$$p_k = \frac{\psi^k}{k!} p_0, \quad k = 1, \dots, n-1 \tag{55}$$

$$p_k = \frac{\psi^k}{n!} \frac{n^n}{n^k} p_0, \quad k \geq n \tag{56}$$

These equations may now be used to derive other important characteristics of the M/M/*n*/*n*/∞ /FIFO system, which include:

1. *Mean number of requirements in the system*:

$$E(N_s) = \overline{n_s} = \sum_{k=0}^{\infty} k \, p_k \tag{57}$$

2. *Mean number of requirements in the queue* (*mean queue length*):

$$E(N_f) = \overline{n_f} = \sum_{k=n}^{\infty} (k-n) p_k \tag{58}$$

3. *Mean number of free service channels*:

$$E(N_c) = \overline{n_c} = \sum_{k=0}^{n-1} (n-k) p_k \tag{59}$$

4. *Mean time spent by a requirement in the system*:

$$E(T_s) = \overline{t_s} = \frac{\overline{n_s}}{\lambda} \tag{60}$$

5. *Mean waiting time of a requirement in the queue*:

$$E(T_f) = \overline{t_f} = \frac{\overline{n_f}}{\lambda} \tag{61}$$

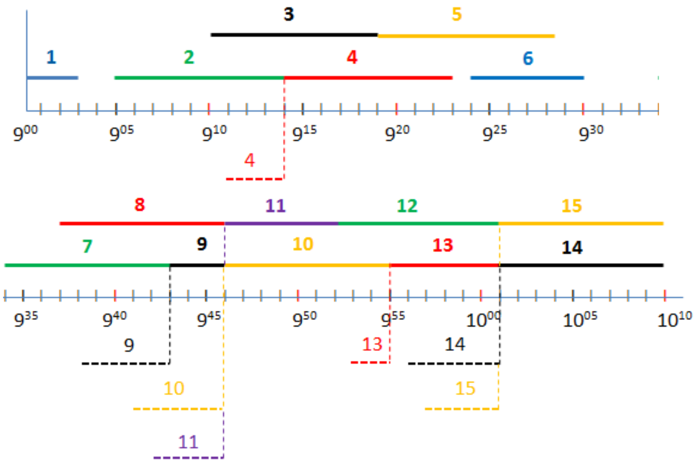

Fig. 3: System with two channels

4 Simulation of Queueing Processes

As, in practice, some assumptions may not be satisfied, particularly the Poisson (Markov) process properties of stationarity and the independence of increases, such as the number of clients in shops and railway stations substantially changing during the daytime, the formulas that we have derived, may not be entirely accurate. However, queueing systems can also be studied by Monte Carlo simulations, which generate random numbers representing the moment of the requirements entering into the system and the service time.

In Fig. 1, a queuing system with two service channels, 15 requirements, and the FIFO queue type is considered. We can see that for 2+4=6 minutes from total 70 minutes there is no requirement in the system, that means

$$p_0 = \frac{\langle 9^{03}, 9^{05}\rangle + \langle 9^{30}, 9^{34}\rangle}{\langle 9^{00}, 10^{10}\rangle} = \frac{2+4}{70} = \frac{6}{70} = 0{,}086$$

$$p_1 = \frac{\langle 9^{00}, 9^{03}\rangle + \langle 9^{05}, 9^{10}\rangle + \langle 9^{23}, 9^{24}\rangle + \langle 9^{28}, 9^{30}\rangle}{\langle 9^{00}, 10^{10}\rangle} +$$
$$+ \frac{\langle 9^{28}, 9^{30}\rangle + \langle 9^{34}, 9^{37}\rangle}{\langle 9^{00}, 10^{10}\rangle} = \frac{3+5+1+2+3}{70} = \frac{14}{70} = 0.2$$

$$p_2 = \frac{\langle 9^{10}, 9^{11}\rangle + \langle 9^{19}, 9^{23}\rangle + \langle 9^{24}, 9^{28}\rangle + \langle 9^{37}, 9^{38}\rangle}{\langle 9^{00}, 10^{10}\rangle} +$$
$$+ \frac{\langle 9^{46}, 9^{53}\rangle + \langle 9^{55}, 9^{56}\rangle + \langle 10^{01}, 10^{10}\rangle}{\langle 9^{00}, 10^{10}\rangle} =$$
$$= \frac{1+4+4+1+7+1+9}{70} = \frac{27}{70} = 0.3857$$

$$p_3 = \frac{\langle 9^{11}, 9^{19}\rangle + \langle 9^{38}, 9^{41}\rangle + \langle 9^{53}, 9^{55}\rangle + \langle 9^{56}, 9^{57}\rangle}{\langle 9^{00}, 10^{10}\rangle} =$$
$$= \frac{8+3+2+1}{70} = \frac{14}{70} = 0.2$$

$$p_4 = \frac{\langle 9^{41}, 9^{42}\rangle + \langle 9^{43}, 9^{46}\rangle + \langle 9^{57}, 10^{01}\rangle}{\langle 9^{00}, 10^{10}\rangle} =$$
$$= \frac{1+3+4}{70} = \frac{8}{70} = 0.1143$$

$$p_5 = \frac{\langle 9^{42}, 9^{43}\rangle}{\langle 9^{00}, 10^{10}\rangle} = \frac{1}{70} = 0.0143$$

