Alpha Fractal Rational Quintic Spline with shape preserving properties

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Abstract: - The intent of this paper is to construct the alpha fractal rational quintic spline. We have considered C^2 rational quintic function, which is of the rational form, where the numerator is a quintic polynomial and denominator is a linear polynomial having two shape parameters i.e. s_m & t_m and deduced the uniform error bound for alpha fractal rational quintic spline. Also constraints have been applied on shape parameters and scaling factors to drive the shape preserving properties.

Key-Words: - Fractal, Iterated Function System, Fractal Interpolation Function, Convergence, Positivity.

Received: July 16, 2022. Revised: October 7, 2023. Accepted: November 11, 2023. Published: December 11, 2023.

1 Introduction

The mathematics behind the fractals started taking shape in 17th century, when Gottfried Leibniz considered recursive self-similarity. And thought that the self-similarity is possessed by straight lines only, which was not true. Then in 1872 an example of function was given by the Karl Weierstrass, which was not differentiable but continuous everywhere. In 1904 a geometrical definition of the Weierstrass example was given by Helge Von Koch and later was considered as the Koch curve. After this many of the mathematicians worked on this concept for years and later in 1975 Mandelbrot [1] gave it a name “Fractals” to define the objects which are too irregular and complex to be defined by the traditional theory. Fractals have several properties but we are mostly concerned with two which are non-integer dimension and self-similarity. When it comes to the real world, we are surrounded by problems or things which are too complex to be understood by traditional theory.

In *1986* fractal interpolation was introduced by the Barnsley [2], which turned out as a solution to the problems we were discussing above. Fractal interpolation has a wide area of use in the real world; it's been used in movies, medical science[3], [4], image compression[5], [6] etc. The knowledge of fractals and ecosystems together is used to determine the acid-rain spread of smoke. we know how stock market changes in a blink of eye sometimes the changes are small but the other-time it can be huge which gives us the sense of irregularity, so here we can see that because of too much irregularity this is not possible to understand and give a map to stock market and hence here comes the use of fractals. Similarly convexity is another fact that is widely used in economics. Ever since the demand from industrial areas, engineering and scientific fields have increased, the construction of shape-preserving interpolants which are also smooth has become one of the major research topics of computer design and approximation theory. A lot of research has been done in the aspect of shape preserving interpolation, among them the rational splines are the most used because of their flexibility see [7]–[16] For the construction of smooth fractal interpolation function, the iterated function system is needed which satisfies the hypothesis given by Barnsley and Harington. By using the base function from [2] and the theorem given by Barnsley and Harington, a method is described by the Navascués in [17] for constructing C^p – fractal interpolation functions, most precisely the FIFs in the form of polynomials. However, for the development of the smooth rational fractal interpolation functions, the method of single-base function is not suitable. So in [18] Puthan and Chand have given the generalization of the construction of C^p – fractal interpolation functions by using α - fractal with the set of finite number of base functions.

In this research article we have constructed an α - fractal quintic spline g^{α} corresponding to the classical rational quintic spline [19]. We have considered fractal spline with linear denominator, in the view that it can give a better approximation of the function being interpolated than the cubic, quadratic denominator. In section 2, we have given the basic theorems, definition and notations which will be used throughout the article and in section 3 we gave the construction of α - fractal quintic spline g^{α} with the family of base functions. The convergence analysis done in section 4 and section 5 includes the shape preserving properties of α - fractal quintic spline.

2 Preliminaries

In this section we are giving a brief introduction of fractal interpolation function and α-fractal function, which will be used in the rest of the paper. Readers are referred to study [2].

Let $\{(y_m, z_m) \in J \times R : m = 1,2,\ldots,M\}$ be the dataset and $J = [y_1, y_M]$ satisfying $y_1 < y_2 < y_3 \ldots < y_M$ and $J_m = [y_m, y_{m+1}]$ for $m \in \wedge = \{1,2,\ldots,M-1\}$.

Consider an affine map $L_m : J \to J_m$ defined as $L_m(y) = c_m y + \ell_m$ such that

$$L_m(y_1) = y_m, L_m(y_M) = y_{m+1}$$

Consider C be the compact subset of R and let $\alpha_m \in [-1,1], m \in \wedge,$ & define $M-1$ continuous maps $F_m : J \times C \to C$ such that,

$$F_m(y,z) - F_m(y, z^*) \le |\alpha_m| |z - z^*|,$$

$$F_m(y_1, z_1) = z_m, F_m(y_M, z_M) = z_{m+1}$$

Define a function $\mu_m : J \times C \to J \times C$ for $m \in \wedge$ by

$$\mu_m(y,z) = (L_m(y), F_m(y,z)).$$

Then $\{J \times D; \mu_m; m \in \wedge\}$ be the iterated function system.

Theorem 2.1 [see[1],[2]]. The iterated function system $\{J \times D; \mu_m; m = 1,2,3,\ldots,M\}$ possesses a set $G \ne \phi$ and $G \subset J \times D$, which is unique and compact, such that $G = \bigcup_{m=1}^{M} \mu_m(G)$, Then G be the graph of a continuous function $f : J \to R$ such that $f(y_m) = z_m; m = 1,2,\ldots,M.$

The graph of function f is the attractor of IFS, and this function f is also known as the FIF corresponding to dataset $\{(y_m, z_m) : m = 1,2,\ldots,M\}$.

