Reccurent-Iterative Scheme to approximate as Desired the Complete Elliptic Integrals

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Abstract. Two sets of closed *analytic* formulas are proposed for the approximate calculus of the complete elliptic integrals K(k) and E(k) in the normal form due to Legendre, their expressions having a remarkable simplicity and accuracy. The special usefulness of the newly proposed formulas consists in they allow performing the *analytic study of variation* of the functions in which they appear, *using derivatives*, being expressed *in terms of elementary (especially algebraic) functions only, without any special function (this would mean replacing one difficulty by another of the same kind)*. Comparative tables of so found approximate values with the exact ones, reproduced from special functions tables, are given (wrt the elliptic integrals' modulus k). The first set of formulas was suggested by *Peano's law on ellipse's perimeter*. *The new functions and their derivatives coincide with the exact ones at* $k = 0$ *only*. As for simplicity, *the formulas in* k / k' *don't need mathematical tables nor advanced calculators*, being purely algebraic. As for accuracy, *the second set, something more intricate, gives more accurate values and extends more closely to* $k = 1$. An *original fast converging recurrent-iterative scheme* to get sets of formulas with the desired accuracy is given in appendix 1. Using the results obtained by applying the newly proposed approximate formulas *a method to approximate the complete elliptic integral* $\Pi(n, k)$ is given in appendix 2.

Key-Words: elliptic integrals' moduli k, k'; special functions tables with Legendre's complete elliptic integrals; Peano's approximate law for the perimeter of an ellipse of low eccentricity k; descending and ascending Landen's transformations

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1 Elliptic integrals – occurrences, definitions

There are many interesting domains in pure and applied mathematics where appear both (or, often, only one) complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre. The arc length of a Bernoulli's lemniscate, as well as the period of oscillations in a vacuum of the simple pendulum, in the dynamics of a constrained heavy particle, are given by a complete elliptic integral of the 1st kind. The perimeter of an ellipse, as well as the lift coefficient of a thin delta wing with subsonic leading edges, in supersonic aerodynamics (small perturbations theory), are given by a complete elliptic integral of the 2nd kind. In electromagnetic theory, the electric and magnetic fields from a circular coil can be expressed using the complete elliptic integrals. The relations below define the integrals of the 1st and 2nd kind, in *canonical* form, K(k) and E(k), respectively:

$\mathrm{K}(k) = \int_0^{\pi/2}(1 - k^2\sin^2\varphi)^{-1/2}d\varphi = \int_0^1[(1 - t^2)(1 - k^2t^2)]^{-1/2}dt;$

$\mathrm{E}(k) = \int_0^{\pi/2}(1 - k^2\sin^2\varphi)^{1/2}d\varphi = \int_0^1[(1 - t^2)(1 - k^2t^2)]^{1/2}dt;$

$k = \sin\theta \geq 0$ is called *modulus*. K(k), E(k) are typical *elliptic* integrals. They do not admit primitive functions (cannot be expressed in terms of elementary functions), being calculated by expanding the integrands into series, integrating term-by-term, and presented wrt $k \in [0, 1]$, or wrt $\theta \in [0, \pi/2]$, in some mathematical tables [1] – [6]. Other examples of such kind of integrals are: Si(x); Ci(x); Ei(x); li(x). Modern mathematics defines an elliptic integral as any function f which can be expressed in the form $f(x) = \int_c^x R[t, P(t)^{1/2}]\, dt$; R is a rational function of its two arguments; P is a polynomial of degree 3 or 4 with no repeated roots; c is a constant. The values given in some special tables allow performing the calculus for a given case (point), but not the analytic study of variation of the functions in which these integrals appear, using the derivatives. Further *two sets* (0; 1) *of closed analytic formulas* to approximate K(k) and E(k) in both algebraic and trigonometric form are given. A *fast converging recurrent-iterative scheme* to get sets of formulas with a desired high accuracy is given in appendix 1. We use *an original purely analytic method* (not some numerical, or sophisticated computer programs, like most authors). There also is a Legendre complete elliptic integral of the 3rd kind. With an appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the *three Legendre canonical forms* (of the 1st, 2nd & 3rd kind).

2 The two sets of newly proposed formulas

The *complementary modulus* is $k' = (1 - k^2)^{1/2} = \cos\theta \geq 0$. The $\mathrm{E}_0(k)$ formula in the 1st set (K_0, E_0) is Peano's law on the perimeter of an ellipse of low eccentricity k; a, b – semiaxes; $k' = b/a$.

$$\mathrm{K}_0(k)=\frac{\pi}{\sqrt[4]{1-k^2}}\left(1-\frac{1}{2\sqrt{2}}\sqrt{\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}}\right)=\pi\left(\frac{1}{\sqrt{k'}}-\frac{1}{2\sqrt{2}}\frac{\sqrt{1+k'}}{k'^{3/4}}\right),$$

$$\mathrm{K}_0(\theta)=\frac{\pi}{\cos^{1/2}\theta}\left[1-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{1/4}\theta}\right]=\pi\left[\frac{1}{\cos^{1/2}\theta}-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{3/4}\theta}\right].$$

$$\mathrm{E}_0(k)=\frac{\pi}{4}\sqrt[4]{1-k^2}\left(\frac{3}{2}\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}-1\right)=\frac{\pi}{4}\left[\frac{3}{2}(1+k')-\sqrt{k'}\right],$$

$$\mathrm{E}_0(\theta)=\frac{\pi}{4}\cos^{1/2}\theta\left[3\frac{\cos^2(\theta/2)}{\cos^{1/2}\theta}-1\right]=\frac{\pi}{4}\left(3\cos^2\frac{\theta}{2}-\sqrt{\cos\theta}\right).$$

Similarly, for the 2nd set (K_1, E_1) we proposed the formulas:

$$\boxed{\begin{aligned}
&\mathrm{K}_1(k)=\frac{\pi\sqrt{2}}{\sqrt{(1+k')\sqrt{k'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k'}}{\sqrt[4]{(1+k')\sqrt{k'}}}\right),\\
&\mathrm{K}_1(\theta)=\frac{\pi}{\cos(\theta/2)\cos^{1/4}\theta}\left[1-\frac{1}{4}\frac{1+\cos^{1/2}\theta}{\cos^{1/2}(\theta/2)\cos^{1/8}\theta}\right].\\
&\mathrm{E}_1(k)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k'}\right)^2-\sqrt{2}\sqrt{1+k'}\sqrt[4]{k'}\right]-k'\cdot \mathrm{K}_1(k),\\
&\mathrm{E}_1(\theta)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{\cos\theta}\right)^2-2\cos\frac{\theta}{2}\sqrt[4]{\cos\theta}\right]-\cos\theta\cdot \mathrm{K}_1(\theta).
\end{aligned}}$$

A 3rd set (K_2, E_2), more accurate than the previous two, can be built (a recurrent-iterative scheme in appendix 1); all set definitions are boxed.

