A Hybrid Method for Assessing Student Mathematical Modelling Skills under Fuzzy Conditions

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Abstract: - Mathematical modelling (MM) appears today as a valuable tool in mathematics education, which connects mathematics with everyday life situations on the purpose of increasing the student interest on the subject. In this paper a hybrid method is applied for the assessment of student groups' MM skills with qualitative grades (i.e. under fuzzy conditions). Namely, soft sets are used as tools for a parametric assessment of a group's performance, the calculation of the GPA index and the Rectangular Fuzzy Assessment Model are applied for evaluating the group's qualitative performance, grey numbers are used as tools for assessing the group's mean performance and neutrosophic sets are utilized when the teacher is not sure about the individual grades assigned to some (or all) students of the group.

Key-Words: - Mathematical Modelling (MM), Fuzzy Logic (FL), Fuzzy Assessment Methods, GPA Index, Rectangular Fuzzy Assessment Model (RFAM), Grey Number (GN), Neutrosophic Set (NS), Soft Set (SS).

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1 Introduction

A *model* is understood to be a simplified representation of a real system including only its characteristics which are related to a certain problem concerning the system (*assumed real system*); e.g. maximizing the system's productivity, minimizing its functional costs, etc. The process of modelling is a fundamental principle of the systems' theory, since the experimentation on the real system is usually difficult (or impossible sometimes) requiring a lot of money and time. Modelling a system involves a deep abstracting process, which is graphically represented in Fig. 1 [1].

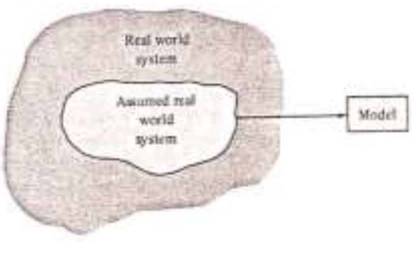

Fig. 1: Graphical representation of the modelling process

There are several types of models used according to the form of the system and of the corresponding problem to be solved [1]. In simple cases *iconic models* may be used, like maps, bas-relief representations, etc. *Analogical models,* such as graphs, diagrams, etc., are frequently used when the corresponding problem concerns the study of the relationship between only two of the system's variables; e.g. speed and time, temperature and pressure, etc. The *mathematical or symbolic models* use mathematical symbols and representations (functions, equations, inequalities, etc.) to describe the system's behavior. This is the most important type of models, because they provide accurate and general (i.e. holding even if the system's parameters are changed) solutions to the corresponding problems. In case of complex systems, however, like the biological ones, where the solution cannot be expressed in solvable mathematical terms or the mathematical solution requires laborious calculations, *simulation models* are often used. These models mimic the system's behavior over a period of time with the help of a well-organized set of logical orders, usually expressed in the form of a computer program. Also, *heuristic models* can be utilized for improving already existing solutions, obtained either empirically or by using other types of models.

Until the middle of the 1970's *Mathematical Modelling (MM)* mainly used to be a tool at hands of the scientists for solving problems related to their disciplines. The failure of the introduction of the "new mathematics" to school education [2], however, turned the attention of the specialists to problem-solving activities as a more effective way for teaching and learning mathematics. MM in particular, has been widely used for connecting mathematics to everyday life situations, on the purpose of increasing the student interest on the subject.

Quality is a desirable characteristic of all human activities. This makes assessment one of the most important components connected to those activities. Assessment takes place in two ways, either with the help of numerical or with the help of qualitative grades, like excellent, good, mediocre, etc.

When numerical grades are used, standard methods are applied for the overall assessment of the skills of a group of objects participating in a certain activity, like the calculation of the mean value of all the individual scores or the *Grade Point Average (GPA)* index, a weighted average in which greater coefficients are assigned to the higher scores.

The use of qualitative grades is usually preferred when more elasticity is desirable (as it frequently happens in case of student assessment), or when no exact numerical data are available. In this case, assessment methods based on principles of *fuzzy logic (FL)* are frequently used.

