Synthesis of Singular Systems Walsh and Walsh-like Functions of Arbitrary Order

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Abstract: - Functionally complete systems of Walsh functions (bases), a particular case of alternating piecewise constant sequential functions, are widely used in various scientific and technological fields. As applied to the tasks of spectral analysis of discrete signals, the most interesting are those Walsh bases that deliver linear coherence of the frequency scales of fast Fourier transform (FFT) processors. By the frequency scales of an FFT processor, we mean the scale on which the normalized frequencies of the input signal are arranged (input scale) and the scale on which the signal's spectral components are arranged (output scale). The frequency scales of the FFT processor are considered linearly coherent if the processor responses with maximum amplitudes and phases of the same sign are located on the bisector of the Cartesian coordinate system formed by the frequency scales of the processor. None of the known Walsh bases ordered by Hadamard, Kaczmage, or Paley provide linear coherence of the frequency scales of the FFT processor. In this study, we develop algorithms to synthesize two systems, called Walsh-Cooley and Walsh-Tukey systems, which turn out to be the only ones in the set of classical Walsh systems and sequents Walsh-like systems, respectively, that deliver linear coherence to the frequency scales of FFT processors.

Key-Words: - Walsh function systems, Sequential Walsh-like functions, Linear connectivity of frequency scales of the DFT processor, Walsh-Cooley and Walsh-Tukey function bases.

Received: July 25, 2023. Revised: May 23, 2024. Accepted: June 22, 2024. Published: July 23, 2024.

1 Introduction

In the theory and practice of spectral analysis of discrete signals on finite intervals [1, 2], noise-resistant coding and compression of audio and video data [3, 4, 5], cryptography protection of information [6, 7], in cellular communication channels [8] and other fields of science and technology [9, 10, 11] functionally complete systems of Walsh functions (for simplicity – Walsh systems), which are a particular case of systems of alternating piecewise constant sequential functions [12], are widely used. These publications cover a range of applications, from signal processing, machine learning, encryption, and medical diagnostics. These can involve advanced mathematical functions like those discussed in synthesizing singular systems using Walsh and Walsh-like functions.

A Walsh system is a set of orthogonal functions taking values $+1$ and -1 over the entire domain of definition $N = 2^n$, where *n* is a natural number. The number of features included in a Walsh system is usually equal to the number of samples *N* of the signal since, in discrete spectral analysis, the number of harmonics must also be equal to *N*. The convenient way to represent these systems is to show them in square matrices (we will also call them

Walsh matrices), in which each row is a Walsh function, and for simplicity, instead of the values of the elements $+1$ and -1, write only their signs $+$ or $-$. The rows of Walsh matrices are numbered from top to bottom, and the columns from left to right, starting from zero. The row number determines the order of the Walsh function in the system and is further denoted by the symbol *k*.

The completeness of Walsh systems implies that on the definition interval *N*, it is impossible to introduce any additional function that would be orthogonal simultaneously to all other functions already included in the system [3]. A complete system of Walsh functions forms a basis of Walsh space, each function being called a basis function of the system of the corresponding order. Walsh systems are formed from basis functions in such a way that they (functions) form a symmetric matrix of order *N*. Matrix symmetry is achieved by the appropriate ordering of Walsh basis functions.

So far, only three orderings have found use: by Hadamard, Walsh, and Paley [13]. The enumerated Walsh systems will be called canonical systems. Hadamard proposed the first ordering in 1893 in connection with studies on the theory of determinants [14]. For example, the Walsh-Hadamard system (matrix) $\boldsymbol{H}_N$ of the eighth order has the form.

$$\boldsymbol{H}_8 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \; \to t \\ 0 & + & + & + & + & + & + & + & + \\ 1 & + & - & + & - & + & - & + & - \\ 2 & + & + & - & - & + & + & - & - \\ 3 & + & - & - & + & + & - & - & + \\ 4 & + & + & + & + & - & - & - & - \\ 5 & + & - & + & - & - & + & - & + \\ 6 & + & + & - & - & - & - & + & + \\ 7 & + & - & - & + & - & + & + & - \\ k & & & & & & & & \end{array} , \qquad (1)$$

where the parameters $k, t = \overline{0, N-1}$, define the order *k* of the basis function $h(k,t)$, coinciding with the matrix's row number and the normalized discrete time *t*, respectively.

The simplicity of forming Hadamard-ordered Walsh function systems [15, 16] has made them particularly convenient for implementing the *fast Fourier transform* (FFT) algorithm using the Cooley-Tukey scheme [17].

In 1923, Walsh proposed a system in which the Hadamard basis functions $\boldsymbol{W}_N = \{w(k,t)\}$, were ordered by the number of sign changes [18]. Later (1948), the $\boldsymbol{W}_N$ systems were called Walsh-Kaczmage systems [19]. In particular

$$\boldsymbol{W}_8 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \; \to t \\ 0 & + & + & + & + & + & + & + & + \\ 1 & + & + & + & + & - & - & - & - \\ 2 & + & + & - & - & - & - & + & + \\ 3 & + & + & - & - & + & + & - & - \\ 4 & + & - & - & + & + & - & - & + \\ 5 & + & - & - & + & - & + & + & - \\ 6 & + & - & + & - & - & + & - & + \\ 7 & + & - & + & - & + & - & + & - \\ k & & & & & & & & \end{array} .$$

Finally, in 1932, Paley developed [20] another way of ordering the Hadamard basis functions, which was reduced to the *binary-inverse permutation* (BIP) of binary numbers of Walsh-Hadamard functions. Subjecting the basis functions of the matrix (1) to BIP, we arrive at the system of Walsh-Paley functions of the eighth order.

$$\boldsymbol{P}_8 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \; \to t \\ 0 & + & + & + & + & + & + & + & + \\ 1 & + & + & + & + & - & - & - & - \\ 2 & + & + & - & - & + & + & - & - \\ 3 & + & + & - & - & - & - & + & + \\ 4 & + & - & + & - & + & - & + & - \\ 5 & + & - & + & - & - & + & - & + \\ 6 & + & - & - & + & + & - & - & + \\ 7 & + & - & - & + & - & + & + & - \\ k & & & & & & & & \end{array} . \qquad (2)$$

Matrices $\boldsymbol{P}_N$ of arbitrary order $N = 2^n$, can be compiled using a simple recurrence rule [3], the essence of which is reduced to the following transformations. Each row of the matrix $\boldsymbol{P}_{N/2}$ is written twice, and then to the first of them (let us call it *even*), the elements of the same row are assigned from the right, i.e., the elements of the right half of the even row repeat the elements of its left half, and to the second (odd row) — the opposite (complementary) elements. Let us call the above method of forming Walsh-Paley function systems a *repeated-complementary algorithm* or, for short, an R.C. algorithm.

In many applications, the Walsh-Paley systems proved preferable to the previously obtained Hadamard and Kaczmage systems. For this reason, the Walsh-Paley function systems $\boldsymbol{P}_N$ are reference systems in the sense that by the appropriate permutation of the numbers k of basis functions of these systems, one can obtain the matrices of all remaining classical Walsh systems [21], the total number of which L_n the relation determines.

$$L_n = \prod_{i=1}^{n} \left(2^i - i (\mathrm{mod}\ 2)\right). \qquad (3)$$

For some values of n, the estimates L_n calculated in formula (3) are given in Table 1.

Table 1. Number of symmetric systems of Walsh functions

n	1	2	3	4	5
N	2	4	8	16	32
	1	4	28	448	13'888

n	6	7	8
N	64	128	256
	888'832	112'881'664	28'897'705'984

Along with the most common way of representing the elements of Walsh matrices in the form + and −, coded matrices are often used. In these matrices, the "plus" sign is replaced by the digit 0 and the "minus" by the digit 1, thus transferring Walsh systems from the space of originals to the space of images. As a result of such substitution, the matrix (2), for example, takes the form (4).

$$\boldsymbol{P}_8 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \; \to t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 6 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 7 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ k & & & & & & & & \end{array} . \qquad (4)$$

The transition from the original space to the image space is accompanied by replacing the bitwise multiplication operations by modulo 2 bitwise addition operations.

The main disadvantage of the R.C. algorithm for synthesizing systems of Paley functions is that the calculation of the matrix of the order 2^{n+1} must precede the computation of the matrix of the order 2^n. An alternative to the recurrent R.C. algorithm is the empirically established [21] algorithm for *direct computation* of basis functions of Paley systems, hereafter referred to as the D.C. algorithm. The essence of the estimation scheme of the D.C. algorithm for the system of Paley basis functions in image space is as follows. Irrespective of the ordering method, the elements of the null order functions in any Walsh system are equal to 0, i.e.,

$$p(0, t) = 0 \ . \qquad (5)$$

The elements of the first-order basis functions, called generating functions of Walsh-Paley systems, are defined by the expression

$$p(1, t) = \begin{cases} 0, & t = \overline{0, N/2 - 1} \\ 1, & t = \overline{N/2, N-1} \end{cases} . \qquad (6)$$

Expressions (5) and (6) are the initial conditions of the Walsh-Paley system synthesis algorithm, which directly follow from the matrix (4), and if $k \geq 1$, then the relations hold:

$$p(2k, t) = p\left(k, (2t)_N\right) \qquad (7)$$

– even and

$$p(2k+1, t) = p(2k, t) \oplus p(1, t)$$

– odd basis functions.