Table 1. Values of the functions K (part one)

θ (°)	$k=\sin\theta$	K(k)	$\mathrm{K}_0(k)$	$\mathrm{K}_1(k)$
0	0.00000	1.5708	1.5708	1.5708
1	0.01745	1.5709	1.5709	1.5709
2	0.03490	1.5713	1.5713	1.5713
3	0.05234	1.5719	1.5719	1.5719
4	0.06976	1.5727	1.5727	1.5727
5	0.08716	1.5738	1.5738	1.5738
6	0.10453	1.5751	1.5751	1.5751
7	0.12187	1.5767	1.5767	1.5767
8	0.13917	1.5785	1.5785	1.5785
9	0.15643	1.5805	1.5805	1.5805
10	0.17365	1.5828	1.5828	1.5828
11	0.19081	1.5854	1.5854	1.5854
12	0.20791	1.5882	1.5882	1.5882
13	0.22495	1.5913	1.5913	1.5913
14	0.24192	1.5946	1.5946	1.5946
15	0.25882	1.5981	1.5981	1.5981
16	0.27564	1.6020	1.6020	1.6020
17	0.29237	1.6061	1.6061	1.6061
18	0.30902	1.6105	1.6105	1.6105
19	0.32557	1.6151	1.6151	1.6151
20	0.34202	1.6200	1.6200	1.6200
21	0.35837	1.6252	1.6252	1.6252
22	0.37461	1.6307	1.6307	1.6307
23	0.39073	1.6365	1.6365	1.6365
24	0.40674	1.6426	1.6426	1.6426
25	0.42262	1.6490	1.6490	1.6490
26	0.43837	1.6557	1.6557	1.6557
27	0.45399	1.6627	1.6627	1.6627
28	0.46947	1.6701	1.6701	1.6701
29	0.48481	1.6777	1.6777	1.6777
30	0.50000	1.6858	1.6857	1.6858
31	0.51504	1.6941	1.6941	1.6941
32	0.52992	1.7028	1.7028	1.7028
33	0.54464	1.7119	1.7119	1.7119
34	0.55919	1.7214	1.7214	1.7214
35	0.57358	1.7312	1.7312	1.7312
36	0.58779	1.7415	1.7415	1.7415
37	0.60182	1.7522	1.7522	1.7522
38	0.61566	1.7633	1.7632	1.7633
39	0.62932	1.7748	1.7748	1.7748
40	0.64279	1.7868	1.7867	1.7868
41	0.65606	1.7992	1.7992	1.7992
42	0.66913	1.8122	1.8121	1.8122
43	0.68200	1.8256	1.8256	1.8256
44	0.69466	1.8396	1.8395	1.8396
45	0.70711	1.8541	1.8540	1.8541
46	0.71934	1.8691	1.8691	1.8691
47	0.73135	1.8848	1.8847	1.8848
48	0.74314	1.9011	1.9009	1.9011
49	0.75471	1.9180	1.9178	1.9180
50	0.76604	1.9356	1.9354	1.9356
51	0.77715	1.9539	1.9536	1.9539
52	0.78801	1.9729	1.9726	1.9729
53	0.79864	1.9927	1.9923	1.9927
54	0.80902	2.0133	2.0128	2.0133
55	0.81915	2.0347	2.0341	2.0347
56	0.82904	2.0571	2.0564	2.0571
57	0.83867	2.0804	2.0795	2.0804
58	0.84805	2.1047	2.1037	2.1047
59	0.85717	2.1300	2.1288	2.1300
60	0.86603	2.1565	2.1551	2.1565
61	0.87462	2.1842	2.1825	2.1842
62	0.88295	2.2132	2.2111	2.2132
63	0.89101	2.2435	2.2410	2.2435
64	0.89879	2.2754	2.2723	2.2754
65	0.90631	2.3088	2.3051	2.3088
66	0.91355	2.3439	2.3394	2.3439
67	0.92050	2.3809	2.3754	2.3809
68	0.92718	2.4198	2.4132	2.4198
69	0.93358	2.4610	2.4530	2.4610
70	0.93969	2.5046	2.4948	2.5045
70.5	0.94264	2.5273	2.5165	2.5273
71	0.94552	2.5507	2.5389	2.5507
71.5	0.94832	2.5749		2.5749
72	0.95106	2.5998		2.5998
72.5	0.95372	2.6256		2.6255
73	0.95630	2.6521		2.6521
73.5	0.95882	2.6796		2.6796
74	0.96126	2.7081		2.7081
74.5	0.96363	2.7375		2.7375
75	0.96593	2.7681		2.7680
75.5	0.96815	2.7998		2.7997
76	0.97030	2.8327		2.8326
76.5	0.97237	2.8669		2.8669
77	0.97437	2.9026		2.9025
77.5	0.97630	2.9397		2.9397
78	0.97815	2.9786		2.9785
78.5	0.97992	3.0192		3.0191
79	0.98163	3.0617		3.0616
79.5	0.98325	3.1064		3.1063
80	0.98481	3.1534		3.1533
80.2	0.98541	3.1729		3.1727
80.4	0.98600	3.1928		3.1927
80.6	0.98657	3.2132		3.2130
80.8	0.98714	3.2340		3.2338
81	0.98769	3.2553		3.2551

Table 1. Values of the functions K (part two)

81.2	0.98823	3.2771		3.2769
81.4	0.98876	3.2995		3.2992
81.6	0.98927	3.3223		3.3221
81.8	0.98978	3.3458		3.3455
82	0.99027	3.3699		3.3696
82.2	0.99075	3.3946		3.3942
82.4	0.99122	3.4199		3.4196
82.6	0.99167	3.4460		3.4456
82.8	0.99211	3.4728		3.4724
83	0.99255	3.5004		3.4999
83.2	0.99297	3.5288		3.5283
83.4	0.99337	3.5581		3.5575
83.6	0.99377	3.5884		3.5877
83.8	0.99415	3.6196		3.6188
84	0.99452	3.6519		3.6510
84.2	0.99488	3.6852		3.6843
84.4	0.99523	3.7198		3.7187
84.6	0.99556	3.7557		3.7545
84.8	0.99588	3.7930		3.7916
85	0.99619	3.8317		3.8302
85.2	0.99649	3.8721		3.8704
85.4	0.99678	3.9142		3.9122
85.6	0.99705	3.9583		3.9560
85.8	0.99731	4.0044		4.0018
86	0.99756	4.0528		4.0498
86.2	0.99780	4.1037		4.1003
86.4	0.99803	4.1574		4.1535
86.6	0.99824	4.2142		4.2097
86.8	0.99844	4.2744		4.2692
87	0.99863	4.3387		4.3325
87.2	0.99881	4.4073		4.4001
87.4	0.99897	4.4811		4.4726
87.6	0.99912	4.5609		4.5507
87.8	0.99926	4.6477		4.6354
88	0.99939	4.7427		4.7277
88.2	0.99951	4.8478		4.8293
88.4	0.99961	4.9654		
88.6	0.99970	5.0988		
88.8	0.99978	5.2527		
89	0.99985	5.4349		
89.1	0.99988	5.5402		
89.2	0.99990	5.6579		
89.3	0.99993	5.7914		
89.4	0.99995	5.9455		
89.5	0.99996	6.1278		
89.6	0.99998	6.3509		
89.7	0.99999	6.6385		
89.8	0.99999	7.0440		
89.9	1.00000	7.7371		
90	1.00000	∞	$-\infty$	$-\infty$

The values strings in the last two columns of table 1 were canceled when each of the two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 1st kind K(k) gives too great relative errors ($|\varepsilon_K| \geq 4$ ‰ – also see chapter 3) for being still accepted in the usual mathematical / technical calculus. The same procedure will be applied in case of the next table (no. 2), for the same reason, concerning the accuracy of the values given by each of the other two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 2nd kind E(k). The accuracy analysis of the two sets of formulas will be performed in the next chapter (no. 3). In chapter 4 some series representations for the exact functions and for both sets of approximation, as well as for their first order derivatives, will be given. For ($K_{0,1}$, $E_{0,1}$) behaviour in the domain's right side see appendix 1.