Let $G = \{f \in C(J) : f(y_1) = z_1 \; \& \; f(y_M) = z_M\}$ and let $T : G \to G$ such that

$$T(f(y)) = F_m(L_m^{-1}(y), f \circ L_m^{-1}(y)), y \in J_m, m \in \wedge$$

T is the contraction on the complete metric space (G, ρ). Consequently, T possess a fixed unique on G, i.e. $f \in G$ such that

$$T(f(y)) = f(y), y \in J$$

Then this function f is known as FIF corresponding to the iterated function system $\{J \times D; \mu_m; m \in \wedge\}$, which satisfies the functional equation:

$$f(y) = F_m(L_m^{-1}(y), f \circ L_m^{-1}(y)) \forall y \in J_m$$

The most used fractal interpolation function in the literature are originate from the following iterated function system

$$\mu_m(y,z) = (L_m(y), F_m(y,z)), \; L_m(y) = c_m y + \ell_m$$

$$\& \; F_m(y,z) = \alpha_m z + p_m(y)$$

Where $r_m(x)$ is a continuous function and can be chosen of any form like linear, quadratic or any polynomial, that satisfy the conditions mentioned earlier in the section. And α_m is the scaling factor of mapping μ_m. In 1986 Barnsley has given a function $p_m(y) = g \circ L_m(y) - \alpha_m b(y); g \ne b \in C(J)$, $b(y)$ is the base function satisfies $b(y_1) = g(y_1) \,\&\, b(y_M) = g(y_M)$, the corresponding fractal interpolation function g^{α} obeys

$$g^{\alpha}(y) = g(y) + \alpha_m (g^{\alpha} - b) \circ L_m^{-1}(y), y \in J_m$$

And

$$G(g^{\alpha}) = \bigcup_{m \in \wedge} \mu_m (G(g^{\alpha})).$$

Theorem2.2. Let $y_1 < y_2 < \ldots < y_M$ be the partition of J. For $m \in \wedge$, let $L_m(y) = c_m y + \ell_m$ be the affine map and let $F_m(y,z) = \alpha_m z + p_m(y)$ satisfying $L_m(y_1) = y_m, L_m(y_M) = y_{m+1}$ and $F_m(y_1, z_1) = z_m$, $F_m(y_M, z_M) = z_{m+1}$ respectively. Suppose for some integer $q \ge 0$ we have $|\alpha_m| \le k c_m^q; 0 \le k < 1$ and $p_m(x) \in C^q(J)$. Let

$$F_{m,r}(y,z) = \frac{\alpha_m z + p_m^{(r)}(y)}{c_m^r}, z_{1,r} = \frac{p_1^{(r)}(y_1)}{c_1^r - \alpha_m}$$

$$z_{M,r} = \frac{p_{M-1}^{(r)}(y_M)}{c_{M-1}^{(r)} - \alpha_{M-1}}, r = 1 : q$$

If $F_{m-1,r}(y_M, z_{M,r}) = F_{m,r}(y_1, z_{1,r})$ for $m \in \wedge$ and $r = 1,2,\ldots,q$, then a fractal interpolation function is determined by an iterated function system $\{J \times R, \mu_m; m \in \wedge\}$ and the IFS $\{J \times R, (L_m(y), F_{m,r}(y,z)); m \in \wedge\}$ for $r = 1,2,\ldots,q$ determines the fractal interpolation function $f^{(r)}$.

Now let $g \in C^q(J)$ and from the above theorem we have $|\alpha_m| \le k c_m^q; 0 \le k < 1 \,\&\, m \in \wedge$. We wish to apply the conditions on the family of functions $B = \{b_m \in C(J); m \in \wedge\}$ such that $F_m(y,z) = \alpha_m z + p_m(y) = \alpha_m z + g \circ L_m(y) - \alpha_m b_m(y)$ and we will do this with the help of a method introduced in [17] however, we have considered a family of base functions instead of single base functions.

From above theorem we have

$$F_{m-1,r}(y_M, z_{M,r}) = F_{m,r}(y_1, z_{1,r})$$

Where, $F_{m,r}(y,z) = \dfrac{\alpha_m z + p_m^{(r)}(y)}{c_m^r}$. For our choices of p_m, we have $p_m^{(r)}(y) = c_m^r g^{(r)}(L_m(y) - \alpha_m b_m^r(y))$, so that

$$c_{m-1}^r F_{m-1,r}(y_M, z_{M,r}) = \frac{\alpha_{m-1}}{c_{M-1}^r - \alpha_{M-1}} \times [c_{M-1}^r g^{(r)}(y_M) - \alpha_{M-1} b_{M-1}^{(r)}(y_M)] + c_{m-1}^r g^{(r)}(y_m) - \alpha_{m-1} b_{m-1}^{(r)}(y_M)$$

$$c_m^r F_{m,r}(y_1, z_{1,r}) = \frac{\alpha_m}{c_1^r - \alpha_1} \times [c_1^r g^{(r)}(y_1) - \alpha_1 b_1^{(r)}(y_1)] + c_m^r g^{(r)}(y_m) - \alpha_m b_m^{(r)}(y_1)$$