In the right part of equality (7), the notation $(2t)_N$ means calculating the remainder of a number $2t$ modulo N.

The systems of Walsh functions, which we denote $\{\varphi(k, t)\}$, are widely used as bases for the *discrete Fourier transform* (DFT). The spectrum $\dot{\boldsymbol{X}}_m(k)$ of the complex discrete signal $\dot{x}_m(t)$, formed by the DFT processor in Walsh function basis, is defined by the relation

$$\boldsymbol{X}_m(k) = \sum_{t=0}^{N-1} \dot{x}_m(t) \ \varphi(k, t), \qquad (8)$$

in which N − sample volume, m − normalized input signal frequency, k − signal harmonic number, and t − signal reference number (discrete time).

The amplitude-frequency and phase-frequency characteristics (AFC and PFC) of DPF processors are usually computed concerning a *discrete exponential signal* (DES)

$$\dot{x}_m(t) = \exp\left(j\frac{2\pi}{N}mt\right), \quad m, t = \overline{0, N-1}. \quad (9)$$

For small values of *N*, spectrum estimation is solved quite simply using the graph-analytical method of calculation, the essence of which is shown in Fig. 1 for the eight-point DPF based on Walsh-Paley functions (2) when calculating, for example, the harmonic $\dot{X}_1(1)$.

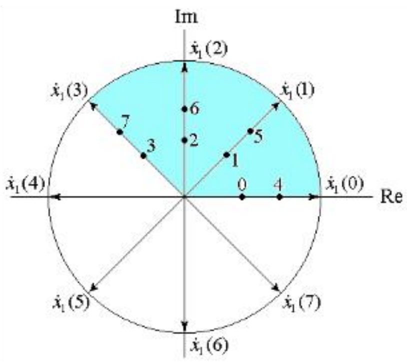


Fig. 1. To the definition of the harmonic $\dot{X}_1(1)$ in the basis of Walsh-Paley functions

The points in Fig. 1 mark the vectors of input signals involved in the formation of the response $\dot{X}_1(1)$ in the first output channel of the processor, and the digits in the vicinity of these points indicate the value of the time reference t of the first order basis function $p(1,t)$. According to the matrix (2), the vectors of the first four samples of signal $\dot{x}_1(t)$ are taken with a + sign and the remaining $(t = \overline{4, 7})$ with a − sign.

FFT processors realized according to the Cooley-Tukey scheme in different Walsh bases provide the following characteristics of the spectrum of signal (9) at integer frequencies. Firstly, the spectrum consists of a set of pairs of harmonics, and in each pair, the harmonics have the same amplitudes but opposite signs of initial phases. Secondly, for all the above Walsh bases, if an "even" signal (i.e., the frequency m is even) is fed to the input of the FFT processor, then only the even output channels of the processor "ring," and vice versa.

The harmonics of the discrete complex-exponential signal, whose sample size is eight, in the Walsh-Paley function basis are given in Table 2 for "odd" and Table 3 for "even" harmonics. The complex values $\dot{z}$ of the harmonics are denoted as follows: $\dot{z} = (a, b)$, where a and b are the real and imaginary components of the vector $\dot{z}$, respectively.

Table 2. Odd harmonics of the complex-exponential signals (9)

m	$\dot{X}_m(1)$	$\dot{X}_m(3)$
1	$2(1, 1+\sqrt{2})$	$2(1, \sqrt{2}-1)$
3	$2(1+\sqrt{2}, -1)$	$2(1-\sqrt{2}, 1)$
5	$2(1, 1-\sqrt{2})$	$2(1, -1-\sqrt{2})$
7	$2(1-\sqrt{2}, -1)$	$2(1+\sqrt{2}, 1)$
m	$\dot{X}_1(5)$	$\dot{X}_1(7)$
1	$2(1, 1-\sqrt{2})$	$2(1, -1-\sqrt{2})$
3	$2(1-\sqrt{2}, -1)$	$2(1+\sqrt{2}, 1)$
5	$2(1, 1+\sqrt{2})$	$2(1, \sqrt{2}-1)$
7	$2(1+\sqrt{2}, -1)$	$2(1-\sqrt{2}, 1)$

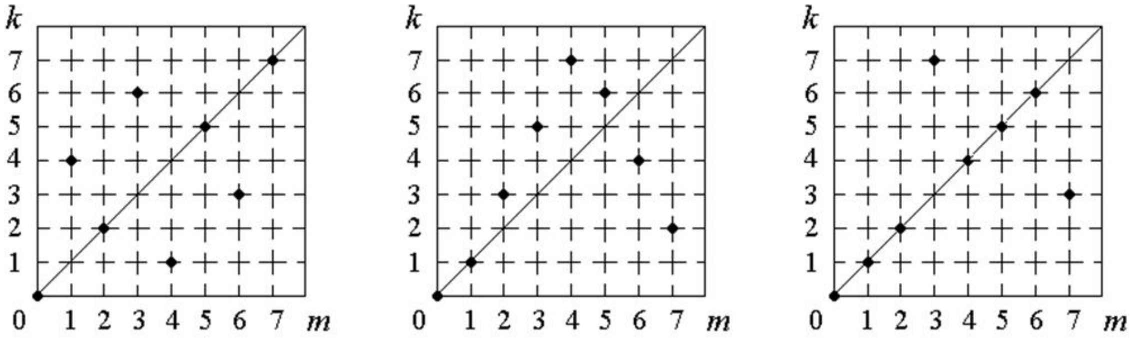
Table 3: The even harmonics of the complex-exponential signal (9)

m	$\dot{X}_m(0)$	$\dot{X}_m(2)$	$\dot{X}_m(4)$	$\dot{X}_m(6)$
0	8	0	0	0
2	0	4(1, 1)	0	4(1, – 1)
4	0	0	8	0
6	0	4(1, – 1) –1)	0	4(1, 1)

Similar discrete complex-exponential signals spectrum calculations can be performed for an arbitrary Walsh basis.

Let us introduce the concepts of the *frequency scales* of the FFT processor (as well as FFT). The abscissa axis *X* of the Cartesian coordinate system, on which we place the normalized frequencies *m* of the input signal $\dot{x}_m(t)$, lets us call the *input* frequency scale of the processor. The ordinate axis *Y*, on which we will put the number of k output channels of the processor, is designed to capture the harmonic of the signal $\dot{x}_m(t)$, we will call the *output* frequency scale of the processor.

Selecting from the tables containing the complex spectrum of the signal (9), harmonics with maximum amplitude and positive phase, we plot the relationship between the frequency scales *m* and *k* of the eight-point DPF processors in Walsh bases ordered by Hadamard, Kaczmage, and Paley (Fig. 2).



Hadamard Basis Kaczmage Basis Paley Basis

Fig. 2. Ratio of frequency scales of DFT processors in canonical Walsh bases

For plotting in Fig. 2, we chose harmonics with positive phases because none of the harmonics with negative phases in the canonical Walsh bases falls on the bisector of the coordinate axes (*m*, *k*), which significantly distorts the trajectory of the location of points on the plane (*X*, *Y*), far from linear.

We will say that a specific basis provides the frequency scales of the DFT processor with linear coherence if the harmonics of the discrete complex-exponential signal with maximum amplitude and unchanging in sign phase are located on the bisector of the right angle formed by the coordinate axes *m* and *k*. Alternatively, in other words, the frequency scales of the DFT processor are linearly coherent if, when samples of a complex-exponential signal are input to the input of the DFT processor, the maximum "ringing" will be in the output channel whose number *k* coincides with the frequency of the input signal *m*. Linear coherence to the frequency scales of the DFT processor is provided, for example, by the basis of *discrete exponential functions* (DEF).

2 Statement of the Research Problem

As follows from the graphs presented in Fig. 2, none of the three canonical Walsh bases, ordered by Hadamard, Kaczmage, or Paley, provide linear coherence to the frequency scales of DFT processors, which complicates their application in spectrum analysis oriented to the measurement of the frequency of input signals. The mentioned disadvantage of the canonical Walsh function systems was the starting point that predetermined the most critical problem solved in this study, namely, the development of algorithms for the synthesis of Walsh and Walsh-like bases (the definition of Walsh-like bases is given in Section 4) that deliver linear coherence to the frequency scales of DFT processors. Such Walsh systems will be called singular.

3 Walsh-Cooley singular systems

Let us try to understand the reason for the nonlinear coherence of the frequency scales of FFT processors in the canonical Hadamard, Kaczmage, and Paley bases. For this purpose, we turn (see Fig. 3) to the tree of an eight-point FFT on the Cooley-Tukey scheme.