Table 2. Values of the functions E (part one)

θ (°)	$k = \sin\theta$	E(k)	$E_0(k)$	$E_1(k)$
0	0.00000	1.5708	1.5708	1.5708
1	0.01745	1.5707	1.5707	1.5707
2	0.03490	1.5703	1.5703	1.5703
3	0.05234	1.5697	1.5697	1.5697
4	0.06976	1.5689	1.5689	1.5689
5	0.08716	1.5678	1.5678	1.5678
6	0.10453	1.5665	1.5665	1.5665
7	0.12187	1.5649	1.5649	1.5649
8	0.13917	1.5632	1.5632	1.5632
9	0.15643	1.5611	1.5611	1.5611
10	0.17365	1.5589	1.5589	1.5589
11	0.19081	1.5564	1.5564	1.5564
12	0.20791	1.5537	1.5537	1.5537
13	0.22495	1.5507	1.5507	1.5507
14	0.24192	1.5476	1.5476	1.5476
15	0.25882	1.5442	1.5442	1.5442
16	0.27564	1.5405	1.5405	1.5405
17	0.29237	1.5367	1.5367	1.5367
18	0.30902	1.5326	1.5326	1.5326
19	0.32557	1.5283	1.5283	1.5283
20	0.34202	1.5238	1.5238	1.5238
21	0.35837	1.5191	1.5191	1.5191
22	0.37461	1.5141	1.5141	1.5141
23	0.39073	1.5090	1.5090	1.5090
24	0.40674	1.5037	1.5037	1.5037
25	0.42262	1.4981	1.4981	1.4981
26	0.43837	1.4924	1.4924	1.4924
27	0.45399	1.4864	1.4864	1.4864
28	0.46947	1.4803	1.4803	1.4803
29	0.48481	1.4740	1.4740	1.4740
30	0.50000	1.4675	1.4675	1.4675
31	0.51504	1.4608	1.4608	1.4608
32	0.52992	1.4539	1.4539	1.4539
33	0.54464	1.4469	1.4469	1.4469
34	0.55919	1.4397	1.4397	1.4397
35	0.57358	1.4323	1.4323	1.4323
36	0.58779	1.4248	1.4248	1.4248
37	0.60182	1.4171	1.4171	1.4171
38	0.61566	1.4092	1.4093	1.4092
39	0.62932	1.4013	1.4013	1.4013
40	0.64279	1.3931	1.3932	1.3931
41	0.65606	1.3849	1.3849	1.3849

Table 2. Values of the functions E (part two)

42	0.66913	1.3765	1.3765	1.3765
43	0.68200	1.3680	1.3680	1.3680
44	0.69466	1.3594	1.3594	1.3594
45	0.70711	1.3506	1.3507	1.3506
46	0.71934	1.3418	1.3419	1.3418
47	0.73135	1.3329	1.3330	1.3329
48	0.74314	1.3238	1.3239	1.3238
49	0.75471	1.3147	1.3148	1.3147
50	0.76604	1.3055	1.3057	1.3055
51	0.77715	1.2963	1.2964	1.2963
52	0.78801	1.2870	1.2872	1.2870
53	0.79864	1.2776	1.2778	1.2776
54	0.80902	1.2681	1.2684	1.2681
55	0.81915	1.2587	1.2590	1.2587
56	0.82904	1.2492	1.2496	1.2492
57	0.83867	1.2397	1.2401	1.2397
58	0.84805	1.2301	1.2307	1.2301
59	0.85717	1.2206	1.2212	1.2206
60	0.86603	1.2111	1.2118	1.2111
61	0.87462	1.2015	1.2024	1.2015
62	0.88295	1.1920	1.1930	1.1920
63	0.89101	1.1826	1.1838	1.1826
64	0.89879	1.1732	1.1745	1.1732
65	0.90631	1.1638	1.1654	1.1638
66	0.91355	1.1545	1.1564	1.1545
67	0.92050	1.1453	1.1475	1.1453
68	0.92718	1.1362	1.1387	1.1362
69	0.93358	1.1272	1.1301	1.1273
70	0.93969	1.1184	1.1217	1.1184
70.5	0.94264	1.1140	1.1176	1.1140
71	0.94552	1.1096	1.1135	1.1096
71.5	0.94832	1.1053		1.1053
72	0.95106	1.1011		1.1011
72.5	0.95372	1.0968		1.0968
73	0.95630	1.0927		1.0927
73.5	0.95882	1.0885		1.0885
74	0.96126	1.0844		1.0844
74.5	0.96363	1.0804		1.0804
75	0.96593	1.0764		1.0764
75.5	0.96815	1.0725		1.0725
76	0.97030	1.0686		1.0686
76.5	0.97237	1.0648		1.0648
77	0.97437	1.0611		1.0611
77.5	0.97630	1.0574		1.0574
78	0.97815	1.0538		1.0538
78.5	0.97992	1.0502		1.0503
79	0.98163	1.0468		1.0468
79.5	0.98325	1.0434		1.0435
80	0.98481	1.0401		1.0402
80.2	0.98541	1.0388		1.0389
80.4	0.98600	1.0375		1.0376
80.6	0.98657	1.0363		1.0364
80.8	0.98714	1.0350		1.0351
81	0.98769	1.0338		1.0339
81.2	0.98823	1.0326		1.0327
81.4	0.98876	1.0314		1.0315
81.6	0.98927	1.0302		1.0303
81.8	0.98978	1.0290		1.0292
82	0.99027	1.0278		1.0280
82.2	0.99075	1.0267		1.0269
82.4	0.99122	1.0256		1.0258
82.6	0.99167	1.0245		1.0247
82.8	0.99211	1.0234		1.0236
83	0.99255	1.0223		1.0226
83.2	0.99297	1.0213		1.0215
83.4	0.99337	1.0202		1.0205
83.6	0.99377	1.0192	false min.	1.0196
83.8	0.99415	1.0182		1.0186
84	0.99452	1.0172		1.0176
84.2	0.99488	1.0163		1.0167
84.4	0.99523	1.0153		1.0158
84.6	0.99556	1.0144		1.0150
84.8	0.99588	1.0135		1.0141
85	0.99619	1.0127		1.0133
85.2	0.99649	1.0118		1.0125
85.4	0.99678	1.0110		1.0118
85.6	0.99705	1.0102		1.0110
85.8	0.99731	1.0094		1.0103
86	0.99756	1.0086		1.0097
86.2	0.99780	1.0079		1.0091
86.4	0.99803	1.0072		1.0085
86.6	0.99824	1.0065		1.0080
86.8	0.99844	1.0059		1.0075
87	0.99863	1.0053		1.0071
87.2	0.99881	1.0047		1.0067
87.4	0.99897	1.0041		1.0064
87.6	0.99912	1.0036		1.0062
87.8	0.99926	1.0031	false min.	1.0060
88	0.99939	1.0026	for $E_1(k)$	1.0060
88.2	0.99951	1.0021		1.0061
88.4	0.99961	1.0017		
88.6	0.99970	1.0014		
88.8	0.99978	1.0010		
89	0.99985	1.0008		
89.1	0.99988	1.0006		
89.2	0.99990	1.0005		
89.3	0.99993	1.0004		
89.4	0.99995	1.0003		
89.5	0.99996	1.0002		
89.6	0.99998	1.0001		
89.7	0.99999	1.0001		
89.8	0.99999	1.0000		
89.9	1.00000	1.0000		
90	1.00000	1.0000	1.1781	1.1781