Now if we have a family of functions $B = \{b_m \in C^q(J), m \in \wedge\}$ such that the derivative of B up to q^{th} order of every member is same as the $g \in C^q(J)$ at the end knots, then the corresponding FIF $g^\alpha \in C^q(J)$ and satisfies $g^\alpha(y_m) = g(y_m)$. Furthermore, $(g^\alpha)^{(r)}$ be the FIF corresponding to the IFS

$$L_m(y) = c_m y + \ell_m,$$

$$F_{m,r}(y,z) = \frac{\alpha_m z + c_m^r g^{(r)}(L_m(y) - \alpha_m b_m^r(y))}{c_m^r}$$

And the functional equation for $(g^\alpha)^{(r)}$ is

$$(g^\alpha)^{(r)}(y) = g^{(r)}(y) + \frac{\alpha_m (g^\alpha - b_m)^{(r)} \circ L_m^{-1}(y)}{c_m^r}$$

This equation specifies that the r[th] derivative of g^α agrees with r[th] derivative of g at end knots.

3 Construction of Alpha Fractal Quintic Spline

In this section we are introducing a new class of α-fractal rational quintic spline $g^\alpha \in C^2$ corresponding to the rational quintic spline $g \in C^2$. Let us consider a dataset $\{(y_m, z_m, d_m) \in J \times R : m = 1,2,3,4,\ldots,M\}$, where $y_1 < y_2 < y_3 < \ldots < y_m$ and here z_m is the function value and d_m be the value of the first derivative at knot y_m. Derivatives can be found by the approximation method, if not given at knots.

A C^2-rational quintic function with two parameters was introduced in [19], we can rewrite it as

$$g(L_m(y)) = \frac{A_1(1-\theta)^5 + A_2(1-\theta)^4\theta + A_3(1-\theta)^3\theta^2 + A_4(1-\theta)^2\theta^3 + A_5(1-\theta)\theta^4 + A_6\theta^5}{s_m(1-\theta) + t_m\theta_m}$$

Where, $\theta = \dfrac{y - y_1}{y_M - y_1}, y \in J$ and s_m, t_m are the non-negative shape parameters. Where,

$A_1 = s_m z_m$ & $A_6 = t_m z_{m+1}$

$A_2 = (4s_m + t_m)z_m + s_m h_m d_m, h_m = y_{m+1} - y_m$

$A_3 = (6s_m + 4t_m)z_m + (3s_m + t_m)h_m d_m + \frac{1}{2}s_m h_m^2 D_m$

$A_4 = (4s_m + 6t_m)z_{m+!} - (s_m + 3t_m)h_m d_{m+1} + \frac{1}{2}t_m h_m^2 D_{m+1}$

$A_5 = (s_m + 4t_m)z_{m+1} - s_m h_m d_{m+1}$

Now $g(L_m(y))$ can rewritten as

$$g(L_m(y)) = \omega_1(s_m,t_m,\theta)z_m + \omega_2(s_m,t_m,\theta)z_{m+1} + \omega_3(s_m,t_m,\theta)d_m - \omega_4(s_m,t_m,\theta)d_{m+1} + \omega_5(s_m,t_m,\theta)D_m + \omega_6(s_m,t_m,\theta)D_{m+1}$$

Here

$$\omega_1(s_m,t_m,\theta) = \frac{(1-\theta)^5 s_m + (1-\theta)^4\theta(4s_m + t_m) + (1-\theta)^3\theta^2(6s_m + 4t_m)}{s_m(1-\theta) + t_m\theta}$$

$$\omega_2(s_m,t_m,\theta) = \frac{\theta^5 t_m + (1-\theta)\theta^4(s_m + 4t_m) + (1-\theta)^2\theta^3(4s_m + 6t_m)}{s_m(1-\theta) + t_m\theta}$$

$$\omega_3(s_m,t_m,\theta) = \frac{(1-\theta)^3\theta^2(3s_m + t_m)h_m + (1-\theta)^4\theta h_m s_m}{s_m(1-\theta) + t_m\theta}$$

$$\omega_4(s_m,t_m,\theta) = -\left(\frac{(1-\theta)^2\theta^3(s_m + 3t_m)h_m + (1-\theta)\theta^4 h_m t_m}{s_m(1-\theta) + t_m\theta}\right)$$

$$\omega_5(s_m, t_m, \theta) = \frac{(1-\theta)^3 \theta^2 \frac{h_m^2}{2} s_m}{s_m(1-\theta)+t_m\theta}$$

$$\omega_6(s_m, t_m, \theta) = \frac{(1-\theta)^2 \theta^3 \frac{h_m^2}{2} t_m}{s_m(1-\theta)+t_m\theta}$$

Now to construct α- fractal rational quintic function corresponding to g, Let us assume that $|\alpha_m| < kc_m$ and consider a family $B = \{b_m \in C(J); m \in \wedge\}$ satisfying the conditions $b_m(y_1) = z_1, b_m(y_M) = z_M$ and $b_m^{(1)}(y_1) = d_1, b_m^{(1)}(y_M) = d_M$ & $b_m^{(2)}(y_1) = D_1, b_m^{(2)}(y_M) = D_M$. For our convenience we will take b_m as the rational quintic function which is similar to the form of interpolant g.