The present author has developed in earlier works several methods for the assessment of human/machine performance under fuzzy conditions including the measurement of *uncertainty* in fuzzy systems, the use of the *Center of Gravity (COG) defuzzification technique*, the use of *fuzzy numbers (FNs)* or of *grey numbers (GNs)*, etc. All these methods are reviewed in [3]. Recently, Voskoglou developed also assessment models using *soft sets (SSs)* and *neutrosophic sets (NSs)* as tools [4, 5]

In this work a hybrid method is applied for the assessment of student groups' MM skills with qualitative grades (i.e. under fuzzy conditions). Namely, SSs are used for the parametric assessment of a group's performance, the calculation of the GPA index and the *Rectangular Fuzzy Assessment Model (RFAM)* are applied for evaluating the group's qualitative performance, GNs are used as tools for assessing the group's mean performance and NSs are utilized when the teacher is not sure about the individual grades assigned to some (or all) students of the group. The paper closes with the final conclusions and some hints for future research.

2. Mathematical Modelling in Education

One of the first who proposed the use of MM as a tool for teaching mathematics was H. O. Pollak [6], who presented in 1976 during the ICME-3 Conference in Karlsruhe the scheme of Fig. 2, known as the *circle of modelling.* In this scheme, given a problem for solution from a topic different from mathematics (other world), the solver, following the direction of the arrows, is transferred to the "universe" of mathematics. There, the solver uses or creates suitable mathematics for the solution of the problem and then returns to the other world to check the validity of the mathematical solution obtained. If the verification of the solution is proved to be non-compatible to the existing real conditions, the same circle is repeated one or more times.

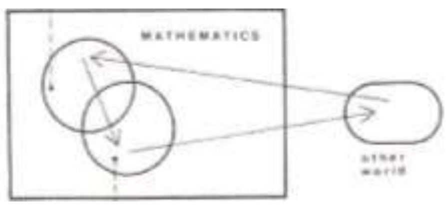

Fig. 2: The Pollak's Circle of Modelling

Following the Pollak's presentation, much effort has been placed by mathematics education researchers to study and analyze in detail the process of MM on the purpose of using it for teaching mathematics. Several models have been developed towards this direction, a brief but comprehensive account of which can be found in [7], including the present author's model (Fig. 3).

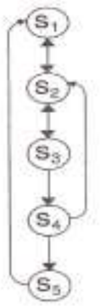

Fig. 3: Flow-diagram of Voskoglou's model for the MM process

In this model [8] Voskoglou described the MM process in terms of a Markov chain introduced on its main steps, which are: S_1 = *analysis* of the problem, S_2 = *mathematization* (formulation and construction of the model), S_3 = *solution* of the model, S_4 = *validation* of the solution and S_5 = *implementation* of the solution to the real system. When the MM process is completed at step S_5, it is assumed that a new problem is given to the class, which implies that the process restarts again from step S_1.

Mathematization is the step of the MM process with the greatest gravity, since it involves a deep abstracting process, which is not always easy to be achieved by a non-expert. A solver who has obtained a mathematical solution of the model is usually able to "translate" it in terms of the corresponding real situation and to check its validity. There are, however, sometimes problems in which the validation of the model and/or the implementation of the final mathematical results to the real system hide surprises, which force solvers to look back to the construction of the model and make the necessary changes to it. A characteristic example is presented in [9].

Models like Voskoglou's (Fig. 3) are useful for describing the solvers' *ideal behavior* when tackling MM problems. Relative researches [10-12], however, report that the reality is not like that. In fact, modelers follow individual routes related to their learning styles and the level of their cognition. Consequently, from the teachers' part there exists an uncertainty about the student way of thinking at each step of the MM process. Those findings inspired the present author to use principles of FL for describing in a more realistic way the process of MM in the classroom on the purpose of understanding, and therefore treating better, the student reactions during the MM process [13]. The steps of the MM process in this model are represented as fuzzy sets on a set of linguistic labels characterizing the student performance in each step.

A complete methodology for teaching mathematics on the basis of MM has been eventually developed, which is usually referred as the *application-oriented teaching of mathematics* [14]. However, as the present author underlines in [9], teachers must be careful, because the extensive use of the application-oriented teaching as a *general method* for teaching mathematics could lead to far-fetched situations, in which more attention is given to the choice of the applications rather than to the mathematical content!

More details about MM from the viewpoint of Education and representative examples can be found in earlier works of the author [9, 14].

3. Mathematical Background

3.1 Fuzzy Sets and Logic

Zadeh, in order to deal with partial truths, introduced in 1965 the concept of *fuzzy set (FS)* as follows [15]:

Definition 1: Let U be the universe, then a FS F in U is of the form

$$F = \{(x, m(x)): x \in U\} \quad (1)$$

In equation (1) m: $U \rightarrow [0,1]$ is the *membership function* of F and m(x) is called the *membership degree* of x in F**.** The greater m(x), the more x satisfies the characteristic property of F. A crisp subset F of U is a FS in U with membership function such that m(x)=1 if x belongs to F and 0 otherwise.