At $\theta = \cos^{-1}(1/9) = 83.62063°$, $E_0(k) = \pi/3 = 1.0472$ – false min.

In the comparative tables 1 and 2, the 4D (four decimal digit) exact values of both Legendre complete elliptic integrals reproduced from special functions tables [6] (tab. 29, p. 117), as well as their 4D approximate values obtained by applying the two sets of closed analytic formulas were given (all wrt the respective elliptic integrals' modulus $k = \sin\theta$). It is to be noticed that both sets of approximate formulas are not given by spline or regression functions, but by asymptotic expressions, these ones having a remarkable simplicity (see, e.g.: the 2nd form of $E_0(k)$, suggested by Peano's law on ellipse's perimeter; *all newly found formulas in k / k′ do not need any mathematical table*, being purely algebraic) and accuracy (see table 3). The identity with the exact functions is satisfied for the domain's left end $k = 0$ ($\theta = 0°$). The 2nd set (K_1, E_1), although a bit more intricate, gives more accurate values than the 1st one (K_0, E_0) and arrives more closely to the domain's right end $k = 1$ ($\theta = 90°$).

3 The accuracy of the two sets of formulas

Let us define the following relative error functions:

$\varepsilon_{K0}(k) = K_0(k)/K(k) - 1$; $\varepsilon_{K1}(k) = K_1(k)/K(k) - 1$,
$\varepsilon_{E0}(k) = E_0(k)/E(k) - 1$; $\varepsilon_{E1}(k) = E_1(k)/E(k) - 1$,

for both sets of approximation of the 1st and 2nd kind integrals, resp. Their values are given in table 3, expressed in thousandths (‰). These errors were calculated for the 1st set (K_0, E_0) only in the field $\theta \in [54°, 71°]$ of the domain, with an increment of 1°, while for the 2nd set (K_1, E_1) only in the field $\theta \in [84°.8, 88°.2]$, with an increment of 0°.2, like in tables 1 & 2.

Table 3. Relative errors ε distribution

θ(°)	$k = \sin\theta$	ε_{K0}(‰)	ε_{K1}(‰)	ε_{E0}(‰)	ε_{E1}(‰)
54	0.80902	− 0.250		+ 0.255	
55	0.81915	− 0.272		+ 0.243	
56	0.82904	− 0.353		+ 0.293	
57	0.83867	− 0.420		+ 0.334	
58	0.84805	− 0.497		+ 0.454	
59	0.85717	− 0.558		+ 0.502	
60	0.86603	− 0.669		+ 0.566	
61	0.87462	− 0.799		+ 0.742	
62	0.88295	− 0.961		+ 0.874	
63	0.89101	− 1.118		+ 0.973	
64	0.89879	− 1.366		+ 1.135	
65	0.90631	− 1.619		+ 1.377	
66	0.91355	− 1.918		+ 1.627	
67	0.92050	− 2.299		+ 1.900	
68	0.92718	− 2.709		+ 2.215	
69	0.93358	− 3.253		+ 2.573	
70	0.93969	− 3.907		+ 2.959	
71	0.94552	− 4.642		+ 3.525	
		-		-	
84.8	0.99588	-	− 0.369	-	+ 0.607
85	0.99619	-	− 0.396	-	+ 0.592
85.2	0.99649	-	− 0.451	-	+ 0.705
85.4	0.99678	-	− 0.500	-	+ 0.748
85.6	0.99705	-	− 0.582	-	+ 0.823
85.8	0.99731	-	− 0.652	-	+ 0.932
86	0.99756	-	− 0.737	-	+ 1.076
86.2	0.99780	-	− 0.832	-	+ 1.160
86.4	0.99803	-	− 0.945	-	+ 1.284
86.6	0.99824	-	− 1.077	-	+ 1.453
86.8	0.99844	-	− 1.214	-	+ 1.571
87	0.99863	-	− 1.421	-	+ 1.743
87.2	0.99881	-	− 1.626	-	+ 1.976
87.4	0.99897	-	− 1.894	-	+ 2.275
87.6	0.99912	-	− 2.234	-	+ 2.553
87.8	0.99926	-	− 2.655	-	+ 2.922
88	0.99939	-	− 3.156	-	+ 3.397
88.2	0.99951	-	− 3.808	-	+ 4.004

The relative errors strings are stopped for values $|\varepsilon| \geq 4$ ‰. One can see that both sets given in chapter 2 have a much lesser relative error for $K(k)$ than the well-known asymptotic expression:
$K(k) \approx \pi/2 + (\pi/8)[k^2/(1 - k^2)] - (\pi/16)[k^4/(1 - k^4)]$,
with a relative precision of $3\cdot 10^{-4}$ for $k < 0.5$ ($\theta < 30°$), only.