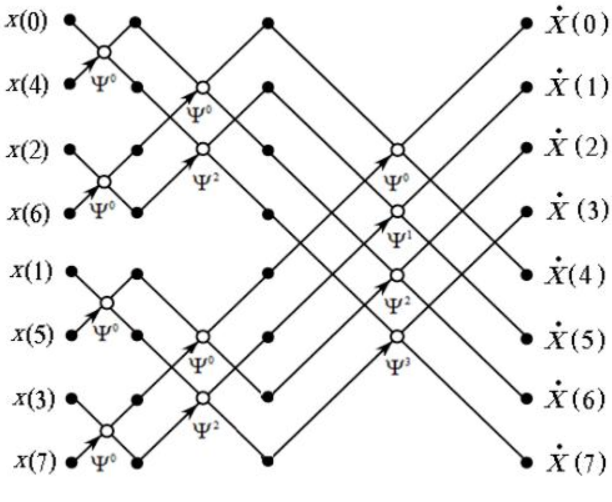


Fig. 3. Directed graph of the eight-point FFT the algorithm in the DEF basis

To give the harmonics *k* of the spectrum $\dot{X}(k)$ a natural linearly increasing ordering, the sample numbers *t* of the input signal $\dot{x}(t)$ must be subjected to BIP.

The symbol Ψ in Fig. 3 denotes the FFT processor's complex *phase multipliers* (PM) in the DEF basis, the degrees of which are defined by the expression

$$\Psi^{l} = \exp\left(-j\frac{2\pi}{N}l\right), \quad l = \overline{0, N-1}.$$

The set of multipliers Ψ^{l}, $l = \overline{0, 3}$, used in the third step of the eight-point FFT tree (Fig. 4), is covered by an arc. The light circle highlights the PM included in the set, and the arc's arrow rests on the vector of the multiplier not included in this set.

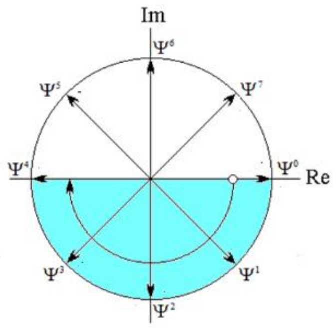


Fig. 4. Phase multipliers of the DEF basis for the eight-point DFT

To ensure the transition to the Walsh function basis, we approximate the PM of the DEF system located in the lower complex half-plane by values $+1$, and in the upper half-plane by values -1, as shown in Fig. 5.

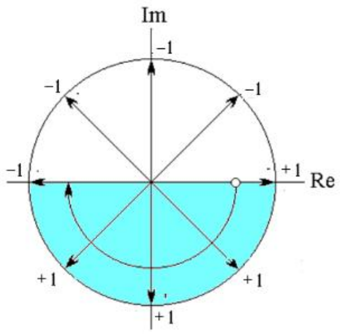


Fig. 5. Approximation of DEF phase multipliers

If the numbers t of time samples of the signal $\dot{x}(t)$, fed to the input of the FFT processor form a natural sequence (see Fig. 6), then the output of the processor's spectrum based on Walsh-Hadamard functions (***H***) is formed. If the numbers t are subjected to BIP, the processor forms a spectrum in the Walsh-Paley function basis (***P***). Finally, if, in addition to BIP, the numbers of samples of the input signal are transformed by the inverse Gray code, then we obtain a spectrum in the Walsh-Kaczmage function basis (***W***).

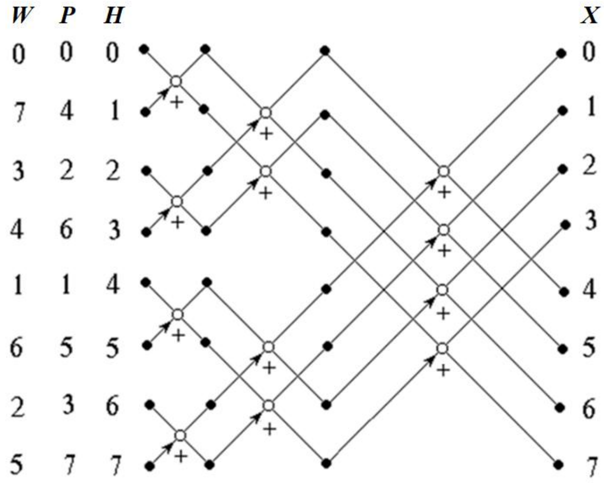


Fig. 6. Directed graph of the eight-point FFT the algorithm

The + sign in the butterfly operators means that the weight of the lower edge of the operator is +1, and thus, the operator performs the transformation shown in Fig. 7.

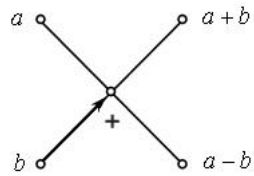


Fig. 7. "Butterfly" operator

From the analysis of the approximation of the complex phase multipliers shown in Fig. 5, from the engineering point of view, the proposed approximation can hardly be recognized as rational. More appropriate should be considered the approximation in which the multipliers located in the right (positive concerning the abscissa axis) complex half-plane are replaced by positive units, and the PMs from the left (negative) half-plane are replaced by negative units.

This kind of PMs approximation can be achieved, as shown in Fig. 8, by rotating the unit circle on the complex plane counterclockwise by the angle $\pi/2$. In this case, the position of the coordinate axes and the arc encompassing the phase multipliers of the last step of the FFT tree should remain unchanged.

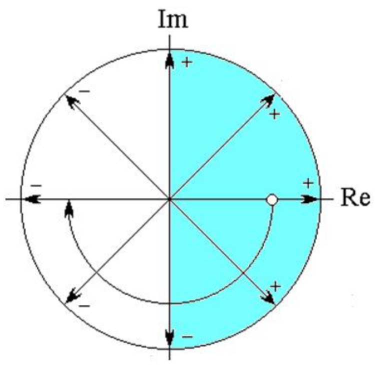


Fig. 8. Modified approximation phase multipliers of the DEF

The butterfly operator with negative PM (let us call it the inverse operator) realizes the transformation shown in Fig. 9.

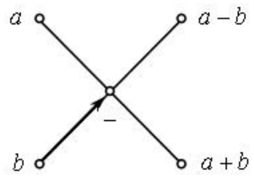


Fig. 9: Inverse operator "butterfly"

Let us make a tree of the eight-point FFT algorithm (Fig. 10), the sequence of phase multipliers in the third stage of the transformation, which $(+,+,-,-)$ is chosen based on the approximation presented in Fig. 8.

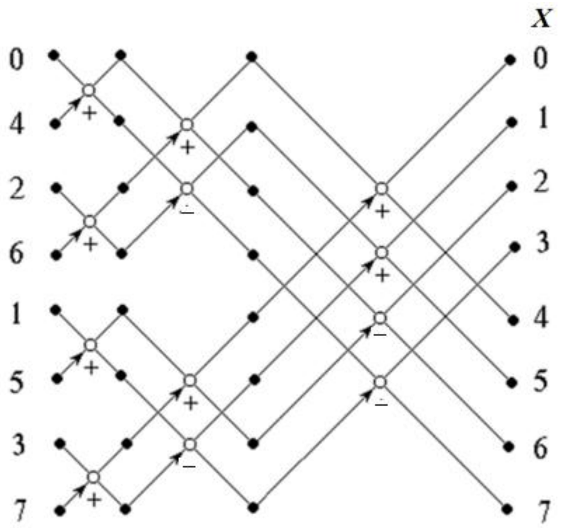


Fig. 10. Tree of the modified FFT algorithm

Following the transformations shown in Figs. 9 and 10, we easily arrive at the Walsh system (denoted by $\boldsymbol{C}_N$), which we will call the Walsh-Cooley system of functions, thus paying tribute to the positive influence of the Cooley-Tukey algorithm on the emergence of a new way of ordering the basis functions of Walsh systems. The Walsh-Cooley system was first reported at the International Conference in Wroclaw [22] in 2000. The eighth-order matrix $\boldsymbol{C}_8$ of the Walsh-Cooley system is represented by the expression (10):

$$\boldsymbol{C}_8 = \begin{array}{c|cccccccc} k \backslash t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 5 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} . \qquad (10)$$

The Walsh-Cooley functions $c(k,t)$ in the image space are calculated by formulas similar to formulas $(4)-(6)$ for the basis functions $p(k,t)$ of Walsh-Paley systems and differ from the latter only by the values of the first order functions. In particular,

$$c(1,t) = \begin{cases} 0, & t = \overline{0, \frac{N}{4}-1} \text{ and } t = \overline{\frac{3N}{4}, N-1}, \\ 1, & t = \overline{\frac{N}{4}, \frac{3N}{4}-1}. \end{cases}$$

And if $k \quad 1$, then

$$c(2k,t) = c(k,(2t)_N) \qquad (11)$$

− for even functions and

$$c(2k+1,t) = c(2k,t) \quad c(1,t)$$

− for odd functions.