$$b_m(y) \frac{T_1(1-\theta)^5 + T_2(1-\theta)^4\theta + T_3(1-\theta)^3\theta^2 + T_4(1-\theta)^2\theta^3 + T_5(1-\theta)\theta^4 + T_6\theta^5}{s_m(1-\theta)+t_m\theta}$$

Where, $\theta = \frac{y - y_1}{y_M - y_1}, y \in J$

$T_1 = s_m z_1 \;\&\; T_6 = t_m z_M$

$T_2 = (4s_m + t_m)z_1 + s_m(y_M - y_1)d_1$

$T_3 = (6s_m + 4t_m)z_1 + (3s_m + t_m)(y_M - y_1)d_1 + \frac{1}{2}s_m(y_M - y_1)^2 D_1$

$T_3 = (4s_m + 6t_m)z_M - (s_m + 3t_m)(y_M - y_1)d_M + \frac{1}{2}t_m(y_M - y_1)^2 D_M$

$T_2 = (s_m + 4t_m)z_M - t_m(y_M - y_1)d_M$

Consider the α-fractal rational quintic spline corresponding to g is

$$g^\alpha(L_m(y)) = \alpha_m g^\alpha(y) + g(L_m(y)) - \alpha_m b_m(y)$$

$$\Rightarrow g^\alpha(L_m(y)) = \alpha_m g^\alpha(y) + \frac{U_m(y)}{V_m(y)}$$

Here,

$U_m(y) = C_1(1-\theta)^5 + C_2(1-\theta)^4\theta + C_3(1-\theta)^3\theta^2 + C_4(1-\theta)^2\theta^3 + C_5(1-\theta)\theta^4 + C_6\theta^5$

$V_m(y) = s_m(1-\theta) + t_m\theta$

And

$C_1 = s_m(z_m - \alpha_m z_1)$

$$C_2 = (4s_m + t_m)z_m + s_m h_m d_m - \alpha_m \begin{Bmatrix} (4s_m + t_m)z_1 + \\ s_m(y_M - y_1)d_1 \end{Bmatrix}$$

$$C_3 = (6s_m + 4t_m)z_m + (3s_m + t_m)h_m d_m + \frac{1}{2}s_m h_m^2 D_m - \alpha_m \begin{Bmatrix} (6s_m + 4t_m)z_1 + (3s_m + t_m)(y_M - y_1)d_1 \\ + \frac{1}{2}s_m(y_M - y_1)^2 D_1 \end{Bmatrix}$$

$$C_4 = (4s_m + 6t_m)z_{m+1} - (s_m + 3t_m)h_m d_{m+1} + \frac{1}{2}t_m h_m^2 D_{m+1} - \alpha_m \begin{Bmatrix} (4s_m + 6t_m)z_M - (s_m + 3t_m)(y_M - y_1)d_M \\ + \frac{1}{2}t_m(y_M - y_1)^2 D_M \end{Bmatrix}$$

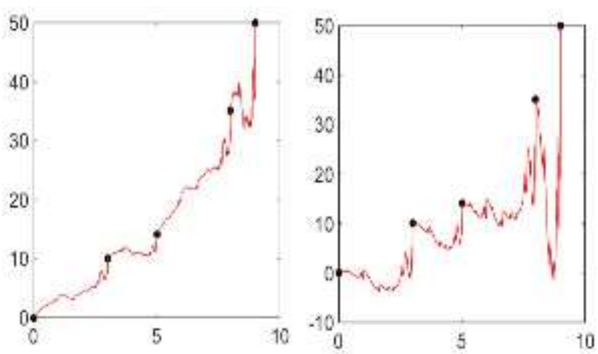
$$C_5 = (s_m + 4t_m)z_{m+1} - t_m h_m d_{m+1} - \alpha_m \begin{Bmatrix} (s_m + 4t_m)z_M - \\ t_m(y_M - y_1)d_M \end{Bmatrix}$$

$C_6 = t_m(z_{m+1} - \alpha_m z_M)$

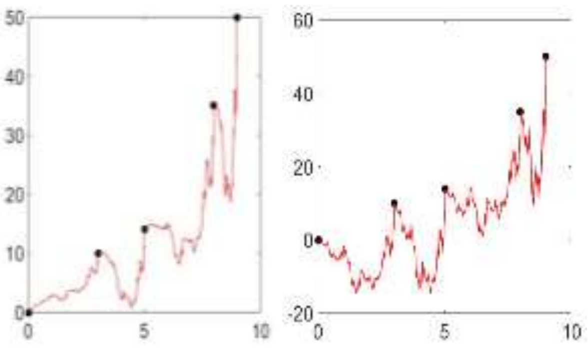
Remark3.1. If we take $\alpha_m = 0 \forall m \in \wedge$, then α- fractal rational quintic spline will become the classical rational quintic function g introduced in [19].