4 Series representations (functions and their derivatives); Legendre's functional relation

Expanding into power series, one obtains for the complete elliptic integrals the set of representations below ([5] – [7]):

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409225}{1073741824}k^{16} + \ldots\right) =$$

$$\frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 k^{2n}\right\} = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}\right\};$$

$$E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760615}{1073741824}k^{16} - \ldots\right) =$$

$$\frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 \frac{k^{2n}}{2n-1}\right\} = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}\right\}.$$

At $k = 0$: $K(0) = E(0) = \pi/2$; at $k = 1$: $K(1)\uparrow\infty$; $E(1) = 1$. Proceeding in the same manner, we get for the 1st set (the most inaccurate) of approximate functions the expansions

$$K_0(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1222}{16384}k^8 + \ldots\right);$$

$$E_0(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{172}{16384}k^8 - \ldots\right),$$

for the 2nd set being *practically identical with the exact ones*

$$K_1(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409222}{1073741824}k^{16} + \ldots\right);$$

$$E_1(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760606}{1073741824}k^{16} - \ldots\right).$$

The difference with respect to the expansions of the exact functions (K, E) begins at the terms in k^8 for the 1st set of approximation (K_0, E_0), and at the terms in k^{16} for the 2nd one (K_1, E_1). For the 1st derivatives of K, E we get

$$\frac{d\mathrm{K}(k)}{dk}=\frac{\mathrm{E}(k)}{k\left(1-k^2\right)}-\frac{\mathrm{K}(k)}{k}=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6\right.$$
$$\left.+\frac{19845}{16384}k^8+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409225}{33554432}k^{14}+\right)$$
$$=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot3\cdot\ldots(2n-1)}{2\cdot4\cdot\ldots\cdot2n}\right]^2 n\,k^{2n-1}=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2 n\,k^{2n-1};$$
$$\frac{d\mathrm{E}(k)}{dk}=\frac{\mathrm{E}(k)-\mathrm{K}(k)}{k}=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\right.$$
$$\left.\frac{2205}{16384}k^8+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{2760615}{33554432}k^{14}+\ldots\right)=$$
$$-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot3\cdot\ldots(2n-1)}{2\cdot4\cdot\ldots\cdot2n}\right]^2\frac{n\,k^{2n-1}}{2n-1}=-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2\frac{n\,k^{2n-1}}{2n-1}.$$

At $k=0$: $d\mathrm{K}/dk=d\mathrm{E}/dk=0$; at $k=1$: $d\mathrm{K}/dk\uparrow\infty$; $d\mathrm{E}/dk\downarrow(-\infty)$. Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions:

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_0=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225.75}{1024}k^6+\ldots\right);$$
$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_0=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{174.25}{1024}k^6+\ldots\right),$$

for the 1st set of approximate functions (K_0, E_0), and resp.

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_1=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6+\frac{19845}{16384}k^8\right.$$
$$\left.+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409226125}{33554432}k^{14}+\ldots\right);$$
$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_1=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\frac{2205}{16384}k^8\right.$$
$$\left.+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{276061425}{33554432}k^{14}+\ldots\right),$$

for the 2nd set of approximate functions (K_1, E_1). The difference with respect to the expansions of the 1st derivatives of the exact functions (K, E) begins at the terms in k^7 for the 1st set, and at the terms in k^{15} for the 2nd one, so much lesser than that for the expansions of the respective sets ($\mathrm{K}_{0,1}$, $\mathrm{E}_{0,1}$). One can also easily find the analytic expressions and series representations for the 2nd derivatives of all K, $\mathrm{K}_{0,1}$, E, $\mathrm{E}_{0,1}$, with similar results, but a lesser precision than for K, E, K′, E′. Besides the above definitions of the derivatives K′(= $d\mathrm{K}/dk$), E′(= $d\mathrm{E}/dk$), there is a useful functional relation (Legendre's): $\mathrm{K}(k)\cdot\mathrm{E}(k')+\mathrm{E}(k)\cdot\mathrm{K}(k')-\mathrm{K}(k)\cdot\mathrm{K}(k')=\pi/2$.

5 Graphic comparison

The variation curves of Legendre complete elliptic integrals, as well as that of the two sets of closed analytic functions are graphically represented in the comparative figures 1 and 2, all wrt θ, in sexagesimal degrees, and given by $\theta=\sin^{-1}k$. In both figures the exact functions K(k), E(k) were represented by solid (continuous) black lines, the 1st set of approximation [$\mathrm{K}_0(k)$, $\mathrm{E}_0(k)$] by dashed black lines, and the 2nd set of approximation [$\mathrm{K}_1(k)$, $\mathrm{E}_1(k)$] by solid red lines. At $k=1$ the graphs of all $\mathrm{K}_{0,1}(k)$ fall to $(-\infty)$; the graphs of all $\mathrm{E}_{0,1}(k)$ pass through $(1, 3\pi/8)$.

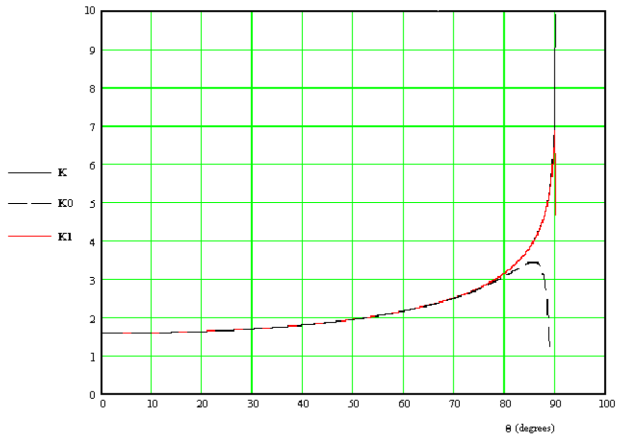

Fig. 1. Comparison of K(k) with the closed analytic functions $\mathrm{K}_0(k)$, $\mathrm{K}_1(k)$; also see the 2nd part of remark 1 in appendix 1

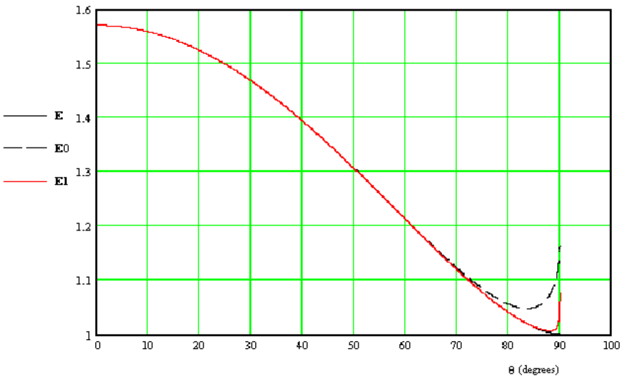

Fig. 2. Comparison of E(k) with the closed analytic functions $\mathrm{E}_0(k)$, $\mathrm{E}_1(k)$; also see the 2nd part of remark 1 in appendix 1

6 Conclusions

As for simplicity, the formulas in $k\,/\,k'$ do not need mathematical tables (are purely algebraic). As for accuracy, in mathematical/technical applications, it must use the 1st set until $\theta=70°.5$ ($k=0.94264$) only, and (for a better accuracy or a greater upper limit of the validity domain) the 2nd set, until $\theta=88°.2$ ($k=0.99951$).

7 Notes; other methods; future research

Without the tables 1 and 2, this work was published previously in a proceedings volume (scientific bulletin), in Romanian [8]. For its first English version see [9], [10]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [11], are developed in [12], [13] (a mixed numerical-analytic method). For newer formulas (using Γ function – not an elementary, but a special one, like K and E, even if these formulas are the most accurate) see [14]. We cite from [14]: "$K[r]$ could be expressed in terms of products of Γ functions, algebraic numbers and powers of π." To find the values of Γ function we need special functions tables. As stated in their abstracts, the works [9], [14] do not have the same goal. To write K and E in terms of Legendre polynomials see [15]. An *original fast converging recurrent-iterative scheme* to get a 3rd (and higher) set of closed analytic formulas (seemingly intricate) with desired accuracy is given in article's appendix 1. For how to get the first two sets (0; 1) see appendix' remark 1. This part of the work is a fully extended version of the article [9].