Perhaps one of the first mathematicians to point out that Walsh function systems are somehow related to Gray's codes was Dr. C. Yen, a professor at the University of Sydney. In particular, he found [23] that if the binary numbers of the rows (basis functions) of the matrix of the Walsh-Paley system (***P***) are rearranged according to the law of Gray's inverse transformation (coding), we come to the Walsh system ordered at Kaczmage (***W***). Naturally, rearranging the strings of the Walsh-Kaczmage system by the direct Gray code restores the Walsh-

Paley system. It was noted above that the DIP of the basis functions of the Walsh-Hadamard system transforms it into the Walsh-Paley system and vice versa. The complete picture of the interrelation of the systems ***P***, ***H***, and ***W*** by Yen is presented in Fig. 11, in which, for simplicity, the DIP operation is denoted by symbol 1, and the operations of the direct and inverse Gray transformations by symbols 2 and 3, respectively.

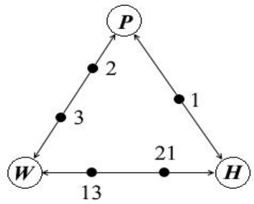


Fig. 11. Relationship of function numbers of Walsh's canonical system

According to Fig. 11, to transform the system ***W*** into ***H***, the numbers of basis functions of the system ***W*** must first be rearranged according to the law of the direct Gray code 2 and thus transform the system ***W*** into the system ***P***, and then perform DIP over the numbers of functions of the system ***P*** using operator 1. Consequently, the system of functions ***W*** is transformed into ***H*** using code 21, which belongs to a subset of the so-called composite Gray codes (CGC). The inverse transformation of the system of functions ***W*** into ***H*** is performed using the code inverse to CGC 21, i.e., using the code .

Gray's codes, proposed in 1953 in response to the requests of engineering practice concerning the construction of optimal converters of the "angle – code" type according to the criterion of minimum ambiguity error [10], at the dawn of their appearance, attracted the attention not only of mathematicians but also a wide range of developers of various electronic equipment. As noted above, the canonical Walsh systems are interconnected by classical Gray operators. In contrast, the new Walsh-Cooley system falls out of this chain of interconnection. The possibility of constructing codes inverse in the direction of formation to the classical Gray codes was out of the researchers' field of view. In the known (classical) scheme (Fig. 12), the formation process of direct and inverse Gray codes develops in the direction from left to right. On this basis, let us call the classical method of Gray transformation left-sided. In this case, the transformed number's high (left) digit remains unchanged in both forward and reverse transformations.

At the same time, it is possible to propose a scheme of the Gray transformation [21], reversed to the direction of the classical (*left-sided*) transformation. In such a class of transformations,

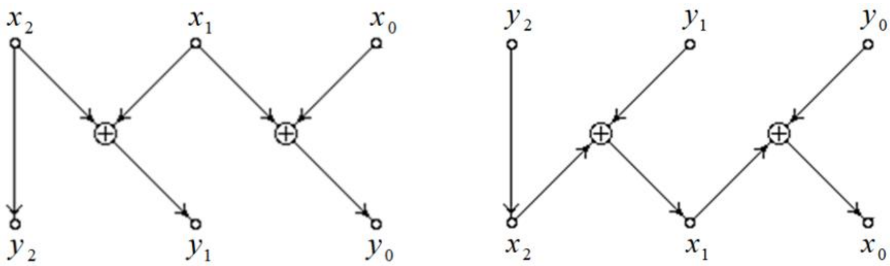


Fig. 12: Graphs of the forward (*a*) and backward (*b*) Gray's left-sided transform

which is called right-sided (Figure 13); the value of the transformed number's lower (right) digit remains unchanged in the forward and reverse transformations, denoted by symbols 4 and 5, respectively.

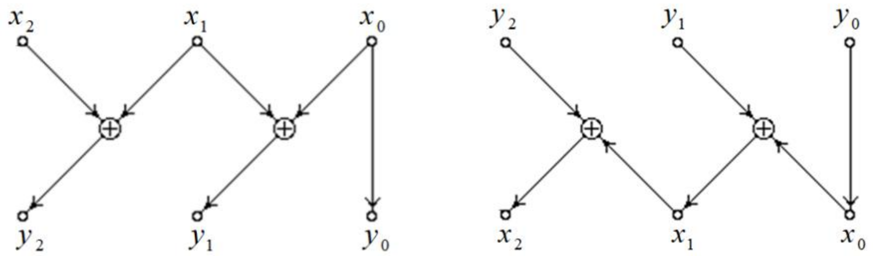


Fig. 13. Graphs of forward (*a*) and backward (*b*) Gray's right-sided transform

From the comparison of Figs. 12 and 13, we see that we arrive at the right-sided Gray transform by expanding the graphs of the left-sided transform by 180° for the vertical axis of symmetry. The fundamental difference between left- and right-sided Gray codes is as follows. If, during the left-sided Gray transformation, the Hemming distance between adjacent code combinations remains constant and equal to one, then, during the right-sided transformation the noted property of classical Gray codes is violated, which can be traced by Table 4.

Table 4: Towards a comparison of Gray's transformations

Dec	Bin	2	4
0	000	000	000
1	001	001	011
2	010	011	110
3	011	010	101
4	100	110	100
5	101	111	111
6	110	101	010
7	111	100	001

Let us summarize Gray's simple operators in Table 5, including also the operator 0, which is a unit matrix, and the operators of cyclic shifts by one digit to the right, denoted by the numerical symbol 6, and to the left, which we will indicate by the symbol 7.

Table 5: The operators of simple Gray transformations

Design. operator	The operation to be performed
0(e)	Saving the source code
1	Inverse code rearrangement
2	Gray's direct left-sided transform
3	Gray's inverse left-sided transform
4	Gray's direct right-sided transform
5	Gray's inverse right-sided transform
6	Cyclic shift one digit to the right
7	Cyclic shift one digit to the left

The listed operators of simple Gray transformations can be represented in matrix form, and the order n of these matrices coincides with the degree of two in the expression $N = 2^n$, by which the order of Walsh matrices is given. In particular, the matrices of simple Gray transform operators accompanying Walsh systems of the eighth order are presented in Table 6.

Table 6. Matrix forms of simple Gray operators

$0 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$1 = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$
$2 = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$	$3 = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$
$4 = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$	$5 = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$
$6 = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$	$7 = \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$

We come to the following conclusions from the data analysis in Table 6. First, the matrices of simple Gray operators are right-sided symmetric, i.e., symmetric concerning the auxiliary diagonal, and the operators 0 and 1 correspond to bilaterally symmetric matrices. Second, the pairs of Gray matrices (operators) located in the rows of Table 6 (except for the operators of the first row) are mutually inverse, which means that $0\ 1 = 1$. The multiplication operation is performed over the field F_2.

According to (7), there are 28 symmetric Walsh systems of the eighth order. In contrast, using classical Gray codes, it is possible to connect, as shown in Fig. 10, only three systems ordered by Hadamard, Kaczmage, and Paley. The reason for the incompleteness of the interconnection graph to cover all Walsh systems of fixed order N is that the approach proposed by S. Yen, based on the use of only classical (left-sided) Gray transformations, turned out to be a dead end.

All Walsh systems of order N consist of the same complete set of basis functions and differ from each other only by the ways of their ordering.

The problems related to the development of algebraic methods for ordering basis functions symmetrizing Walsh systems became solvable only after the introduction of right-sided Gray transformations [21]. Fig. 14, a development of the scheme for transforming Walsh systems according to C. Yen (see Fig. 11), is the simplest example illustrating the application of the direct right-sided Gray transform operator (operator 4).

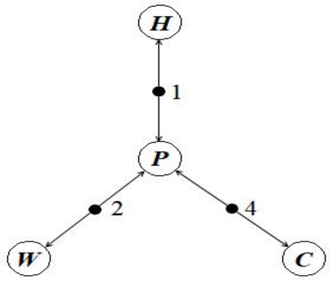


Fig. 14. Tree of the interrelation
of the fourth order Walsh systems

The complete graph of the systems of Walsh functions of the eighth order is shown in Fig. 15.

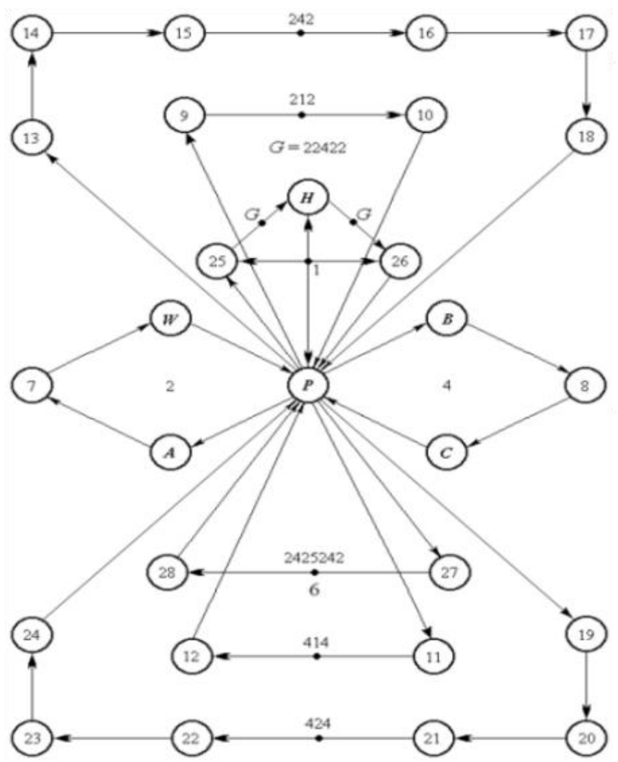


Fig. 15: The complete graph of Paley-connected
of the eighth order Walsh systems

The large Latin letters in the graph nodes denote the invariant fundamental Walsh systems, named so because the algorithm for their synthesis does not depend on the order N of the Walsh systems. The numbers in the nodes of the contours denote the numbers of systems that, together with the invariant systems, constitute the complete set of classical Walsh systems of the eighth order. The directed edges connect the tree nodes, each assigned a weight equal to a simple or symmetric composite Gray code.