Now, from these estimations, characteristics (57)-(61) may be computed.

In [5], the M/M/n/n/∞/FIFO system was implemented in MATLAB using simulation data from a supermarket.

It makes it possible to enter λ (*mean intensity of the input*), μ (*mean service intensity*), the number of service lines n (then it is checked if $\lambda/n\mu < 1$), and the number of requirements. For these data, the probabilities $p_k(t)$ and the above-mentioned characteristics are computed.

To understand the behaviour of the system, the program also offers graphical output of simulations. In Fig. 3, four graphs are shown for a system with three lines, which show requirement arrival times, requirement service times, requirement waiting times for service and finishing times of services for these requirements.

Now we compare the analytical solution with the values obtained by simulation for different numbers of requirements. In the analytical part, we use formulas (8), (9) and (10), and the corresponding characteristics (11), etc. Simulations were run for 50, 200, and 500 requirements (clients). Table 1 sums up the results of the analytical formulas and simulations.

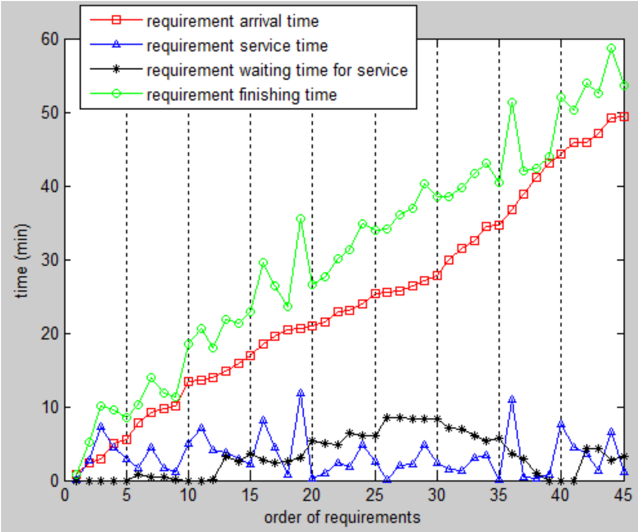

Fig. 4: Simulation of the M/M/n/n/∞ system for λ=45, μ=18, n=3, and 45 requirements.

	Analytical evaluation mean values	Simulation results number of requirements		
		50	200	500
$E(T_s)-E(T_f)$	3.33333	3.64353	3.5118	3.45478
$E(T_f)$	4.68165	3.06941	4.17567	5.21177
$E(T_s)$	8.01498	6.71293	7.68747	8.66655
$E(N_s)$	6.01124	5.09125	5.62803	6.23332
$E(N_f)$	3.51124	2.30994	3.05703	3.74851
$E(N_s)-E(T_f)$	2.5	2.78131	2.57101	2.48481

Table 1: A comparison of analytical and simulation results

We can see that, if the number of requirements increases, then the difference between the analytical and the simulation results decreases. For 50 requirements, the difference is about 12%, but for 500 requirements, only 5%. Based on these achievements, we can conclude that the computer implementation of the simulation model reasonably approximates the M/M/n/n/∞/FIFO system.

5 Conclusion

This paper describes an approach to modelling a queuing system with the use of Markov process properties and, for M/M/1/1/∞/FIFO and *M/M/n/n/∞/FIFO* systems, derives their characteristics in detail. These derivations are based on the assumptions of stationarity, regularity, and independence of Markov processes.

In real situations, some of the assumptions may not be satisfied, particularly the stationarity and the independence of increases, or even the distribution of stochastic variables may not be known at all. For these reasons, the calculations of transition probabilities that do not take this fact into account may give imprecise results.

Therefore, we propose a simulation approach, a strategy of requirement processing implemented in MATLAB based on the number of service lines, and a way of computing the characteristics from time intervals with the same number of requirements. However, the approximation of a theoretical model by the simulation model using real or randomly generated data depends on the number of requirements (and, therefore, on the time horizon length of the whole processing). The higher the number of requirements, the more the simulation results match the theoretical ones.

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