References:

[1] Legendre, A. M., *Tables of the complete and incomplete elliptic integrals*. Reissued (from tome II of Legendre's *Traité des fonctions elliptiques*, Paris, 1825) by K. Pearson, London, 1934.
[2] Heuman, C. A., *Tables of complete elliptic integrals*, ***J. Math. Physics***, **20**, pp. 127 – 206, 336, 1941; https://onlinelibrary.wiley.com/doi/epdf/10.1002/sapm1941201127.
[3] Hayashi, K., *Tafeln der Besselschen, Theta-, Kugel- und anderen Funktionen*, Berlin, 1930; Table errata no. 518 (pp. 670 – 672) by: O. Skovgaard and M. Helmer Petersen; (A. Fletcher, J. C. P. Miller, L. Rosenhead & L. J. Comrie), ***Math. Comp.***, Vol. 29, No. 130 (Apr. 1975).
[4] Hayashi, K., *Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, elliptischen und anderen Funktionen*, Berlin, 1933; Table errata no. 517 (p. 670) by: O. Skovgaard and M. Helmer Petersen; (A. Fletcher), ***Math. Comp.***, Vol. 29, No. 130 (Apr. 1975).
[5] Jahnke, E., Emde, F., *Tables of Functions with Formulae and Curves*, Dover Publications, New York, 1943; Fourth Edition, 1945; (translated into Russian: Е. Янке и Ф. Эмде, *Таблицы функии с формулами и кривыми*, Физматгиз, Москва – Ленинград, 1959;)
[6] Jahnke, E., Emde, F., Lösch, F., *Tafeln höherer Funktionen*, sechste Auflage. Neubearbeitet von F. Lösch, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960, 1961; https://doi.org/10.1002/zamm.19610410619; (translated into Russian: Е. Янке, Ф. Эмде, Ф. Лёш, *Специальные функции – формулы, графики, таблицы*, ред.: Л. И. Седов, Наука, Москва, 1964; https://ikfia.ysn.ru/wp-content/uploads/2018/01/JankeEmdeLyosh1964ru.pdf).
[7] Gradshteyn, I. S., Ryzhik, I. M., *Table of Integrals, Series, and Products*. Fourth Edition Prepared by Yu. V. Geronimus / M. Yu. Tseytlin, Academic Press, New York, London, 1965; Translated from Russian by Scripta Technica, Inc.; Seventh Edition, 2007; Eds.: A. Jeffrey, D. Zwillinger; (Russian, German, Polish, English, Japanese & Chinese eds.;) http://fisica.ciens.ucv.ve/~svincenz/TISPISGIMR.pdf.
[8] Selescu, R., *Formule analitice închise pentru aproximarea integralelor eliptice complete de speţa întâia şi a doua ale lui Legendre*, Buletinul Ştiinţific al Sesiunii Naţionale de Comunicări Ştiinţifice, Academia Forţelor Aeriene "Henri Coandă" & Centrul Regional pentru Managementul Resurselor de Apărare, Editura Academiei Forţelor Aeriene "Henri Coandă", Braşov, 1 – 2 Noiembrie 2002; Vol. MATEMATICA – INFORMATICA, Anul III, Nr. 2 (14), (ISSN 1453-0139), pp. 37 – 44; (in Romanian).
[9] Selescu, R., *Closed Analytic Formulas for the Approximation of the Legendre Complete Elliptic Integrals of the First and Second Kinds*, ***International Journal of Pure Mathematics – NAUN***, Vol. 8, pp. 23 – 28, DOI: 10.46300/91019.2021.8.2, 29 April 2021; https://www.naun.org/cms.action?id=23293.
[10] Selescu, R., *Closed Analytic Formulas for the Approximation of the Legendre Complete Elliptic Integrals of the First and Second Kinds*, ***International Journal of Mathematical and Computational Methods***, Vol. 21, pp. 49 – 55, 21 May 2021; http://www.iaras.org/iaras/journals/ijmcm.
[11] Luke, Y. L., *Simple formulas for the evaluation of some higher transcendental functions*, ***J. Math. Physics***, v. 34, pp. 298 – 307, 1956, MR **17**, # 1138.
[12] Luke, Y. L., *Approximations for Elliptic Integrals*, ***Math. Comp.***, Vol. 22, No. 103 (Jul. 1968), pp. 627 – 634, MR **17**, # 2412; AMS; https://ams.org/journals/mcom/1968-22-103/S0225-5718-1968-0226825-3/S0025-5718-1968-0226825-3.pdf.
[13] Luke, Y. L., *Further Approximation for Elliptic Integrals*, ***Math. Comp.***, Vol. 24, No. 109 (Jan. 1970), pp. 191 – 198, AMS; https://ams.org/journals/mcom/1970-24-109/S0025-5719-1970-0258243-5/S0025-5719-1970-0258243-5.pdf.
[14] Bagis, N., *Formulas for the approximation of the complete Elliptic Integrals*, https://arxiv.org/abs/1104.4798v1 **[math.GM]**, 6 pages (pp. 1 – 6), 25 April 2011, Cornell University, preprint.
[15] González, M. O., *Elliptic integrals in terms of Legendre polynomials*, Proc. Glasgow Math. Assoc. 2, pp. 97 – 99, 1954, https://www.cambridge.org/core/terms. https://doi.org/10.1017/S2040618500033104.

Appendix 1 – A fast converging recurrent-iterative scheme to get a third (and higher) set of analytic formulas with desired accuracy

The formulas for transforming the modulus ([16], [17]) are:

$$\mathrm{K}(k)=\frac{2}{1+\sqrt{1-k^2}}\mathrm{K}\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)=\frac{2}{1+k'}\mathrm{K}\left(\frac{1-k'}{1+k'}\right),$$

$$\text{or}: \mathrm{K}(\theta)=\mathrm{K}[\tan^2(\theta/2)]/\cos^2(\theta/2), \text{ and, respectively}:$$

$$\mathrm{E}(k)=\left(1+\sqrt{1-k^2}\right)\mathrm{E}\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)-\sqrt{1-k^2}\,\mathrm{K}(k)=$$

$$(1+k')\mathrm{E}\left[(1-k')/(1+k')\right]-k'\mathrm{K}(k), \text{ with } k'=(1-k^2)^{1/2},$$

$$\text{or}: \mathrm{E}(\theta)=2\cos^2(\theta/2)\cdot\mathrm{E}[\tan^2(\theta/2)]-\cos\theta\cdot\mathrm{K}(\theta),$$