Let us briefly explain how, based on the eighth-order Walsh-Paley matrix $\boldsymbol{P}$ given by expression (4), we can obtain, for example, the Walsh matrix number 15 (denoted by $\boldsymbol{M}_{15}$). The matrices $\boldsymbol{P}$ and $\boldsymbol{M}_{15}$ are related by the CGC $\boldsymbol{G} = (242)^3$. Based on the data in Table 6, we have

$$242 = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \quad \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} =$$

$$= \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{matrix} ,$$

which leads to the value

$$\boldsymbol{G} = \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix} .$$

The numbers k_{15} of the basis functions of the system $\boldsymbol{M}_{15}$ are related to the numbers k_p of the basis functions of the Walsh-Paley system by the relation

$$k_{15} = k_p \;\; \boldsymbol{G} .$$

By successively searching the states of three-bit numbers k_p , we arrive at the values of the row numbers of the matrix $\boldsymbol{M}_{15}$ (see Table 7), in which rows of matrix $\boldsymbol{P}$ are moved.

Table 7. Rearrangement of rows
of the matrix $\boldsymbol{P}$ to the rows of the matrix $\boldsymbol{M}_{15}$

k_p	0	1	2	3	4	5	6	7
k_{15}	0	6	4	2	1	7	5	3

The matrix $\boldsymbol{M}_{15}$ has the form

$$P_{15} = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \rightarrow t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 5 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 7 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ k & & & & & & & & \end{array} .$$

The considered example explaining the algorithm of calculating the matrix M_{15} leads to the following constructive conclusion. Each matrix W of a Walsh system of order $N = 2^n$ corresponds to a uniquely related matrix of order *n*, which we will call an *indicator matri*x (IM) and denote by ***J***. The IMs of Walsh systems are binary right-sided symmetric matrices that are non-degenerate over the field F_2. The number L_n of IMs of size *n* coincides with the number of Walsh systems of order N and is determined by expression (3). Knowing the IM ***J***, one can easily find the matrix ***W*** corresponding to it. For this purpose, it is enough to use the formula

$$k_w = k_p \;\; J \, , \tag{12}$$

where $k_p -$ rows of the Walsh-Paley matrix ***P*** are transferred to $k_w -$ rows of the matrix ***W.***

Naturally, the Cooley-Tukey FFT tree remains unchanged regardless of the basis realized by the FFT processor. At the same time, for the processor to form the signal spectrum on one or another basis, the time samples of the signal at the input of the processor must be subjected to a permutation depending on the chosen transformation basis (see, for example, Fig. 6). Using the relation (12), let us show what permutations should be subjected to the numbers of the signal samples at the input of the FFT processor to obtain the spectrum of the signal in the Walsh-Cooley function basis. For this purpose, let us refer to Fig. 15. As we can see, the transition from the Walsh-Paley system (***P***) to the Walsh-Cooley system (***C***) is carried out as a result of the rearrangement of row numbers of the Walsh-Paley matrix by the inverse Gray's transformation right-sided (whose numerical symbol is 5). The latter means that the IM of the Walsh-Cooley system J_c is (see Table 6) the matrix

$$J_c = 5 = \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} . \tag{13}$$

Substituting matrix (13) into formula (12) and taking into account that the sequence of numbers k_p is formed by DIP of the sequence of natural numbers k_h, we come to the summary Table 8 of permutations of the signal samples at the input of the eight-point FFT processors, providing the formation of DES spectra (8) in the corresponding bases.

Table 8: Rearrangement of signal count numbers at the input of FFT processors in basis ***H***, ***P***, and ***C***

k_h	0	1	2	3	4	5	6	7
k_p	0	4	2	6	1	5	3	7
k_c	0	4	6	2	7	3	1	5

Among the numerous Walsh systems, the Walsh-Cooley systems are the only ones that provide delivers linear coherence to the frequency scales of the DFT processor. As a consequence, the frequency of the input signal can be unambiguously determined by the number of processor output channels in which the response with maximum modulus and negative phase is observed. None of the remaining Walsh systems provides similar linear coherence to the frequency scales of DFT processors. This is the uniqueness of Walsh-Cooley systems.

Let us use the graph-analytical method of calculating the harmonics of the complex-exponential signal (11) in the Walsh-Cooley function basis (10), assuming the sample size to be eight. An example of harmonic calculation $\dot{X}_1(1)$ is shown in Fig. 16.

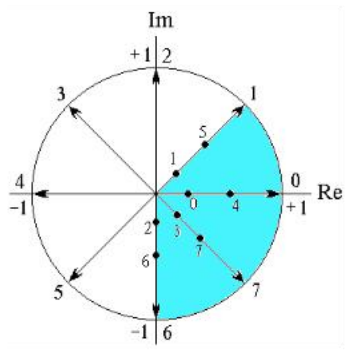


Fig. 16. Harmonic definition $\dot{X}_1(1)$

Summing up the marked vectors and distinguishing the real and imaginary parts, we obtain:

$$\dot{X}_1(1) = 2(1+\sqrt{2}, -1) .$$

The results of similar calculations of harmonics of "even" signals are summarized in Table 9 and of "odd" signals − in Table 10.

Table 9: Harmonics of "even" signals in the Walsh-Cooley basis

m	$\dot{X}_m(0)$	$\dot{X}_m(2)$	$\dot{X}_m(4)$	$\dot{X}_m(6)$
0	8	0	0	0
2	0	$4(1,\ -1)$	0	$4(1,\ 1)$
4	0	0	8	0
6	0	$4(1,\ 1)\quad 1)$	0	$4(1,\ -1)$

Table 10. Harmonics of "odd" signals in Walsh-Cooley basis

m	$\dot{X}_m(1)$	$\dot{X}_m(3)$
1	$2(1+\sqrt{2}, -1)$	$2(1-\sqrt{2},\ 1)$
3	$2(1,\ 1-\sqrt{2})$	
5	$2(1-\sqrt{2}, -1)$	$2(1+\sqrt{2}, 1)$
7	$2(1,\ 1+\sqrt{2})$	$2(1, \sqrt{2}-1)$
m	$\dot{X}_1(5)$	$\dot{X}_1(7)$
1	$2(1-\sqrt{2},\ -1)$	$2(1+\sqrt{2}, 1)$
3	$2(1,\ 1+\sqrt{2})$	$2(1, \sqrt{2}-1)$
5	$2(1+\sqrt{2}, -1)$	$2(1-\sqrt{2},\ 1)$
7	$2(1,\ 1-\sqrt{2})$	$2(1, -(1+\sqrt{2}))$

The analysis of Tables 9 and 10 shows that on the main diagonals of the matrices of spectra, there are harmonics with maximum modulo and negative phases. The harmonics $\dot{X}_0(0)$ and $\dot{X}_4(4)$ are natural, and their phases are equal to zero.

4 Walsh-Tukey singular systems

At least these properties are characteristic of classical Walsh systems:

1. The weight of each basis function of the system, except for the null order function (let us call it the zero function), is equal to $N/2$;

2. All non-zero basis functions in both the left and right halves contain the same number, equal to $N/4$, of 0 and 1 elements except for the function whose left half consists of only zeros and whose right half consists of only ones;

3. The Hemming distance between any pair of basis functions included in the system is $N/2$.

The so-called *Walsh-like systems* of (0,1)-sequential functions are alternatives to the classical Walsh systems. We will refer to Walsh-like systems that preserve the first and third properties of classical Walsh systems mentioned above but in which the second property is not satisfied for all basis functions. Or, in other words, to Walsh-like systems, we will refer to such (0, 1)-sequent systems [12], in whose basis functions the number of zeros and ones in each half of the definition interval is not necessarily the same as in the classical Walsh functions.

Let us estimate the number of symmetric orthogonal Walsh-like systems $(0,1)$ – of the eighth order [24]. Let us form the set of sequential functions, including only those that begin with zero, i.e., the sequential left digit contains the digit 0, and the remaining lower seven digits contain three zeros and four ones. Hence, the complete L_8 set contains 35 non-zero eighth-order sequents, which is determined by the number of seven-by-three combinations. All these functions are summarized in Table 11.