Based on the concept of FS Zadeh developed the infinite-valued FL [16], in which truth values are modelled by numbers in the unit interval [0, 1]. FL is an extension of the classical *bivalent logic (BL)* of Aristotle embodying the Lukasiewicz's "Principle of Valence" [17]. In contrast to the Aristotle's principle of the "Excluded Middle", Lukasiewicz's principle states that propositions are not only either true or false, but they can have intermediate truth-values too.

It was only in a second moment that FS theory and FL were used to embrace *uncertainty* modelling [18, 19]. This happened when membership functions were reinterpreted as possibility distributions. *Possibility theory* is an uncertainty theory devoted to the handling of incomplete information [20]. Zadeh [18] articulated the relationship between possibility and *probability*, noticing that what is probable must preliminarily be possible. For general facts on FSs and the connected to them uncertainty we refer to the book [21].

3.2 Neutrosophic Sets

Following the introduction of FSs, various generalizations and other related to FSs theories have been proposed enabling a more effective management of all types of the existing in real world uncertainty. A brief description of the main among those generalizations and theories can be found in [22].

Atanassov added in 1986 to Zadeh's membership degree the *degree of non-membership* and introduced the concept of *intuitionistic fuzzy set (IFS)* [23] as the set of the ordered triples

$$A = \{(x, m(x), n(x)): x \in U, 0 \leq m(x) + n(x) \leq 1\} \ (2)$$

Smarandache, motivated by the various neutral situations appearing in real life - like <friend, neutral, enemy>, <positive, zero, negative>, <small, medium, high>, <male, transgender, female>, <win, draw, defeat>, etc. – introduced in 1995 the degree of *indeterminacy/neutrality* of the elements of the universal set U in a subset of U and defined the concept of *neutrosophic set (NS)* [24]. The term neuttrosophic is the result of the synthesis of the words "neutral" and "sophia" which means in Greek language "wisdom". In this work we need only the simplest version of the concept of NS, which is defined as follows:

Definition 2: A *single valued NS (SVNS)* A in U is of the form

$$A = \{(x,T(x),I(x),F(x)): x\in U,\ T(x),I(x),F(x)\in[0,1],\ 0\leq T(x)+I(x)+F(x)\leq 3\} \quad (3)$$

In (3) T(x), I(x), F(x) are the degrees of *truth* (or membership), indeterminacy and *falsity* (or non-membership) of x in A respectively, called the *neutrosophic components* of x. For simplicity, we write A<T, I, F>.

For example, let U be the set of the players of a basketball team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in [0, 1]. For instance, x(0.7, 0.1, 0.4) ∈ A means that there is a 70% belief that x is a good player, a 10% doubt about it and a 40% belief that x is not a good player. In particular, x(0,1,0) ∈ A means that we do not know absolutely nothing about x's affiliation with A.

In an IFS the indeterminacy coincides by default to 1- T(x) – F(x). Also, in a FS is I(x)=0 and F(x) = 1 – T(x), whereas in a crisp set is T(x)=1 (or 0) and F(x)= 0 (or 1). In other words, crisp sets, FSs and IFSs are special cases of SVNSs.

When the sum T(x) + I(x) + F(x) of the neutrosophic components of x ∈ U in a SVNS A on U is <1, then x leaves room for *incomplete* information, when is equal to 1 for *complete* information and when is greater than 1 for *paraconsistent* (i.e. contradiction tolerant) information. A SVNS may contain simultaneously elements leaving room to all the previous types of information. For general facts on SVNSs we refer to [25].