Remark3.2. If $s_m = 1 = t_m$, then the α-fractal rational quintic spline will be reduced to an α- fractal non-rational quintic spline.

Example3.1. Let $(0,0),(3,10),(5,14),(8,35),(9,50)$ be the data set, through which we will verify the diversity and flexibility of α- fractal rational quintic spline over the rational quintic fractal interpolation function. Throughout Fig.1, both the shape parameters $s_m \;\&\; t_m$ are considered $(50,50,50,50)$. We have considered the vertical scale $\alpha = (0.2, 0.15, 0.40, 0.74)$ in Fig.1 (a) and Fig.1 (b), Fig.1 (a) is the graph of RQFIF and Fig.1 (b) is the graph of α- fractal rational quintic spline. We can see the difference between these two graphs with the same vertical scaling for $\alpha = (0.2, 0.15, 0.40, 0.74)$ the RQFIF giving us the positive graph while the α - fractal rational quintic spline gives a negative graph. Fig.1 (c) and Fig.1 (d) shows the graph for vertical scaling $\alpha = (0.1, 0.2, 0.4, 0.45)$ and $\alpha = (0.3, 0.37, 0.42, 0.5)$ respectively



(a) **(b)**



(c) **(d)**

Fig.1. α **- fractal rational quintic spline** g^{α}

4 Convergence

To show that the convergence of α -fractal rational quintic spline g^{α} toward the φ, which is the data generating function belongs to $C^3[y_1, y_M]$, we need to find the uniform distance between them. Now as we know that g^{α} is an implicit expression, so it will be difficult to calculate $\left\|\varphi - g^{\alpha}\right\|_{\infty}$ by using the standard techniques. Now let g^{α} & g be the α -fractal rational quintic spline and classical quintic function respectively. We will derive an upper bound of the error by using g^{α} & g .

By the triangular inequality we've

$$\left\|\varphi - g^{\alpha}\right\|_{\infty} \le \left\|\varphi - g\right\|_{\infty} + \left\|g - g^{\alpha}\right\|_{\infty}$$

Now we use the convergence result of the classical rational quintic interpolant g.

Proposition4.1. [19] Let $\varphi \in C^3[y_1, y_M]$ and $\{(y_m, z_m) \in J \times R : m = 1,2,3,4,\ldots, M\}$ be the dataset. Let g be the rational quintic spline corresponding to φ, then the error between φ & g in every subinterval $[y_m, y_{m+1}]$ is,

$$|\varphi(y) - g(y)| \le e_m h_m^3 \left\|\varphi^{(3)}(y)\right\|$$

Where, $e_m = \max\limits_{0\le\theta\le1} \sigma(s_m, t_m, \theta)$

$$\sigma(s_m, t_m, \theta) = \begin{cases} \max \sigma_1(s_m, t_m, \theta); 0 \le \theta \le \dfrac{2s_m + t_m}{3(s_m + t_m)} \\ \max \sigma_2(s_m, t_m, \theta); \dfrac{2s_m + t_m}{3(s_m + t_m)} \le \theta \le 1 \end{cases}$$

Theorem4.2. Let $\varphi \in C^3[y_1, y_M]$ and $\Delta = \{y_1, y_2, \ldots, y_M\}$ be the partition J satisfying the condition $y_1 < y_2 < \ldots < y_M$. Let g be the rational quintic fractal interpolation function which interpolate the values of function φ on the points of Δ , then

$$\left\|\varphi - g\right\|_{\infty} \le e_m h_m^3 \left\|\varphi^{(3)}\right\|_{\infty} + \frac{|\alpha|}{1 - |\alpha|}\left\{p(h) + p^*(h)\right\}$$

Where,

$$p(h) = \|\varphi\|_{\infty} + h_m\left(\theta(1-\theta)\left(1 + \theta - \theta^2\right)\right)|d|_{\infty} + \frac{h_m^2}{2}\left(\theta^2(1-\theta)^2\right)|D|_{\infty}$$

$$p^*(h) = \|z\|_{1,m} + h_m\left(\theta(1-\theta)\left(1 + \theta - \theta^2\right)\right)|d|_{1,m} + \frac{h_m^2}{2}\left(\theta^2(1-\theta)^2\right)|D|_{1,m}$$

And

$$e_m = \max_{0\le\theta\le1} \sigma(s_m, t_m, \theta); 0 \le \theta \le 1,$$

$$|d|_{\infty} = \max\left\{|d|_m; 1 \le m \le M\right\},$$

$$|D|_{\infty} = \max\left\{|D|_m; 1 \le m \le M\right\}, |d|_{1,m} = \max\left\{|d|_1, |d|_m\right\}$$

$$|D|_{1,m} = \max\left\{|D|_1, |D|_m\right\}$$

Proof: For the considered data set and α_m satisfying $|\alpha_m| \le c_m; m \in \wedge$, the α - fractal rational quintic spline g^{α} is the fixed point of R-B operator T_{α} defined on metric space $G^* := \{f \in C(J, R) : f(y_1) = z_1, f(y_M) = z_M, f'(y_1) = d_1, f^{(1)}(y_M) = d_M, f^{(2)}(y_1) = D_1, f^{(2)}(y_M) = D_M\}$ such that

$$(T_{\alpha} f)(y) = \alpha_m(y) f\left(L^{-1}(y)\right) + \frac{U_m\left(\alpha_m, s_m, t_m, L^{-1}(y)\right)}{V_m\left(s_m, t_m, L^{-1}(y)\right)}$$