Table 11. Set of sequential functions of the eighth order

#	Discharge number									#	Discharge number							
	0	1	2	3	4	5	6	7			0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0		18	0	1	0	1	1	0	0	1
1	0	1	1	1	1	0	0	0		19	0	0	1	1	1	0	0	1
2	0	1	1	1	0	1	0	0		20	0	1	1	0	0	1	0	1
3	0	1	1	0	1	1	0	0		21	0	1	0	1	0	1	0	1
4	0	1	0	1	1	1	0	0		22	0	0	1	1	0	1	0	1
5	0	0	1	1	1	1	0	0		23	0	1	0	0	1	1	0	1
6	0	1	1	1	0	0	1	0		24	0	0	1	0	1	1	0	1

7	0	1	1	0	1	0	1	0		25	0	0	0	1	1	1	0	1
8	0	1	0	1	1	0	1	0		26	0	1	1	0	0	0	1	1
9	0	0	1	1	1	0	1	0		27	0	1	0	1	0	0	1	1
10	0	1	1	0	0	1	1	0		28	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	1	0		29	0	1	0	0	1	0	1	1
12	0	0	1	1	0	1	1	0		30	0	0	1	0	1	0	1	1
13	0	1	0	0	1	1	1	0		31	0	0	0	1	1	0	1	1
14	0	0	1	0	1	1	1	0		32	0	1	0	0	0	1	1	1
15	0	0	0	1	1	1	1	0		33	0	0	1	0	0	1	1	1
16	0	1	1	1	0	0	0	1		34	0	0	0	1	0	1	1	1
17	0	1	1	0	1	0	0	1		35	0	0	0	0	1	1	1	1

Let us match to each sequential function from Table 11 a set of sequents (highlighted by shading) separated from the forming sequents located in the diagonal elements of Fig. 17 by the Hamming distance $d = 4$.

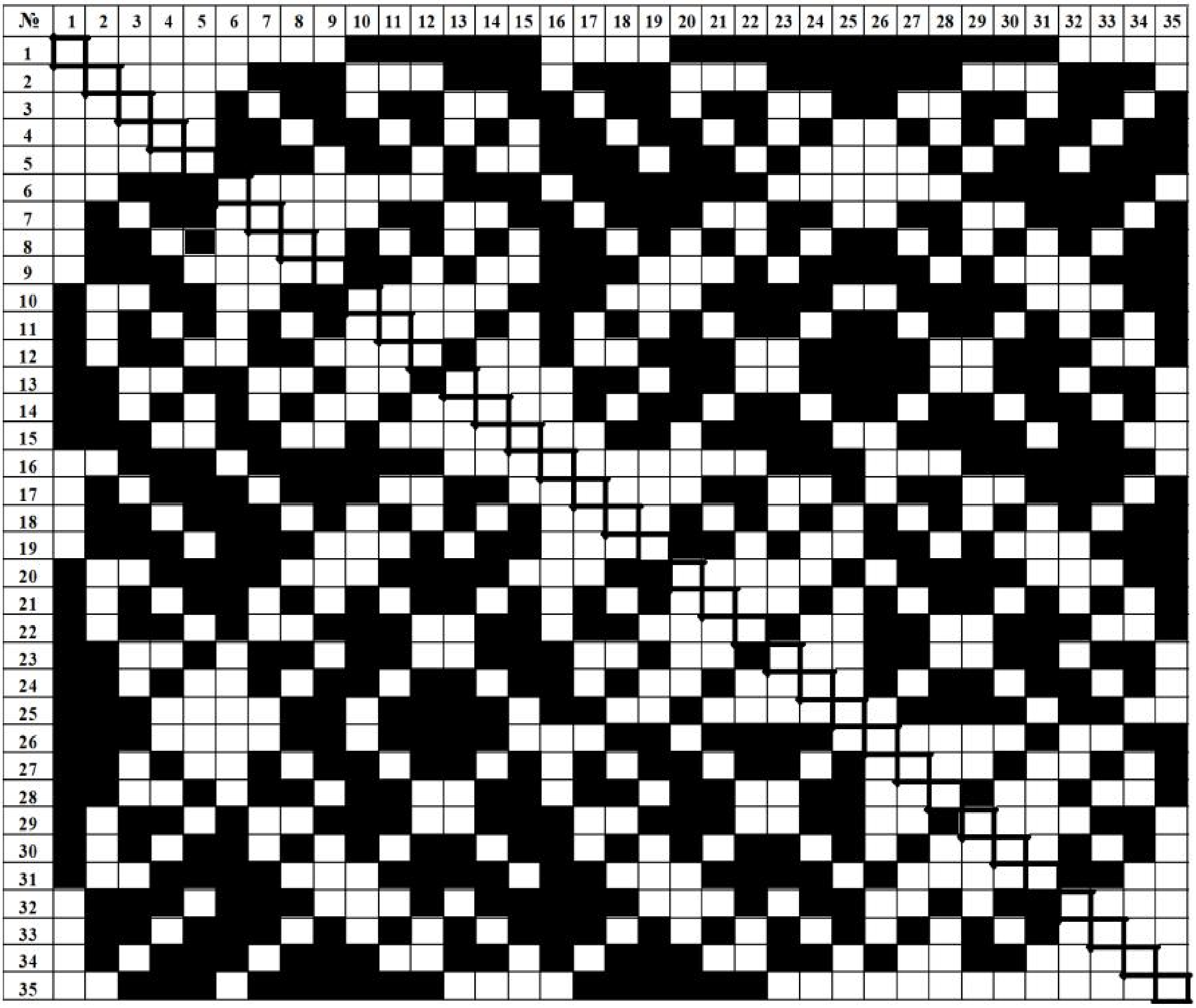


Fig. 17: Set of functions distant from the forming sequent at Hamming distance $d = 4$

The left column of Fig. 17 shows the numbers of the forming sequents, and the top row shows the numbers of sequents that are the Hamming distances away from the forming sequents, equal to 4.

In the following, we will stick to the following notations: the lowercase letter $\boldsymbol{s}_j$ will denote a specific j-th sequent, $j = \overline{0, L_N}$, representing an N-bit vector (in this article, it is a byte), and the uppercase letter $\boldsymbol{S}_i$ – the set of all sequents located in the shaded elements of the i-th row of the table in Fig. 17, $i = \overline{1, 35}$, including the zero sequent not shown in this figure $\boldsymbol{s}_0$. Let, in addition, $€_i$ – be a sequent forming j – the complete group of functions for which we introduce the symbol $\boldsymbol{SF}_{i,j}$ (*Sequence Full*).

Let us note such features of Fig. 17. Firstly, the table is symmetric concerning the main diagonal. Second, each table row, except for the forming sequent $€_k$ (the light diagonal element of the table, highlighted by the bold frame), includes 18 sequents $\boldsymbol{s}_j$, separated from the forming element $€_k$ by the Hamming distance $d(€_k, \boldsymbol{s}_j) = 4$. Finally, third, the entire set $\boldsymbol{\Omega}$ of rows $\boldsymbol{S}_i$ of the table can be partitioned into ten non-overlapping subsets $\boldsymbol{\Omega}_l$, $l = \overline{1, 10}$, with l-th subset including consecutive rows $\boldsymbol{S}_i$ containing the same number n_l of sequents to the left of the sequent $€_k$. For example, the subset $\boldsymbol{\Omega}_1$ is generated by the sequents $€_j$, $j = \overline{1, 5}$, with $n_1 = 0$; a second subset $\boldsymbol{\Omega}_2$ is formed by the sequents $€_j$, $j = \overline{6, 9}$, for which $n_2 = 3$, etc.

Information about the subsets $\boldsymbol{\Omega}_l$ is contained in Table 12

Table 12. Composition of subsets of $\boldsymbol{\Omega}_l$ sequential functions

	Numbers of l subsets of the sequent $\boldsymbol{\Omega}_l$									
	1	2	3	4	5	6	7	8	9	10
# sequent $€_k$	1-5	6-9	10-12	13-15	16-19	20-22	23-25	26-28	29-31	32-35
n_l	0	3	5	6	9	11	12	15	16	18

The forming sequents $€_k$, together with the zero sequential function $\boldsymbol{s}_0$ and 18 sequents located in the rows of the table in Fig. 17, form six complete groups of pairwise equidistant code combinations. The groups of sequential functions $\boldsymbol{SF}_{i,j}$, formed by the forming sequents $€_i$ of the subset $\boldsymbol{\Omega}_1$, are given in Table 13.