Summation of neutrosophic triplets is equivalent to the neutrosophic union of sets. That is why the neutrosophic summation and implicitly its extension to neutrosophic scalar multiplication can be defined in many ways, equivalently to the known in the literature neutrosophic union operators [26]. Here, writing the elements of a SVNS A in the form of neutrosophic triplets we define addition and scalar product in A as follows:

Let (t_1, i_1, f_1), (t_2, i_2, f_2) be in A and let k be a positive number. Then:

- The *sum* $(t_1, i_1, f_1) + (t_2, i_2, f_2) = (t_1+ t_2,\ i_1+ i_2, f_1+ f_2)$ (4)
- The *scalar product* $k(t_1, i_1, f_1) = (kt_1, k i_1, kf_1)$ (5)

3.3 Soft Sets

A disadvantage connected to the concept of FS is that there is not any exact rule for defining properly the membership function. The methods used for this are usually empirical or statistical and the definition of the membership function is not unique depending on the "signals" that each observer receives from the environment, which are different from person to person. For example, defining the FS of "tall men" one may consider as tall all men having heights more than 1.90 meters and another all those having heights more than 2 meters. As a result, the first observer will assign membership degree 1 to men of heights between 1.90 and 2 meters, in contrast to the second one, who will assign membership degrees <1. Consequently, analogous differences is logical to appear for all the other heights. The only restriction, therefore, for the definition of the membership function is to be compatible to the common sense; otherwise the resulting FS does not give a reliable description of the corresponding real situation. This could happen for instance, if in the FS of "tall men", men with heights less than 1.60 meters have membership degrees ≥0.5.

The same difficulty appears to all generalizations of FSs in which membership functions are involved (e.g. IFSs, NSs, etc.). For this reason, the concept of *interval-valued FS (IVFS)* [27] was introduced in 1975, in which the membership degrees are replaced by sub-intervals of the unit interval [0, 1]. Alternative to FS theories were also proposed, in which the definition of a membership function is either not necessary (*grey systems/GNs* [28]), or it is overpassed by considering a pair of sets which give the lower and the upper approximation of the original crisp set (*rough sets* [29]).

Molodstov, in order to tackle the uncertainty in a parametric manner, initiated in 1999 the concept of *soft set (SS)* as follows [30]:

Definition 3: Let E be a set of parameters, let A be a subset of E, and let f be a map from A into the power set P(U) of all subsets of the universe U. Then the SS (f, A) in U is defined to be the set of the ordered pairs

$$(f, A) = \{(e, f(e)): e \in A\} \quad (6)$$

The term "soft" is due to the fact that the form of (f, A) depends on the parameters of A. For example, let U= {C_1, C_2, C_3} be a set of cars and let E = {e_1, e_2, e_3} be the set of the parameters e_1=cheap, e_2=hybrid (petrol and electric power) and e_3= expensive. Let us further assume that the cars C_1, C_2 are cheap, C_3 is expensive and C_2, C_3 are hybrid cars. Then, a map f: $E \rightarrow P(U)$ is defined by $f(e_1)=\{C_1, C_2\}$, $f(e_2)=\{C_2, C_3\}$ and $f(e_3)=\{C_3\}$. Therefore, the SS (f, E) in U is the set of the ordered pairs (f, E) = {(e_1, {C_1, C_2}), (e_2, {C_2, C_3}, (e_3, {C_3}}.

A FS in U with membership function y = m(x) is a SS in U of the form (f, [0, 1]), where $f(\alpha)=\{x \in U: m(x) \geq \alpha\}$ is the corresponding $\alpha - cut$ of the FS, for each α in [0, 1]. For general facts on SSs we refer to [31].

Obviously, an important advantage of SSs is that, by using the parameters, they pass through the need of defining membership functions. The theory of SSs has found many and important applications to several sectors of human activity like decision making, parameter reduction, data clustering and data dealing with incompleteness, etc. One of the most important steps for the theory of SSs was to define mappings on SSs, which was achieved by A. Kharal and B. Ahmad and was applied to the problem of medical diagnosis in medical expert systems [32]. But fuzzy mathematics has also significantly developed at the theoretical level providing important insights even into branches of classical mathematics like algebra, analysis, geometry, topology etc.

3.4 Grey Numbers

Approximate data are frequently used nowadays in many problems of everyday life, science and engineering, because many constantly changing factors are usually involved in large and complex systems. Deng introduced in 1982 the *grey system (GS)* theory as an alternative to the theory of FSs for tackling such kind of data [27]. A GS is understood to be a system that lacks information such as structure message, operation mechanism and/or behaviour document. The GS theory, which has been mainly developed in China, has recently found many important applications [33].