Clearly the interpolants g^{α} is a fixed point of T_{α} with $\alpha \ne 0$ and g is a fixed point of T_{α} with $\alpha = 0$, then by the contraction property of R-B operator, we have

$$\left\|T_{\alpha} g^{\alpha} - T_{\alpha} g\right\|_{\infty} \le |\alpha|_{\infty} \left\|g^{\alpha} - g\right\|_{\infty}$$

By the mean value theorem of a function with multiple variables, we'll get

$$\|T_\alpha g(y)-T_0 g(y)\|_\infty \le \left|\alpha_m g\left(L^{-1}(y)\right)+\frac{U_m\left(\alpha_m,s_m,t_m,L^{-1}(y)\right)}{V_m\left(s_m,t_m,L^{-1}(y)\right)}-\frac{U_m\left(0,s_m,t_m,L^{-1}(y)\right)}{V_m\left(s_m,t_m,L^{-1}(y)\right)}\right|_\infty$$

$$\le |\alpha_m|_\infty\left(\|g\|_\infty+\left|\frac{\partial\left\{\frac{U_m\left(\tau_m,s_m,t_m,L^{-1}(y)\right)}{V_m\left(s_m,t_m,L^{-1}(y)\right)}\right\}}{\partial\alpha_m}\right|\right)$$

$$;|\tau_m|\in(0,\alpha_m)$$

Now we will calculate the R.H.S of the above inequality. So from the classical rational quintic interpolation function we have

$$g(L_m(y))=\omega_1(s_m,t_m,\theta)z_m+\omega_2(s_m,t_m,\theta)z_{m+1}+\omega_3(s_m,t_m,\theta)d_m-\omega_4(s_m,t_m,\theta)d_{m+1}+\omega_5(s_m,t_m,\theta)D_m+\omega_6(s_m,t_m,\theta)D_{m+1}$$

We note that

$$\omega_1(s_m,t_m,\theta)+\omega_2(s_m,t_m,\theta)=1$$

Also

$$\omega_3(s_m,t_m,\theta)d_m-\omega_4(s_m,t_m,\theta)=\theta(1-\theta)\left(1+\theta-\theta^2\right)h_m$$

And,

$$\omega_5(s_m,t_m,\theta)+\omega_6(s_m,t_m,\theta)=\frac{1}{2}\theta^2(1-\theta)^2h_m^2.$$

So,

$$g(y)\le \max_{j=m,m+1}\{z_j|\}+h_m\theta(1-\theta)\left(1+\theta-\theta^2\right)\max_{j=m,m+1}\{d_j|\}+\frac{h_m^2}{2}\theta^2(1-\theta)^2\max_{j=m,m+1}\{D_j|\}$$

$$\le |z_j|_{1,m}+h_m\theta(1-\theta)\left(1+\theta-\theta^2\right)|d|_\infty+\frac{1}{2}h_m^2\theta^2(1-\theta)^2|D|_\infty$$

Now that above expression is true for m, we can also get the following expression

$$\|g\|_\infty\le p(h)=\|\varphi\|_\infty+h_m\left(\theta(1-\theta)\left(1+\theta-\theta^2\right)\right)|d|_\infty+\frac{1}{2}h_m^2\left(\theta^2(1-\theta)^2\right)|D|_\infty$$

As we know that $V_m(y)$ is independent of alpha, so

$$\frac{\partial\left\{\frac{U_m\left(\tau_m,s_m,t_m,L^{-1}(y)\right)}{V_m\left(s_m,t_m,L^{-1}(y)\right)}\right\}}{\partial\alpha_m}=\omega_1(s_m,t_m,\theta)z_1+\omega_2(s_m,t_m,\theta)z_m+\omega_3(s_m,t_m,\theta)d_1-\omega_4(s_m,t_m,\theta)d_m+\omega_5(s_m,t_m,\theta)D_1+\omega_6(s_m,t_m,\theta)D_m$$

By using the same arguments we used for the first part we'll get,

$$\frac{\partial\left\{\frac{U_m\left(\tau_m,s_m,t_m,L^{-1}(y)\right)}{V_m\left(s_m,t_m,L^{-1}(y)\right)}\right\}}{\partial\alpha_m}\le p^*(h)=\|z\|_{1,m}+h_m\left(\theta(1-\theta)\left(1+\theta-\theta^2\right)\right)|d|_{1,m}+\frac{1}{2}h_m^2\left(\theta^2(1-\theta)^2\right)|D|_{1,m}$$

By substituting these values we get,

$$\|T_\alpha g(y)-T_0g(y)\|_\infty\le|\alpha_m|_\infty\left(p(h)+p^*(h)\right), y\in[y_1,y_m].$$