Table 13. Composition of groups formed by the forming sequences of the subset $\boldsymbol{\Omega}_1$

$\boldsymbol{SF}_{1,j}$	Group Sequences																			
	0	1	10	11	12	13	14	15	20	21	22	23	24	25	26	27	28	29	30	31
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{2,j}$	0	2	7	8	9	13	14	15	17	18	19	23	24	25	26	27	28	32	33	34
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	
6	■	■			■	■				■			■		■					■
$\boldsymbol{SF}_{3,j}$	0	3	6	8	9	11	12	15	16	18	19	21	22	25	26	29	30	32	33	35
1	■	■	■					■		■			■				■	■		
2	■	■	■					■			■	■				■			■	
3	■	■		■			■		■					■			■	■		
4	■	■		■			■				■	■			■					■
5	■	■			■	■			■					■		■			■	

6																				
$SF_{4,j}$	0	4	6	7	9	10	12	14	16	17	19	20	22	24	27	29	31	32	34	35
1																				
2																				
3																				
4																				
5																				
6																				
$SF_{5,j}$	0	5	6	7	8	10	11	13	16	17	18	20	21	23	28	30	31	33	34	35
1																				
2																				
3																				
4																				
5																				
6																				

Sequent s_j, included in the groups $SF_{i,j}$, are marked with gray cells in the rows of Table 13, and their corresponding j function numbers are in the black rows of the table above the sequences, with the row element containing the number i of the forming sequent $\tilde{s}_i$, lightened. The left column of Table 13 contains the numbers of $j=\overline{1,6}$ groups $SF_{i,j}$, formed by the sequents $\tilde{s}_i$, $i=\overline{1,5}$, constituting a subset of $\mathbf{\Omega}_1$. The 30 groups summarized in Table 13, corresponding to the subset of sequential functions $\mathbf{\Omega}_1$, constitute the complete set of groups of pairwise equidistant sequential functions. That means, in particular, that the group of functions formed by any sequent $\tilde{s}_j$, $6 \quad j \quad 35$, is absorbed by one of the groups of the subset $SF_{i,j}$ of $\mathbf{\Omega}_1$. Let us confirm the above statement. For this purpose, let us choose, for example, as forming sequents $\tilde{s}_{17}$ and $\tilde{s}_{33}$, whose complete groups of equidistant functions are presented in Table 14.

Table 14. Composition of sequential function groups formed by the elements $\tilde{s}_{17}$ and $\tilde{s}_{33}$

Groups	Group Sequences																			
$SF_{17,j}$	0	2	4	5	6	8	9	10	13	14	**17**	21	22	25	27	28	31	32	33	35
1																				
2																				
3																				
4																				
5																				
6																				
$SF_{33,j}$	0	2	3	5	6	7	9	11	13	15	16	17	19	21	23	25	27	29	31	**33**
1																				
2																				
3																				
4																				
5																				
6					0															

From the data comparison, we can easily see that any group from Table 14 is in one of the rows of Table 13. The correspondence between the groups $SF_{17,j}$, $SF_{33,j}$, $j=\overline{1,6}$, and the groups (i,j) of the subset $\mathbf{\Omega}_1$ is shown in Table 15.

Table 15. Correspondence of groups of sequential functions, formed by the elements $\tilde{s}_{17}$ and $\tilde{s}_{33}$

$SF_{17,j}$	1	2	3	4	5	6
(i,j)	2,3	2,5	4,1	4,6	5,1	5,6
$SF_{33,j}$	1	2	3	4	5	6
(i,j)	2,2	2,5	3,2	3,5	5,1	5,3

Let us notice the mosaic of rectangular squares (Table 13), in which the groups of equidistant sequents of the subset $\boldsymbol{\Omega}_1$ are placed. The coloring of all the squares turns out to be the same, which provides an opportunity to significantly reduce the labor intensity of the computation of the composition of the groups of the subset $\boldsymbol{\Omega}_1$. Indeed, suppose that a site mosaic is composed for groups $\boldsymbol{SF}_{1,j}$ for which the constituent is a sequent of €$_1$. To compute the sequents of any groups $\boldsymbol{SF}_{i,j}$, i 1, is enough to replace the top line $\boldsymbol{S}_1$ in the group pad $\boldsymbol{SF}_{1,j}$ by line $\boldsymbol{S}_1$, composed of the sequents of the i-th line of Table 13, excluding all its empty elements.

Generating basis function $\tau(1,t)$ of Walsh-like systems of order N is defined by the expression

$$\tau(1,t) = 01^{(N/2)}0^{(N/2-1)}, \quad t = \overline{0, N-1}\,, \qquad (14)$$

where $a^{(m)} = \underbrace{a\ a\ \ldots\ a}_{m \text{ times}}$.

When forming the Walsh-Tukey systems, we will consider such features of these systems.

First, the even basis functions $\tau(2k,t)$ are defined in a way similar to the way of calculating the even basis functions in Walsh-Cooley systems, i.e.

$$\tau(2k,t) = \tau(k,(2t)_N)\,. \qquad (15)$$

Second, the elements of the left and right halves of odd basis functions are mutually complementary, i.e.

$$\begin{array}{c}\tau(2k+1, t+N/2) \leftrightarrow \overline{\tau}(2k+1,t), \\ t = \overline{0, N/2-1}\end{array}, \qquad (16)$$

where $\overline{\varphi}$ denotes the function complementary to the function φ.

Based on expressions (14) and (15), let us compose the partially unfilled matrix $\boldsymbol{T}_8$ of the Walsh-Tukey system of the eighth order.

$$\boldsymbol{T}_8^{(1)} = \begin{array}{c|cccccccc}
 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 & \mathbf{4} & 5 & 6 & 7 \;\; \to t \\
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\mathbf{2} & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
3 & 0 & 1 & 0 & & 1 & & & \\
\mathbf{4} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
5 & 0 & 0 & 1 & & 1 & & & \\
6 & 0 & 0 & 1 & & 0 & & & \\
7 & 0 & 0 & 0 & & 1 & & & \\
k & & & & & & & &
\end{array} . \qquad (17)$$

The numbers of rows and columns to be filled in at the first stage of matrix formation are marked in bold in (18) $\boldsymbol{T}_8^{(1)}$. Using the relation (16), we restore some of the missing elements in the matrix (17) and, thus, come to the matrix, in which the numbers of those rows and columns are highlighted, which underwent changes during the second stage of transformation.

$$\boldsymbol{T}_8^{(2)} = \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} \;\; \to t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\mathbf{3} & 0 & 1 & 0 & & 1 & 0 & 1 & \\
4 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\mathbf{5} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
\mathbf{6} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\mathbf{7} & 0 & 0 & 0 & & 1 & 1 & 1 & \\
k & & & & & & & &
\end{array} . \qquad (18)$$

As a result of the second stage of transformation, the fifth and sixth rows (as well as columns) of matrix (18) turned out to be filled. The uncertainty concerning the empty elements in the third and seventh rows of the matrix (18) is easily resolved in this way. Let us construct (see Fig. 18) a system consisting of eight vectors corresponding to the input signal samples on the complex plane, the normalized frequency of which equals three. Let us mark the vectors with the elements of the third row of the matrix (18), thus calculating the harmonic $\dot{\boldsymbol{X}}_3(3)$.

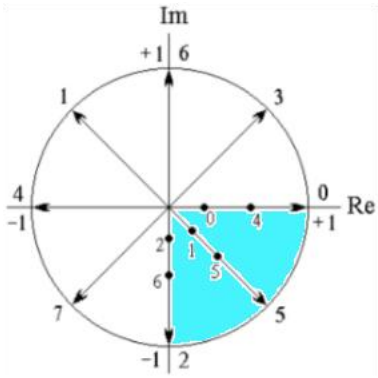


Figure 18: To calculate the harmonic $\dot{X}_3(3)$

According to Fig. 18, the maximum of the harmonic $\dot{X}_3(3)$ can only be under the condition that the third and seventh elements of the third row of the matrix (19) are 0 and 1, respectively, which completes the formation of the matrix $\boldsymbol{T}_8$, which will take the form

$$\boldsymbol{T}_8 = \begin{array}{c|cccccccc} & 0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} \; \to t \\ 0 & 0&0&0&0&0&0&0&0 \\ 1 & 0&1&1&1&1&0&0&0 \\ 2 & 0&1&1&0&0&1&1&0 \\ \mathbf{3} & 0&1&0&0&1&0&1&1 \\ 4 & 0&1&0&1&0&1&0&1 \\ \mathbf{5} & 0&0&1&0&1&1&0&1 \\ \mathbf{6} & 0&0&1&1&0&0&1&1 \\ \mathbf{7} & 0&0&0&1&1&1&1&0 \\ k & & & & & & & & \end{array} . \quad (19)$$

From the comparison of the Walsh-Cooley (10) and Walsh-Tukey (19) matrices, one can easily establish both similarities and differences in the algorithms forming the non-zero basis functions of the systems (see Table 16).

Table 16. Comparison of methods of formation of the eighth-order basis functions

k	$c(k,t)$	$\tau(k,t)$
1	00111100	01111000
2	$f(2,t) = f(1,(2t)_8)$	
3	(1 2)	(1 6)
4	$f(4,t) = f(2,(2t)_8)$	
5	(1 4)	(1 4)
6	$f(6,t) = f(3,(2t)_8)$	
7	(1 6)	(1 2)

In Table 16, the symbol f denotes the basis functions c or τ, and the expression (1 l) denotes the bitwise sum of functions of the first- and l-th orders. The same for the Walsh-Cooley and Walsh-Tukey systems is the method of recurrent computation of even order basis functions, since their definition formulas (11) and (16) are equivalent.