An interesting application of the closed intervals of real numbers is their use in the GS theory for handling approximate data. In fact, a numerical interval I = [x, y], with x, y real numbers, x<y, can be considered as representing a real number with known range, whose exact value is unknown. The closer x to y, the better I approximates the corresponding real number. When no other information is given about this number, it looks logical to consider as its representative approximation the real value

$$V(I) = \frac{x+y}{2} \quad (7)$$

Moore et al. [34] introduced the basic arithmetic operations on closed real intervals. In the present work we shall make use only of the addition and scalar product defined as follows: Let $I_1 = [x_1, y_1]$ and $I_2 = [x_2, y_2]$ be closed intervals, then their *sum* $I_1 + I_2$ is the closed interval

$$I_1 + I_2 = [x_1+ x_2, y_1+ y_2] \quad (8)$$

Further, if k is a positive number then the *scalar product* kI_1 is the closed interval

$$kI_1 = [kx_1, ky_1] \quad (9)$$

When the closed real intervals are used for handling approximate data, are usually referred as *grey numbers (GNs)*. A GN [x, y], however, may also be connected to a *whitenization function* f: [x, y] → [0, 1], such that, ∀ a ∈ [x, y], the closer f(a) to 1, the better a approximates the unknown number represented by [x, y].

We close this subsection with the following definition, which will be used in the assessment method that will be presented later in this work.

Definition 4: Let $I_1, I_2, \ldots, I_n$ be a finite number of GNs, n≥2, then the *mean value* of these GNs is defined to be the GN

$$I = \frac{1}{n}(I_1 + I_2 + \ldots + I_k) \quad (10)$$

3.5 GPA Index and the Rectangular Fuzzy Assessment Model

The calculation of the *Grade Point Average (GPA) Index* is a classical method, very popular in the USA and other western countries, for evaluating a group's *qualitative performance*, where greater coefficients are assigned to the higher grades. For this, let n be the total number of the objects of the group under assessment and let n_X be the number of the group's objects obtaining the grade X, X =A, B, C, D, F, where A=excellent, B=very good, C=good, D=mediocre and F=unsatisfactory. Then, the GPA index is calculated by the formula

$$GPA = \frac{0n_F + n_D + 2n_C + 3n_B + 4n_A}{n} \quad (11)$$

[35] (Chapter 6, p. 125)

In the worst case ($n=n_F$) equation (11) gives that GPA=0, whereas in the best case ($n=n_A$) it gives that GPA=4. We have in general, therefore, that $0\leq GPA\leq 4$, which means that values of $GPA\geq 2$ indicate a satisfactory qualitative performance.

Setting $y_1 = \frac{n_F}{n}$, $y_2 = \frac{n_D}{n}$, $y_3 = \frac{n_C}{n}$, $y_4 = \frac{n_B}{n}$ and $y_5 = \frac{n_A}{n}$, equation (11) can be written as

$$GPA = y_2 + 2y_3 + 3y_4 + 4y_5 \quad (12)$$

Voskoglou developed a fuzzy model for representing mathematically the process of learning a subject matter in the classroom [36]. Later, considering a student class as a fuzzy system, he calculated the existing in it *total possibilistic uncertainty* for assessing the student mean performance [37]. Subbotin et al., based on Voskoglou's model, adapted properly the *Center of Gravity (COG) defuzzification technique* for use as an assessment method of student learning skills [38]. Since then, Subbotin and Voskoglou applied, jointly or separately, the COG technique, termed by them as the *Rectangular Fuzzy Assessment Model (RFAM)*, in many other types of assessment problems; e.g. see [35] (Chapter 6).

There is a commonly used in FL approach to represent the fuzzy data by the coordinates (x_c, y_c) of the COG of the level's area between the graph of the corresponding membership function and the OX axis [39]. In our case, keeping the same notation as for the GPA index, it can be shown that the coordinates of the COG are calculated by the formulas

$$x_c = \frac{1}{2}(y_1+3y_2+5y_3+7y_4+9y_5) \quad (13)$$

$$y_c = \frac{1}{2}(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2) \quad (14)$$

[3] (Section 4)

It can be also shown the following result [3] (Section 4**):**

Assessment Criterion:

- Between two groups, the group with the greater x_c demonstrates the better performance.
- For two groups with the same value of x_c, if $x_c\geq 2.5$ the group with the greater value of y_c performs better, and if $x_c<2.5$ the group with the lower value of y_c performs better.

Combining equations (12) and (13) one finds that

$x_c = \frac{1}{2}(2GPA + 1)$ or

$$x_c = GPA + \frac{1}{2} \quad (15)$$

Thus, with the help of the first case of the previous criterion, one concludes that, if the GPA value of two student groups is different, then the RFAM and the GPA index give the same outcomes concerning the assessment of the qualitative performance of the two groups. If the GPA index, however, is the same for the two groups, then one MUST apply the RFAM to see which group performs better.

