Algebrain Independence Measure of Some Contnued Fractions

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Abstract: - In the present paper, we prove the algebraic independence of a finite number of real continued fractions which have partial quotients that increase rapidly. Then, we give an algebraic independence measure of these continued fractions.

Key-Words: - Continued fraction, measure, algebraic independence.

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1 Introduction

The theory of transcendental numbers has a long history. We know since J. Liouville in 1844 that the very rapidly converging sequences of rational numbers provide examples of transcendental numbers. So, in his famous paper, Liouville showed that a real number admitting very good rational approximation can note be algebraic, then he constructed explicitly the first example of transcendental numbers, [8].

Let $A = [a_0; a_1, a_2, \cdots, a_n, \cdots]$ and $B = [b_0; b_1, b_2, \cdots, b_n, \cdots]$ be two real continued fractions. In 1984, P. Bundscuh, [1], proved that A and B are algebraically independents if there exists a real number $r > 1$ such that $r^{-1}a_n \geq a_{n-1}^{n-1}$ for all $n \geq 2$.

In particular, the six numbers $A, B, A \pm B$, and $AB^{\pm 1}$ are transcendental over $\mathbb{Q}$.

In [3] we have improved Bundschuh result by replacing the exponent $(n-1)$ of b_{n-1} by any increasing sequence α_{n-1} of real numbers >1 which tends towards infinity. To prove this, we have used Roth's Theorem, [10], which is direct and sample than Durand criteria which is used by Bundschuh. In this paper, we first give the algebraic independence of a finite family of k real numbers $A_1, \cdots, A_k$ with $k \geq 2$.

Then we give an algebraic measure of this finite family of k real numbers. It is a generalization of our result in, [4].

We give some Lemmas which will be used in our results.

Theorem of Liouville, [11]. *Let ξ an algebraic number with degree d, there exists a real constant $C > 0$ such that for each $\frac{p}{q} \in \mathbb{Q}^*$, we hav*

$$\left|\xi - \frac{p}{q}\right| > \frac{C}{q^d}.$$

We then state the Roth's Theorem which gives a sufficient condition criterion of transcendence.

Theorem of Roth [10]. *Let A and α be two real numbers such that $\alpha > 2$. If the inequality*

$$\left|A - \frac{p}{q}\right| < \frac{1}{q^\alpha}$$

has infinitely many solutions of rational numbers $\frac{p}{q}$ such that $gdc(p, q) = 1$, then A is a transcendental number.

2 Main results

2.1 Algebraic independence

We fixe following notations. Let $A_1, \cdots, A_k$ be k real numbers with continued fraction expansions $A_j = [a_{0,j}; a_{1,j}, a_{2,j}, \cdots]$

Denote the convergent of A_j by $p_{n,j}/q_{n,j}$ for $1 \leq j \leq k$. The first main result of our manuscript is.

Theorem 1. *Let r be a real number > 1, (β_n) an increasing sequence of real numbers > 1 which tends to infinity such that for all $n \geq 2$, we have*

$$\begin{cases} r^{-1}a_{n,j} > a_{n,j-1}, & 2 \leq j \leq k \\ a_{n,1} > a_{n-1,k}^{\beta_{n-1}} \end{cases}.$$

Then, $A_1, \cdots, A_k$ are algebraically independent.

For the proof of Theorem 1, the further Lemma is needed.

Lemma 1. (i) *For all* $1 < j < k$, *we have*

$$|A_j - A_{n,j}| < \frac{1}{q_{n,j}q_{n-1,j}}$$

(ii) *If* $r^{-1}a_{n,j} > a_{n,j-1}$ *for* $j = 2, \cdots, k$ *and* $n \geq 2$ *then we have*

$$q_{n,j} > r^{\frac{n}{2}}q_{n,j-1} > q_{n,j-1}. \quad (1)$$

(iii) *Further the hypothesis* $a_{n,j} > a_{n-1,j}^{\beta_{n-1}}$ *for* $n \geq 2$ *implies that* $q_{n,j} < 2a_{n,j}^{\frac{\beta_1}{\beta_1-1}}$. (2)

Proof Lemma 1. (i) See, [3].
(ii) We verify it as in Lemma 2 of Bundschuh, [1].
(iii) We prove it with the same method as in Lemma 1 of, [4].