The difference manifests itself in the fact that if in the Walsh-Cooley system, when calculating odd basis functions $c(2k+1,t)$, $k>0$, the first order function $c(1,t)$ is bitwise summed with the functions $c(2k,t)$ of even order, then in the Walsh-Tukey system the sequence of even functions participating in bitwise addition operations at the stages of function definition $\tau(2k+1,t)$, is inverse to the sequence of even functions in the Walsh-Cooley system.

Based on relations (15) and (16) and relying on the rules given in Table 16, it is no problem to arrive at an algorithm (see Fig. 19) for synthesizing systems of Walsh-Tukey functions of arbitrary orders.

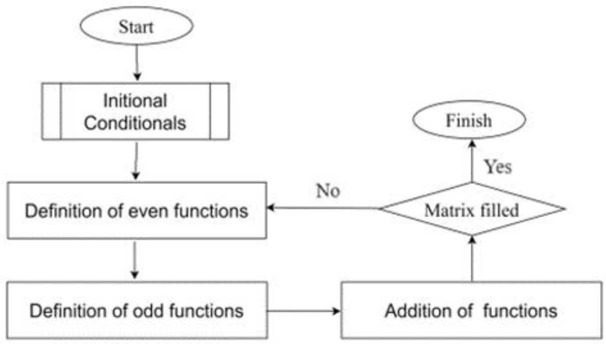


Fig. 19. The block diagram of the algorithm synthesis of Walsh-Tukey function systems

Following the proposed synthesis algorithm, the matrix of the Walsh-Tukey system of arbitrary degree can be easily generated. As empirically established, the formation of matrices $\boldsymbol{T}_N$ of order $N = 2^n$ takes n-2 computation cycles.

The amplitude and phase characteristics of the 16-point discrete Fourier transform processor in the Walsh-Cooley and Walsh-Tukey function bases are shown in Fig. 20 and Table 17, respectively.

Table 17. Phase-frequency characteristics of 16-point
DFT processors in Walsh-Cooley and Walsh-Tukey bases

Basis	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\to k$
Cooley	-0.20	-0.39	-0.65	-0.79	-0.92	-1.18	-1.37	0	-0.20	-0.39	-0.65	-0.79	-0.92	-1.18	-1.37	
Tukey	-1.37	-1.18	-0.92	-0.79	-0.65	-0.39	-0.20	0	-1.37	-1.18	-0.92	-0.79	-0.65	-0.39	-0.20	

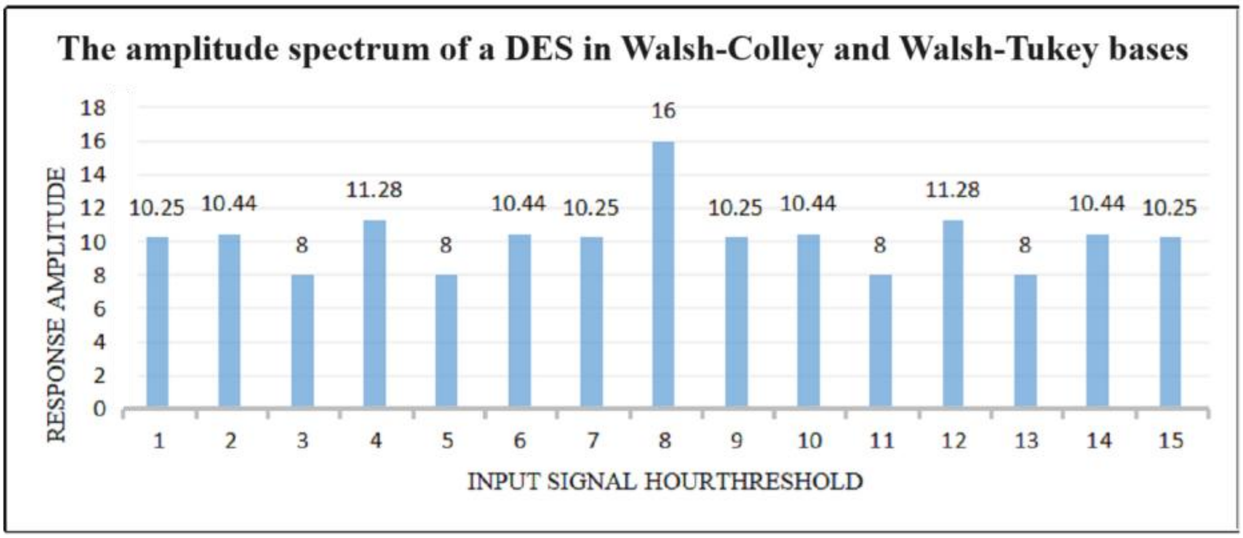


Fig. 20: Amplitude-frequency characteristics of 16-point
DFT processors in Walsh-Cooley and Walsh-Tukey bases

As Fig. 20 and Table 17 show, the amplitude spectra of discrete complex-exponential signals in the Walsh-Cooley and Walsh-Tukey bases coincide. In contrast, the phase spectra are opposite: so if in some k-th output channel of an N-point DFT processor, the phase response in the Walsh-Cooley basis is $\varphi_c(k)$, then in the Walsh-Tukey basis $\varphi_\tau(k) = \varphi_c(H-k)$.

5 Discussion and Further Research

Let us note the peculiarities that can be drawn from comparing the data in Table 13 and Figure 17.

First, the number of groups of equidistant (according to Hamming) code combinations placed in Table 13 (and there are five groups for N = 8) coincides with the number of diagonal elements in the upper part of Fig. 17, to the left of which there is not a single non-zero sequent.

Second, the Hamming distance between the fifth boundary sequent €_5, which closes Table 11, and the nearest to it sixth sequent €_6 is four, i.e., $d(\text{€}_5, \text{€}_6) = 4$. The above numerical characteristics make it possible to outline approaches to constructing an algorithm for synthesizing the complete set of sequent Walsh-like systems of arbitrary order $N = 2^n$, $n \quad 4$, which includes such basic computational steps.

At the first stage of the sliding window mode for the chosen order N of the matrices of the Walsh-like function systems, we find the minimum value of M at which equality $d(\text{€}_M, \text{€}_{M+1}) = N/2$ is achieved. The parameter M determines the number of sequents €_i, $i = \overline{1, M}$, each of which forms functions $\boldsymbol{SF}_{i,j}$, $j = \overline{1, R}$, where R is the number of complete sets of equidistant sequents generated by each forming sequent €_i. In particular, for N = 8, according to Table 13, M = 5 and R = 6. All remaining sequents €_i, $i > M$, can be excluded from further consideration, since, as it was shown earlier, none of them gives rise to any new Walsh-like systems.

In the second step, all sequents that are distant from any selected parent sequent $\tilde{S}_i$ by the Hamming distance $d = N / 2$ are determined. Finally, in the third step, the composition of pairwise equidistant sequents generated by sequents $\tilde{S}_i$, $i = \overline{1, M}$, is instantiated from the set of sequents selected in the second computational step.

Given the above, it seems appropriate to outline such directions for further research.

1. Obtain analytical estimates of the number of symmetric systems of Walsh-like functions of arbitrary order *N*.

2. Find out whether there are indicator matrices for the systems of Walsh-like functions similar to those by which the classical Walsh systems are interconnected.

3. Develop methods for constructing FFT algorithms on the basis of Walsh-like functions.

4. Identify areas of practical applications of systems of Walsh-like functions.

6 Conclusion

The main scientific results achieved by the present study are as follows.

First, based on the Cooley-Tukey FFT algorithm, an FFT basis called the Walsh-Cooley basis has been developed, which is unique in that it is the only one in the set of classical Walsh bases that delivers linear coherence to the frequency scales of FFT processors. None of the canonical Walsh bases ordered by Hadamard, Kaczmage, or Paley provides such coherence to frequency scales, which brings specific difficulties in realizing such devices, for example, spectrometers.

Second, an algorithm has been developed to synthesize new Walsh-like systems that retain all the properties of classical Walsh systems but significantly exceed the latter in power. For example, there are 28 classical Walsh systems of the eighth order in total, whereas there are 840 Walsh-like systems. The basis functions forming Walsh-like (and classical) systems constitute bases with all the properties necessary for constructing fast Fourier transform algorithms. Namely, the systems are complete, orthogonal, symmetric, and involution, i.e., such that they allow fast and efficient transformation of signals between time and frequency domains and provide good separation of different signal frequency components.

Third, in the set of Walsh-like bases of arbitrary degree order, a single basis exists called the Walsh-Tukey basis. Like the Walsh-Cooley basis, it provides linear coherence of frequency scales to FFT processors. The spectra of discrete complex-exponential signals in these bases are such that their AFCs coincide, and their PFCs are inverted relative to each other.

The developed Walsh-Cooley and Walsh-Tukey functions systems have certain advantages over the canonical Walsh systems and on this basis, in our opinion, can replace the latter in many applications.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.