(passing from k to $k_1=(1-k')/(1+k')\le k$ and from θ to $\theta_1=\sin^{-1}[\tan^2(\theta/2)]\le\theta$; $k_1=k\,(\theta_1=\theta)$, for: $k=0;\ 1\ (\theta=0;\ \pi/2)$), which can be transcribed in recurrent form, as follows:

$$\boxed{\begin{aligned}&\mathrm{K}_2(k)=\frac{2}{1+\sqrt{1-k^2}}\mathrm{K}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)=\frac{2}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right),\\&\text{or:}\quad \mathrm{K}_2(\theta)=\mathrm{K}_1[\tan^2(\theta/2)]/\cos^2(\theta/2),\quad\text{and,}\quad\text{resp.:}\\&\mathrm{E}_2(k)=\left(1+\sqrt{1-k^2}\right)\mathrm{E}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)-\sqrt{1-k^2}\,\mathrm{K}_2(k)=\\&=(1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right)-\frac{2k'}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right),\quad\text{or:}\quad\mathrm{E}_2(\theta)=\\&2\cos^2(\theta/2)\,\mathrm{E}_1[\tan^2(\theta/2)]-[\cos\theta/\cos^2(\theta/2)]\mathrm{K}_1[\tan^2(\theta/2)],\end{aligned}}$$

expressing the 3rd set (K_2, E_2) in terms of the 2nd one (K_1, E_1), so starting a recurrent-iterative scheme ([18], [19]); it allows writing for the (n + 1)th set: $\boxed{\mathrm{K}_n(k)=\frac{2}{1+k'}\mathrm{K}_{n-1}\left(\frac{1-k'}{1+k'}\right)}$, and:

$$\boxed{\mathrm{E}_n(k)=(1+k')\mathrm{E}_{n-1}\left(\frac{1-k'}{1+k'}\right)-\frac{2k'}{1+k'}\mathrm{K}_{n-1}\left(\frac{1-k'}{1+k'}\right)}, \text{ resp.}$$

$K_n(\theta)$, $E_n(\theta)$ are given by recurrences similar to $K_2(\theta)$, $E_2(\theta)$. Starting from the newly found closed analytic formulas, which connect the 3rd set (K_2, E_2) with the 2nd one (K_1, E_1), and applying the new recurrent-iterative scheme, the comparative tables 1 & 2 were remade, inserting the new columns "$K_2(k)$" and "$E_2(k)$" with 4*D* approximate values. Besides the 1st set (K_0, E_0), there is a better version, (K_{01}, E_{01}), based on Peano's optimized law on ellipse's perimeter (see appendix' 1 remark 1), leading to:

$$E_{01}(k)=\frac{\pi\sqrt[4]{1-k^2}}{100}\left(33\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}-16\right)=\frac{\pi}{4}\left[1.32(1+k')-0.64\sqrt{k'}\right].$$

Table 4. Values of the functions K (part one)
(this table completes and replaces table 1)

θ(°)	$k=\sin\theta$	$K(k)$	$K_0(k)$	$K_1(k)$	$K_2(k)$
0	0.00000	1.5708	1.5708	1.5708	1.5708
1	0.01745	1.5709	1.5709	1.5709	1.5709
2	0.03490	1.5713	1.5713	1.5713	1.5713
3	0.05234	1.5719	1.5719	1.5719	1.5719
4	0.06976	1.5727	1.5727	1.5727	1.5727
5	0.08716	1.5738	1.5738	1.5738	1.5738
6	0.10453	1.5751	1.5751	1.5751	1.5751
7	0.12187	1.5767	1.5767	1.5767	1.5767
8	0.13917	1.5785	1.5785	1.5785	1.5785
9	0.15643	1.5805	1.5805	1.5805	1.5805
10	0.17365	1.5828	1.5828	1.5828	1.5828
11	0.19081	1.5854	1.5854	1.5854	1.5854
12	0.20791	1.5882	1.5882	1.5882	1.5882
13	0.22495	1.5913	1.5913	1.5913	1.5913
14	0.24192	1.5946	1.5946	1.5946	1.5946
15	0.25882	1.5981	1.5981	1.5981	1.5981
16	0.27564	1.6020	1.6020	1.6020	1.6020
17	0.29237	1.6061	1.6061	1.6061	1.6061
18	0.30902	1.6105	1.6105	1.6105	1.6105
19	0.32557	1.6151	1.6151	1.6151	1.6151
20	0.34202	1.6200	1.6200	1.6200	1.6200
21	0.35837	1.6252	1.6252	1.6252	1.6252
22	0.37461	1.6307	1.6307	1.6307	1.6307
23	0.39073	1.6365	1.6365	1.6365	1.6365
24	0.40674	1.6426	1.6426	1.6426	1.6426
25	0.42262	1.6490	1.6490	1.6490	1.6490
26	0.43837	1.6557	1.6557	1.6557	1.6557
27	0.45399	1.6627	1.6627	1.6627	1.6627
28	0.46947	1.6701	1.6701	1.6701	1.6701
29	0.48481	1.6777	1.6777	1.6777	1.6777
30	0.50000	1.6858	1.6857	1.6858	1.6858
31	0.51504	1.6941	1.6941	1.6941	1.6941
32	0.52992	1.7028	1.7028	1.7028	1.7028
33	0.54464	1.7119	1.7119	1.7119	1.7119
34	0.55919	1.7214	1.7214	1.7214	1.7214
35	0.57358	1.7312	1.7312	1.7312	1.7312
36	0.58779	1.7415	1.7415	1.7415	1.7415
37	0.60182	1.7522	1.7522	1.7522	1.7522
38	0.61566	1.7633	1.7632	1.7633	1.7633
39	0.62932	1.7748	1.7748	1.7748	1.7748
40	0.64279	1.7868	1.7867	1.7868	1.7868
41	0.65606	1.7992	1.7992	1.7992	1.7992
42	0.66913	1.8122	1.8121	1.8122	1.8122
43	0.68200	1.8256	1.8256	1.8256	1.8256
44	0.69466	1.8396	1.8395	1.8396	1.8396
45	0.70711	1.8541	1.8540	1.8541	1.8541
46	0.71934	1.8691	1.8691	1.8691	1.8691
47	0.73135	1.8848	1.8847	1.8848	1.8848
48	0.74314	1.9011	1.9009	1.9011	1.9011
49	0.75471	1.9180	1.9178	1.9180	1.9180
50	0.76604	1.9356	1.9354	1.9356	1.9356
51	0.77715	1.9539	1.9536	1.9539	1.9539
52	0.78801	1.9729	1.9726	1.9729	1.9729
53	0.79864	1.9927	1.9923	1.9927	1.9927
54	0.80902	2.0133	2.0128	2.0133	2.0133
55	0.81915	2.0347	2.0341	2.0347	2.0347
56	0.82904	2.0571	2.0564	2.0571	2.0571
57	0.83867	2.0804	2.0795	2.0804	2.0804
58	0.84805	2.1047	2.1037	2.1047	2.1047
59	0.85717	2.1300	2.1288	2.1300	2.1300
60	0.86603	2.1565	2.1551	2.1565	2.1565
61	0.87462	2.1842	2.1825	2.1842	2.1842
62	0.88295	2.2132	2.2111	2.2132	2.2132
63	0.89101	2.2435	2.2410	2.2435	2.2435
64	0.89879	2.2754	2.2723	2.2754	2.2754
65	0.90631	2.3088	2.3051	2.3088	2.3088
66	0.91355	2.3439	2.3394	2.3439	2.3439
67	0.92050	2.3809	2.3754	2.3809	2.3809
68	0.92718	2.4198	2.4132	2.4198	2.4198
69	0.93358	2.4610	2.4530	2.4610	2.4610
70	0.93969	2.5046	2.4948	2.5045	2.5046
70.5	0.94264	2.5273	2.5165	2.5273	2.5273
71	0.94552	2.5507	2.5389	2.5507	2.5507
71.5	0.94832	2.5749		2.5749	2.5749
72	0.95106	2.5998		2.5998	2.5998
72.5	0.95372	2.6256		2.6255	2.6256
73	0.95630	2.6521		2.6521	2.6521
73.5	0.95882	2.6796		2.6796	2.6796
74	0.96126	2.7081		2.7081	2.7081
74.5	0.96363	2.7375		2.7375	2.7375
75	0.96593	2.7681		2.7680	2.7681
75.5	0.96815	2.7998		2.7997	2.7998
76	0.97030	2.8327		2.8326	2.8327
76.5	0.97237	2.8669		2.8669	2.8669
77	0.97437	2.9026		2.9025	2.9026
77.5	0.97630	2.9397		2.9397	2.9397
78	0.97815	2.9786		2.9785	2.9786
78.5	0.97992	3.0192		3.0191	3.0192
79	0.98163	3.0617		3.0616	3.0617
79.5	0.98325	3.1064		3.1063	3.1064
80	0.98481	3.1534		3.1533	3.1534
80.2	0.98541	3.1729		3.1727	3.1729
80.4	0.98600	3.1928		3.1927	3.1928
80.6	0.98657	3.2132		3.2130	3.2132
80.8	0.98714	3.2340		3.2338	3.2340
81	0.98769	3.2553		3.2551	3.2553