Proof of Theorem 1. We will note $\beta_n = \beta(n)$. We prove the result of Theorem 1 by induction in k.
For $k = 2$, see chapter 2 of, [4].
For $k = 3$, assume that for all $j \leq k - 1$ the numbers $A_{l_1}, \cdots, A_{l_j}$, are algebraically independent.
Now, assume that the k real numbers $(A_1, \cdots, A_k)$ are algebraically dependent. Then, there exists a non-constant and minimal polynomial $P \in Z[X_1, \cdots, X_k]$ of total degree d such that

$$P(A_1, \cdots, A_k) = 0.$$

By applying Taylor's formula there exists a real constant $C_v = C(v_1, \cdots, v_k) > 0$ such that

$$P(x_1, \cdots, x_k) = \sum_{v_1+\cdots v_k \geq 1} C_v(x_1 - A_1)^{v_1} \cdots (x_k - A_k)^{v_k}. \quad (3)$$

Let $C_i = C_v$ for $i = 1,2, \cdots, k$ with $C_i = C(0, \cdots, 0,1,0, \cdots, 0)$, where "1" is in the itch place.
Since the polynomial P is minimal, we can put.

$$C_1 = C(1,0, \cdots, 0) = \frac{\partial P}{\partial x_1}(A_1, \cdots, A_k) \neq 0.$$

From (3), we obtain

$$P(A_{n,1}, \cdots, A_{n,k}) = (A_{n,1} - A_1)\left\{C_1 + C_2\left(\frac{A_{n,2}-A_2}{A_{n,1}-A_1}\right) + \cdots + C_k\left(\frac{A_{n,k}-A_k}{A_{n,1}-A_1}\right) + 0(|A_{n,1} - A_1|)\right\}.$$

Remark 1.
The choice of $C_1 \neq 0$ *and* $|A_{n,1} - A_1|$ *as a factor in the last equality come from the fact that* $|A_{n,1} - A_1|$ *is the largest of* $|A_{n,j} - A_j|$ *for* $j = 1, \cdots, k$.

Thus using (i) of Lemma 1, one find

$$\frac{|A_j - A_{n,j}|}{|A_1 - A_{n,1}|} < \frac{q_{n,1}q_{n+1,1}}{q_{n,j}q_{n+1,j}} < r^{-n-1/2},$$

which tends to zero.
Further, $|A_1 - A_{n,1}| \neq 0$ implies that for all sufficiently large n, we have
$P(A_{n,1}, \cdots, A_{n,k}) \neq 0$.
Let d_j denotes degree of P in X_j, there exists a real constant $C_2' > 0$ such that

$$\frac{1}{q_{n,1}^{d_1} \cdots q_{n,k}^{d_k}} \leq |P(A_{n,1}, \cdots, A_{n,k})|. \leq C_2'|A_{n,1} - A_1|. \quad (4)$$

From the hypotheses of Theorem 1, one has

$$|A_1 - A_{n,1}| < \frac{1}{a_{n+1,1}} < \frac{1}{a_{n,k}^{\beta_n}}. \quad (5)$$

Therefore by (5) and $q_{n,j} < q_{n,k}$ for all $1 \leq j \leq k - 1$, one sees

$$\frac{1}{q_{n,k}^d} < |P(A_1, \cdots, A_k)| < \frac{1}{a_{n,k}^{\beta_n}}. \quad (6)$$

This together with (iii) of Lemma 1, give

$$a_{n,k}^{\beta_n} < 2^d a_{n,k}^{\frac{\beta_1}{\beta_1-1}d}$$

For all sufficiently large n contradicting the fact that $\beta_n \to +\infty$. So, $A_1, \cdots, A_k$ are algebraically independent.

2.2. Algebraic independence measure

In this section, we give the second main result of this article and express an algebraic independence measure of the above numbers $A_1, \cdots, A_k$.