Now by using these inequalities

$$\|g^\alpha-g\|_\infty=\|T_\alpha g^\alpha(y)-T_0g(y)\|_\infty$$

$$\le\|T_\alpha g^\alpha(y)-T_\alpha g(y)\|_\infty+\|T_\alpha g(y)-T_0g(y)\|_\infty$$

$$\le|\alpha_m|_\infty\|g^\alpha-g\|_\infty+|\alpha_m|_\infty\left(p(h)+p^*(h)\right)$$

$$\Rightarrow\|g^\alpha-g\|_\infty\le\frac{|\alpha_m|_\infty\left(p(h)+p^*(h)\right)}{1-|\alpha_m|_\infty}.$$

We have

$$\|\varphi-g^\alpha\|_\infty\le\|\varphi-g\|_\infty+\|g-g^\alpha\|_\infty$$

So by using the Proposition4.1,

$$\|\varphi-g^\alpha\|_\infty\le e_mh_m^3\|\varphi^{(3)}\|_\infty+\frac{|\alpha|}{1-|\alpha|}\left\{p(h)+p^*(h)\right\}$$

Convergence Result: Since we have $|\alpha_m|\le c_m^2$, this implies $|\alpha|_\infty\le\frac{(y_M-y_1)^2}{h^2}$. From the above theorem we can say that the α-fractal rational quintic spline converges uniformly to the original function φ as $h^2\to 0$ and if we take $|\alpha_m|\le c_m^3$, then $\|\varphi-g^\alpha\|_\infty=O(h^3)$ as $h\to 0$.

4 Positivity Conditions for Alpha Rational Quintic Spline

Let $\{(y_m,z_m)\in J\times R: m=1,2,\ldots,M\}$ be the data set with $z_m>0$. Now we will see that the parameters in the α-fractal rational quintic spline can be chosen in a way that $g^\alpha(y)>0\forall y\in J$. Let $\alpha_m\ge 0\forall m\in\wedge$. Then from the functional equation of α- fractal rational quintic spline, it follows that $g^\alpha(y)>0$. Also $V_m(y)>0$ as s_m and t_m are strictly positive. So it will be sufficient to show that $U_m(y)>0$ and condition $U_m(y)>0$ holds if $s_m(z_m-\alpha_mz_1)>0\ \&\ t_m(z_{m+1}-\alpha_mz_M)>0$

$$(4s_m+t_m)z_m+s_mh_md_m-\alpha_m\begin{Bmatrix}(4s_m+t_m)z_1+\\ s_m(y_M-y_1)d_1\end{Bmatrix}>0$$

$$(6s_m+4t_m)z_m+(3s_m+t_m)h_md_m+\frac{1}{2}s_mh_m^2D_m$$
$$-\alpha_m\begin{Bmatrix}(6s_m+4t_m)z_1+(3s_m+t_m)(y_M-y_1)d_1\\ +\frac{1}{2}s_m(y_M-y_1)^2D_1\end{Bmatrix}>0$$

$$(4s_m+6t_m)z_{m+1}-(s_m+3t_m)h_md_{m+1}+\frac{1}{2}t_mh_m^2D_{m+1}$$
$$-\alpha_m\begin{Bmatrix}(4s_m+6t_m)z_M-(s_m+3t_m)(y_M-y_1)d_M\\ +\frac{1}{2}t_m(y_M-y_1)^2D_M\end{Bmatrix}>0$$

$$(s_m+4t_m)z_{m+1}-t_mh_md_{m+1}-\alpha_m\begin{Bmatrix}(s_m+4t_m)z_M-\\ t_m(y_M-y_1)d_M\end{Bmatrix}>0$$

First two inequalities hold if $\alpha_m<\min\left\{\frac{z_m}{z_1},\frac{z_{m+1}}{z_M}\right\}$, and from the third & fourth inequalities we get

$$t_mz^*_{m,1}+s_mh_m\left(d_m-\frac{\alpha_m}{h_m}(y_M-y_1)d_1\right)>0$$
&
$$s_mz^*_{m+1,M}-t_mh_m\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)>0.$$

We know that $s_m\ \&\ t_m>0$ so, to find the solution for these inequalities, we will add a condition over the derivative that is $d_m\geq 0$. By assuming $d_0\ \&\ d_M>0$ these two inequalities can satisfy by taking

$$\alpha_m<\min\left\{\frac{h_md_m}{(y_M-y_1)d_1},\frac{h_md_{m+1}}{(y_M-y_1)d_M}\right\},$$
$$\frac{s_m}{t_m}>\frac{h_m\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)}{z_{m+1}-\alpha_mz_M}\ \forall m\in\wedge$$

Similarly from the fifth and sixth inequalities, we have

$$t_mh_m\left(d_m-\frac{\alpha_m}{h_m}(y_M-y_1)d_1\right)+$$
$$\frac{1}{2}s_mh_m^2\left(D_m-\frac{\alpha_m}{h_m^2}(y_M-y_1)^2D_1\right)>0$$
And
$$-s_mh_m\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)$$
$$+t_m\frac{h_m^2}{2}\left(D_{m+1}-\frac{\alpha_m}{h_m^2}(y_M-y_1)^2D_M\right)>0$$
$$\alpha_m<\min\left\{,\frac{h_m^2D_m}{(y_M-y_1)^2D_1},\frac{h_m^2D_{m+1}}{(y_M-y_1)^2D_M}\right\}$$
$$,\frac{s_m}{t_m}<\frac{h_m\left(D_{m+1}-\frac{\alpha_m}{h_m^2}(y_M-y_1)^2D_M\right)}{2\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)}$$