4. The Hybrid Assessment Model

A hybrid method is applied in this Section for the assessment of a student group's MM skills with qualitative grades. Namely, SSs are used as tools for a parametric assessment of the group's performance, the calculation of the GPA index and the RFAM are applied for evaluating the group's qualitative performance, GNs are used as tools for assessing the group's mean performance and NSs are used when the teacher is not sure about the individual grades assigned to some (or all) students.

4.1 Parametric Assessment Using Soft Sets

Assume that a mathematics teacher wants to assess the MM skills of a group $U = \{S_1, S_2, \ldots, S_n\}$ of n students, $n\geq 2$. Let E = {A, B, C, D, E} be the set of the parameters A=excellent, B=very good, C=good, D=mediocre and F=unsatisfactory. Assume further that the first four students of the group demonstrated excellent performance, the next five very good, the following 7 good, the next eight mediocre and the rest of them unsatisfactory performance. Let f be the map assigning to each parameter of E the subset of students whose performance was assessed by this parameter. Then, the overall student performance can be represented mathematically by the SS

$(f, E) = \{(A, \{S_1, S_2, S_3\}), (B, \{S_4, S_5, \ldots, S_8\}), (C, \{S_9, S_{10}, \ldots, S_{15}\}), (D, \{S_{16}, S_{17}, \ldots, S_{23}\}), (F, \{S_{24}, S_{25}, \ldots, S_n\})\}$ (16)

The use of SSs also enables the representation of each student's individual performance at each step of the MM process. For this, let $V = \{S_1, S_2, S_3, S_4, S_5\}$ be the set of the steps of the MM process according to Voskoglou's model presented in Section 2 (Fig. 3). Consider a particular student of the group U and define a map f: $E \rightarrow \Delta(V)$ assigning to each parameter of E the subset of V consisting of the steps of the MM process assessed by this parameter with respect to the chosen student. For example, the SS (f, E) = {(A, {S_1, S_3}), (B, {S_5}), (C, {S_4}), (D, {S_2}), (F, ∅)} (17)

represents the profile of a student who demonstrated excellent performance at the steps of analysis of the problem and solution of the model, very good performance at the step of implementation of the solution, good performance at the step of validation and mediocre performance at the step of mathematizing (he/she faced difficulties, but he/she finally came through).

4.2 Use of the COG Technique and the RFAM for Assessing a Group's Qualitative Performance

The following example illustrates this method:

Example 1: The students of two classes obtained the following grades in a test involving MM problems: Class I: A=5 students, B=3, C=7, D=0, F=5, Class II: A=4, B=4, C=7, D=1, F=4. Which class demonstrated the better qualitative performance?

Solution: Equation (11) gives that $GPA_I = GPA_2 = \frac{43}{20}$. The RFAM model must be used, therefore, for comparing the two classes' qualitative performance. Thus, by equation (13) one gets that $x_{C_I} = x_{C_{II}} = \frac{53}{20} > \frac{5}{2}$. But equation (14) gives that y_{C_I} = 54 and $y_{C_{II}}$ = 49, therefore, by the second case of the RFAM assessment criterion, one concludes that Class I demonstrated a better qualitative performance. Further, since $GPA_I = GPA_2 = \frac{43}{20} > 2$, both groups demonstrated satisfactory qualitative performance.

4.3 Use of Grey Numbers for Evaluating a Group's Mean Performance.

When the student individual assessment is realized with qualitative grades, a student group's mean performance cannot be assessed with the classical method of calculating the mean value of the student scores. To overcome this difficulty, using the numerical climax 1-100 we assign to each of the student qualitative grades a closed real interval (GN), denoted for simplicity with the same letter, as follows: A = [85, 100], B = [75, 84], C = [60, 74], D = [50, 59] and F= [0, 49].

It is of worth noting that, although the GNs assigned to the qualitative grades satisfy commonly accepted standards, the previous assignment is not unique, depending on the teacher's personal goals. For a more strict assessment, for example, the teacher could choose A = [90, 100], B = [80, 89], C = [70, 79], D = [60, 69], F= [0, 59], etc.

The estimation of a group's mean performance with the help of the previously defined GNs is illustrated with the following example:

Example 2: Reconsider Example 1. Which class demonstrated the better mean performance?