Definition: Let $\theta_1, \theta_2, \cdots, \theta_k$ be real umbers. The function $g: \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ is an algebraic independence measure of $\theta_1, \theta_2, \cdots, \theta_k$, if for any polynomial $P \in \mathbb{Z}[X_{1,\ldots,}X_k]\backslash\{0\}$ of total degree d and height $\leq H$ one has

$$|P(\theta_1, \cdots, \theta_k)| \geq g(d, H).$$

Theorem 2. *Let* $(\beta_n)_n$ *be an increasing sequence of real numbers* > 1 *which tends to* $+\infty$, l, r *two real numbers* > 1 *and* $P \in \mathbb{Z}[X_{1,\ldots,}X_k]$ *a non-vanish polynomial of total degree* $d = \sum_{i=1}^{k} degX_i$, $d \geq 2$ *and height* $H \geq q_{2,k}^{\beta_2}$. *If for all* $n \geq 2$, *we have*

$$\begin{cases} r^{-1}a_{n,j} > a_{n,j-1}, \\ a_{n,1} > a_{n-1,k}^{\beta_{n-1}}, \\ a_{n,k} < a_{n-1,1}^{l\beta_{n-1}} \end{cases}$$

Then, there exists a real constant $C = C(A_1, \cdots, A_k, d, l, \beta_1) > 0$, *such that*

$$|P(A_1, \cdots, A_k)| > CH^{-2ld\left(\frac{\beta_1}{\beta_1-1}\right)^2\left(1+\frac{d \ln H}{\ln 2}\right)}.$$

In order to prove Theorem 2, we need the following Lemma.

Lemma 2. (i) *If* $a_{n-1,k}^{\beta_{n-1}} < a_{n,1}$ *for all* $n \geq 2$, *then there exists a real constant* $C_3' = C(\beta_1) > 0$ *such that*

$$q_{n,k} < C_3' a_{n,k}^{\frac{\beta_1}{\beta_1-1}}.$$

(ii) *If* $a_{n,k} < a_{n-1,k}^{l\beta_{n-1}}$ *for all* $n \geq 2$, *then there exists a positive real constant* C_4' *such that*

$$q_{n,k} < C_4' q_{n-1,k}^{\frac{\beta_1}{\beta_1-1}\beta(n-1)}.$$

Proof of Lemma 2. (i) Since

$$\begin{aligned} q_{n,k} &= a_{n,k}q_{n-1,k} + q_{n-2,k} \\ &< (a_{n,k}+1)q_{n-1,k} \\ (8) \quad &< \prod_{i=1}^{n}(1 + a_{i,k}) \end{aligned}$$

Then, we get

$$\begin{aligned} q_{n,k} &< \prod_{i=1}^{n}\left(1 + \frac{1}{a_{i,k}}\right)\prod_{i=1}^{n} a_{i,k} \\ (9) \quad &< C(\beta_1)\prod_{i=1}^{n} a_{i,k}. \end{aligned}$$

So, we obtain

$$(10) \quad q_{n,k} < C(\beta_1) a_{n,k}^{1+\frac{1}{\beta_1}+\frac{1}{\beta_1\beta_2}+\cdots+\frac{1}{\beta_1\beta_2\cdots\beta_{n-1}}}.$$

From the fact that the sequence (β_n) is increasing, we deduce that
$\beta_1\beta_2 \cdots \beta_i \geq \beta_1^i$ for all $1 \leq i \leq n-1$.