The sufficient condition for α-fractal rational quintic spline for preserving the positivity of the given dataset in each subinterval is:

$$0\leq\alpha_m<\min\left\{c_m,\frac{z_m}{z_1},\frac{z_{m+1}}{z_M},\frac{h_md_m}{(y_M-y_1)d_1},\right.$$
$$\left.\frac{h_md_{m+1}}{(y_M-y_1)d_M},\frac{h_m^2D_m}{(y_M-y_1)^2D_1},\frac{h_m^2D_{m+1}}{(y_M-y_1)^2D_M}\right\}$$
$$\frac{h_m\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)}{z_{m+1}-\alpha_mz_M}<\frac{s_m}{t_m}$$
$$<\frac{h_m\left(D_{m+1}-\frac{\alpha_m}{h_m^2}(y_M-y_1)^2D_M\right)}{2\left(d_{m+1}-\frac{\alpha_m}{h_m}(y_M-y_1)d_M\right)}\ \forall m\in J$$

Example5.1 Let $(0,0.5),(0.5,1),(1,2),(1.5,3),(2,5)$ be the positive interpolating points and with these data points we will be generating the positive α - fractal rational quintic spline having different shape parameters and scaling vector. In Fig.2 (a), we have taken arbitrary values of shape parameters i.e. $s_m=(34,87,89,78)\ \&\ t_m=(56,23,43,67)$ and the scaling vector $\alpha=(0.3,0.37,0.49,0.65)$ respectively, the corresponding graph is non-positive. Now for the positive α - fractal rational quintic we have calculated the shape parameters according to the above theorem. For the rest of the graphs we have taken shape parameters $s_m=(2,6,8,4)\ \&\ t_m=(3,5,4,6)$, $s_m=(20,76,0.2,0.4)$ $\&\ t_m=(15,89,0.1,0.5)$ and $s_m=(289,88,0.1,0.9)$ $\&\ t_m=(89,60,0.1,0.8)$ and the vertical scaling is $\alpha=(0.1,0.2,0.40,0.45)$, $\alpha=(0.12,0.24,0.34,0.45)$ and $\alpha=(0.2,0.25,0.39,0.5)$, which is calculated according to the above theorem and their graphical representation is given in Fig.2 (b), Fig.2 (c) and Fig.2 (d) respectively. We can easily witness the change in shape due to the different values for vertical scale and shape parameters.

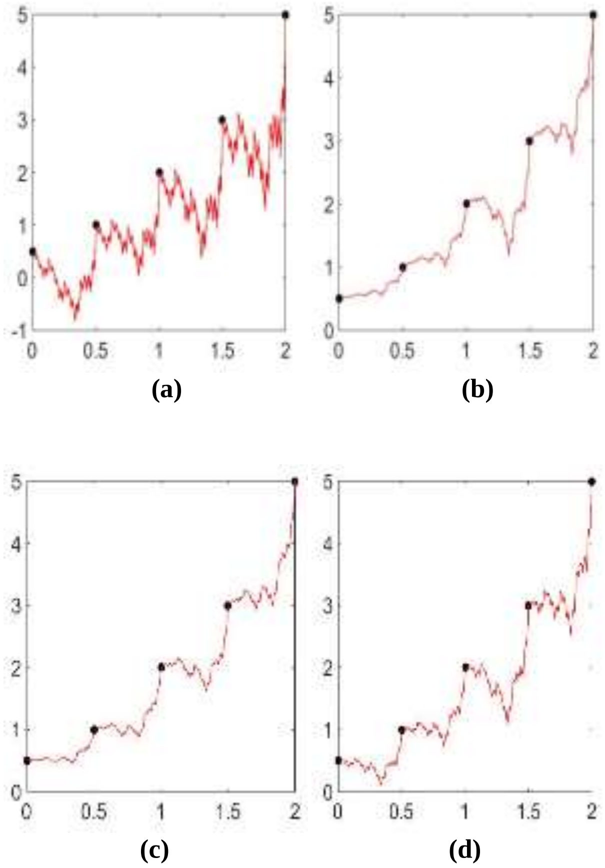


Fig.2. Positive α - fractal rational quintic spline g^{α}

4 Conclusion

The techniques of approximations have been advanced by using the smooth fractal interpolation function. By modifying the methods given in [2], [17], a smooth α-fractal function is constructed in this paper.

By using this procedure we have given the construction of α-fractal rational quintic spline with two families of shape parameters. By applying the conditions over scaling factor, the convergence of α-fractal rational quintic spline is discussed. The scaling factor and the shape parameters have been constrained to preserve the positivity of α-fractal rational quintic spline.

Acknowledgement:
We are thankful to the anonymous reviewers and the editors for their constructive comments and suggestions, which helped us to improve the manuscript considerably.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Conceptualization, methodology and writing, editing are done by Shamli Sharma Gautam, review and supervision is done by Kuldip Katiyar.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.