Solution: Under the light of equation (10), it is logical to accept that the GNs

$M_I = \frac{1}{20}(5A+3B+7C+0D+5F)$ and

$M_{II} = \frac{1}{20}(4A+4B+7C+1D+4F)$ respectively can be used for estimating the two classes' mean performance. Straightforward calculations with the help of equations (8) and (9) give that

$M_I = \frac{1}{20}[1070, 1515] = [53.5, 75.75]$ and

$M_{II} = \frac{1}{20}[1110, 1509] = [55.5, 75.45]$.

Equation (7) gives, therefore, that $V(M_I) = 64.625$ and $V(M_{II}) = 64.75$. Thus, both classes demonstrated good (C) mean performance, with the mean performance of Class II being slightly better.

4.4 Using Neutrosophic Sets for Student Assessment

In many cases the teacher has doubts about the grades assigned to some (or all) students of the group under assessment. In such cases the use of NSs is more appropriate for estimating the student group overall performance. This process is illustrated in the following example:

Example 3: Let {s_1, s_2, …. , s_{20}} be a class of 20 students. The teacher of the class is not sure about the grades obtained by them in a test involving MM problems, because some of the students did not give proper explanations about their solutions. The teacher decides, therefore, to characterize the students who demonstrated excellent performance in the test by using neutrosophic triplets as follows: s_1(1, 0, 0), s_2(0.9, 0.1, 0.1), s_3(0.8, 0.2, 0.1), s_4(0.4, 0.5, 0.8), s_5(0.4, 0.5, 0.8), s_6(0.3, 0.7, 0.8), s_7(0.3, 0.7, 0.8), s_8(0.2, 0.8, 0.9), s_9(0.1, 0.9, 0.9), s_{10}(0.1, 0.9, 0.9} and for all the other students (0, 0, 1). This means that the teacher is absolutely sure that s_1 demonstrated excellent performance, 90% sure that s_2 demonstrated excellent performance too, but at the same time has a 10% doubt about it and also a 10% belief that s_2 did not demonstrate excellent performance, etc. For the last 10 students the teacher is absolutely sure that they did not demonstrate excellent performance. What should be the teacher's conclusion about the class's mean performance in this case?

Solution: It is logical to accept that the class's mean performance can be estimated by the neutrosophic triplet $\frac{1}{20}$[(1, 0, 0)+(0.9, 0.1, 0.1)+(0.8, 0.2, 0.1)+2(0.4, 0.5, 0.8)+2(0.3, 0.7, 0.8)+(0.2, 0.8, 0.9)+2(0.1, 0.9, 0.9)+10(0, 0, 1)], which by equations (8) and (9) is equal to

$\frac{1}{20}$(4.5, 5.3, 16.3) = (0.225, 0.265, 0.815). This means that the performance of a random student of the class has a 22.5% probability to be characterized as excellent, however, there exist also a 26.5% doubt about it and an 81.5% probability to be characterized as not excellent. Obviously this conclusion is characterized by inconsistency, which is an expected outcome due to the teacher's uncertainty for the grades assigned to students.

The teacher could work in the same way by considering the NSs of students who demonstrated very good, good, mediocre and unsatisfactory performance in the test, thus obtaining analogous conclusions.

5. Discussion and Conclusions

A hybrid assessment method was applied in this work for assessing student MM skills under fuzzy conditions (with qualitative grades). The whole process followed leads to the following conclusions:

- SSs can be used for realizing a parametric assessment of the student group's overall performance.
- The qualitative performance of a student group (where greater coefficients are assigned to the higher grades) can be measured either by the classical method of calculating the GPA index, or by applying the RFAM, which is based on the COG defuzzification technique. When two groups have the same GPA index, however, then the RFAM model must be applied to find which group demonstrates the better performance.
- In case of using qualitative grades for assessing the student performance, the assessment of a student group's mean performance cannot be realized by the classical way of calculating the mean value of the student individual scores. The student mean performance in this case can be estimated by using GNs (closed real intervals).
- When the teacher has doubts for the grades assigned to some (or all) students, NSs is more appropriate to be used for assessing the overall performance of a student group.

Our experience from the present and earlier works implies that hybrid methods, like the previous one, usually give better and more complete results, not only in the assessment processes, but also in decision-making, in tackling the existing in real world uncertainty and possibly in various other human or machine activities. This is, therefore, an interesting area for further research.

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