Therefore, the inequality (10) becomes

$$\begin{aligned} q_{n,k} &< C(\beta_1) a_{n,k}^{1+\frac{1}{\beta_1}+\frac{1}{\beta_1^2}+\cdots+\frac{1}{\beta_1^{n-1}}} \\ &< C(\beta_1) a_{n,k}^{\frac{1}{1-\frac{1}{\beta_1}}} = C(\beta_1) a_{n,k}^{\frac{\beta_1}{\beta_1-1}}. \end{aligned}$$

(ii) It follows from the inequality

$$\begin{aligned} q_{n,k} &< C(\beta_1) a_{n,k}^{\frac{\beta_1}{\beta_1-1}} \\ &< C(\beta_1) a_{n-1,k}^{\frac{\beta_1}{\beta_1-1}l\beta(n-1)} \\ &< C(\beta_1) q_{n-1,k}^{\frac{\beta_1}{\beta_1-1}l\beta(n-1)}. \end{aligned}$$

Proof of Theorem 2. In order to prove Theorem 2, we must minus $|P(A_1, \cdots, A_k)|$ by a positive function of the height H and a total degree d. From the inequality,

$$(11) \quad |P(A_1, \cdots, A_k)| \geq |P(A_{n,1}, \cdots, A_{n,k})| - |P(A_1, \cdots, A_k) - P(A_{n,1}, \cdots, A_{n,k})|$$

And $q_{n,j} < q_{n,k}$ for all $1 \leq j \leq k-1$, one has

$$(12) \quad |P(A_{n,1}, \cdots, A_{n,k})| \geq \frac{1}{q_{n,1}^{d_1} \cdots q_{n,k}^{d_k}} > \frac{1}{q_{n,k}^{d}}.$$

On the other hand, there exists a constant $C_5' = C(A_1, \cdots, A_k, d) > 0$ such that

$$\begin{aligned} (13) \quad &|P(A_1, \cdots, A_k) - P(A_{n,1}, \cdots, A_{n,k})| \leq \\ &C_5' max|A_j - A_{n,j}| < C_5' H|A_1 - A_{n,1}| \\ &< \frac{C_5' H}{q_{n,1}q_{n+1,1}} < \frac{C_5' H}{a_{n+1,1}}. \end{aligned}$$

By using the inequality (12) and (13), the relationship (11) becomes

$$|P(A_1, \cdots, A_k)| > \frac{1}{q_{n,k}^{d}} - \frac{C_5' H}{a_{n+1,1}}.$$

Since $a_{n+1,1} > a_{n,k}^{\beta_n}$, one get

$$|P(A_1, \cdots, A_k)| > \frac{1}{q_{n,k}^{d}} - \frac{C_5 H}{a_{n,k}^{\beta_n}}.$$

This together with (i) of Lemma 2 give

$$|P(A_1, \cdots, A_k)| > \frac{1}{q_{n,k}^{d}} - C_5 H\left(\frac{C(\beta_1)}{q_{n,k}}\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)}.$$

Now, we look for an integer n from which the quantities

$\frac{1}{q_{n,k}^{d}}$ and $C_5 H\left(\frac{C(\beta_1)}{q_{n,k}}\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)}$ are of the same order.

In order to have

$$|P(A_1,\cdots,A_k)| > \frac{1}{2q_{n,k}^d},$$

It is sufficient that n satisfies

$$\frac{1}{q_{n,k}^d} - \frac{C_5 H\left(C(\beta_1)\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)}}{q_{n,k}^{\frac{\beta_1-1}{\beta_1}\beta(n)}} > \frac{1}{2q_{n,k}^d},$$

This is equivalent to the following inequality,

$$\frac{1}{2q_{n,k}^d} > C_5 H\left(\frac{C(\beta_1)}{q_{n,k}}\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)}.$$

That is

$$(14) \qquad q_{n,k}^{\frac{\beta_1-1}{\beta_1}\beta(n)} > 2C_5 H\left(C(\beta_1)\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)} q_{n,k}^d.$$

To satisfy condition (14), one has only to require

$$(15) \qquad \begin{cases} q_{n,k}^{\frac{\beta_1-1}{\beta_1}\beta(n)} > 2C_5 H\left(C(\beta_1)\right)^{\frac{\beta_1-1}{\beta_1}\beta(n)} \\ q_{n,k}^{\frac{\beta_1-1}{\beta_1}\beta(n)} > H q_{n,k}^d. \end{cases}$$

The first inequality of (15) is easy to achieve because $q_{n,k} > C(\beta_1)^2$ for all sufficiently large n. Therefore, the second one is equivalent to

$$(16) \qquad q_{n,k}^{\frac{\beta_1-1}{\beta_1}\beta(n)-d} > H.$$

Let n_1 be the smallest integer ≥ 2 such that

$$(17) \qquad q_{n_1-1,k}^{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d} \leq H < q_{n_1,k}^{\frac{\beta_1-1}{2\beta_1}\beta(n_1)-d}.$$

The integer n_1 exists because

$$H \geq q_{2,k}^{\beta_2} > q_{2,k}^{\frac{\beta_1-1}{2\beta_1}\beta_2-2},$$

the sequence $\left(q_{n,k}^{\frac{\beta_1-1}{2\beta_1}\beta(n)-d}\right)_n$ is increasing and tend to $+\infty$.

Further, by (ii) of Lemma 2, one has

$$q_{n_1,k} < C(\beta_1) q_{n_1-1,k}^{\frac{l\beta_1}{\beta_1-1}\beta(n_1-1)}$$

This gives

$$|P(A_1,\cdots,A_k)| > \frac{1}{2q_{n_1,k}^d}$$

$$(18) \qquad > \frac{1}{2C(\beta_1)^d} \frac{1}{q_{n_1-1,k}^{\frac{\beta_1}{\beta_1-1}\beta(n_1-1)ld}}.$$

By using the left hand side of (15), (18) becomes

$$(19) \qquad |P(A_1,\cdots,A_k)| > \frac{1}{2}\left(C(\beta_1)\right)^{-d} H^{\frac{\beta_1 ld}{\beta_1-1}\frac{\beta(n_1-1)}{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d}}$$

Then one can remark that

$$(20) \qquad \frac{\beta(n_1-1)}{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d} = \frac{2\beta_1}{\beta_1-1}\left(1+\frac{d}{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d}\right).$$

To complete the proof of Theorem 2, since $\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)d > 1,\ 2 \leq q_{n_1-1}$ and (17), We obtain

$$\frac{1}{2^{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d}} < 2^{\frac{\beta_1-1}{2\beta_1}\beta(n_1-1)-d} \leq H.$$

Then, we find

$$(21) \qquad \frac{\beta_1-1}{2\beta_1}\beta(n_1-1) - d < \frac{\ln H}{\ln 2}.$$

Consequently, by using (20) and (21) in the inequality (19), there exists a constant $C_6 > 0$ such that

$$|P(A_1,\cdots,A_k)| \geq \frac{C_6}{H^{2\left(\frac{\beta_1}{\beta_1-1}\right)^2\left(1+\frac{d\ln 2}{\ln H}\right)dl}},$$

which completes the proof of Theorem 2.

Remark 2. *We note that this measure of algebraic independence is better than the classical measures which are of the form, ln(ln) because it involves only one logarithm.*

Example. Let

$$\begin{cases} a_n = 2^{\frac{(2n)!}{2^{n+1}}}, \quad n \geq 2, \\ b_n = 2^{\frac{(2n)!}{2^{n+2}}}, \quad n \geq 4 \ \ and\ n\ even \\ b_n = 2^{\frac{1}{2}\frac{(2n)!}{2^{n+n}}}, \quad n \geq 3 \ \ and \ \ n \ \ odd, \\ \beta_n = (n+1)(n+1/2), \quad n \geq 2, \end{cases}$$

We chose a_0, b_0, a_1, b_1 and b_2 such that $q_{1,1} = q_1(B) \leq q_{1;2} = q_1(A)$ and
$q_{2,1} = q_2(B) \leq q_{2;2} = q_2(A)$.
By applying Theorem 1, we conclude that the continued fractions A and B algebraically independent.
Further, for any quadratic polynomial $P \in \mathbb{Z}[X,Y]$, with the height $H \geq q_2(A)^{15/2}$, there exists a constant $C = C(A,B) > 0$ such that

$$|P(A,B)| > CH^{-\frac{200}{9}\left(1+\frac{\ln H}{\ln 2}\right)}.$$

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Kacem Belhroukia carried out the proof of Theorem 1.

Ali Kacha proved the Theorem 2.

Brahim Ounir carried out the example.

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