Table 4. Values of the functions K (part two)

81.2	0.98823	3.2771		3.2769	3.2771
81.4	0.98876	3.2995		3.2992	3.2995
81.6	0.98927	3.3223		3.3221	3.3223
81.8	0.98978	3.3458		3.3455	3.3458
82	0.99027	3.3699		3.3696	3.3699
82.2	0.99075	3.3946		3.3942	3.3946
82.4	0.99122	3.4199		3.4196	3.4199
82.6	0.99167	3.4460		3.4456	3.4460
82.8	0.99211	3.4728		3.4724	3.4728
83	0.99255	3.5004		3.4999	3.5004
83.2	0.99297	3.5288		3.5283	3.5288
83.4	0.99337	3.5581		3.5575	3.5581
83.6	0.99377	3.5884		3.5877	3.5884
83.8	0.99415	3.6196		3.6188	3.6196
84	0.99452	3.6519		3.6510	3.6519
84.2	0.99488	3.6852		3.6843	3.6852
84.4	0.99523	3.7198		3.7187	3.7198
84.6	0.99556	3.7557		3.7545	3.7557
84.8	0.99588	3.7930		3.7916	3.7930
85	0.99619	3.8317		3.8302	3.8317
85.2	0.99649	3.8721		3.8704	3.8721
85.4	0.99678	3.9142		3.9122	3.9142
85.6	0.99705	3.9583		3.9560	3.9583
85.8	0.99731	4.0044		4.0018	4.0044
86	0.99756	4.0528		4.0498	4.0528
86.2	0.99780	4.1037		4.1003	4.1037
86.4	0.99803	4.1574		4.1535	4.1574
86.6	0.99824	4.2142		4.2097	4.2142
86.8	0.99844	4.2744		4.2692	4.2744
87	0.99863	4.3387		4.3325	4.3387
87.2	0.99881	4.4073		4.4001	4.4073
87.4	0.99897	4.4811		4.4726	4.4811
87.6	0.99912	4.5609		4.5507	4.5609
87.8	0.99926	4.6477		4.6354	4.6477
88	0.99939	4.7427		4.7277	4.7427
88.2	0.99951	4.8478		4.8293	4.8478
88.4	0.99961	4.9654			4.9654
88.6	0.99970	5.0988			5.0987
88.8	0.99978	5.2527			5.2527
89	0.99985	5.4349			5.4349
89.1	0.99988	5.5402			5.5402
89.2	0.99990	5.6579			5.6579
89.3	0.99993	5.7914			5.7913
89.4	0.99995	5.9455			5.9454
89.5	0.99996	6.1278			6.1276
89.6	0.99998	6.3509			6.3506
89.7	0.99999	6.6385			6.6380
89.8	0.99999	7.0440			7.0428
89.9	1.00000	7.7371			7.7336
90	1.00000	∞	$-\infty$	$-\infty$	$-\infty$

The values string in the last column is given by:

$$\mathrm{K}_2(k) = \frac{2}{1+\sqrt{1-k^2}}\mathrm{K}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right) = \frac{2}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right),$$

$$\text{with:}\quad \mathrm{K}_1(k_1) = \frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right) = \frac{\pi\sqrt{2}}{\sqrt{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}{\sqrt[4]{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\right),$$

and finally the algebraic formula: $\boxed{\mathrm{K}_2(k) = 2\mathrm{K}_1(k_1)/(1+k')}$.

Table 5. Values of the functions E (part one)
(this table completes and replaces table 2)

$\theta(°)$	$k=\sin\theta$	$\mathrm{E}(k)$	$\mathrm{E}_0(k)$	$\mathrm{E}_1(k)$	$\mathrm{E}_2(k)$
0	0.00000	1.5708	1.5708	1.5708	1.5708
1	0.01745	1.5707	1.5707	1.5707	1.5707
2	0.03490	1.5703	1.5703	1.5703	1.5703
3	0.05234	1.5697	1.5697	1.5697	1.5697
4	0.06976	1.5689	1.5689	1.5689	1.5689
5	0.08716	1.5678	1.5678	1.5678	1.5678
6	0.10453	1.5665	1.5665	1.5665	1.5665
7	0.12187	1.5649	1.5649	1.5649	1.5649
8	0.13917	1.5632	1.5632	1.5632	1.5632
9	0.15643	1.5611	1.5611	1.5611	1.5611
10	0.17365	1.5589	1.5589	1.5589	1.5589
11	0.19081	1.5564	1.5564	1.5564	1.5564
12	0.20791	1.5537	1.5537	1.5537	1.5537
13	0.22495	1.5507	1.5507	1.5507	1.5507
14	0.24192	1.5476	1.5476	1.5476	1.5476
15	0.25882	1.5442	1.5442	1.5442	1.5442
16	0.27564	1.5405	1.5405	1.5405	1.5405
17	0.29237	1.5367	1.5367	1.5367	1.5367
18	0.30902	1.5326	1.5326	1.5326	1.5326
19	0.32557	1.5283	1.5283	1.5283	1.5283
20	0.34202	1.5238	1.5238	1.5238	1.5238
21	0.35837	1.5191	1.5191	1.5191	1.5191
22	0.37461	1.5141	1.5141	1.5141	1.5141
23	0.39073	1.5090	1.5090	1.5090	1.5090
24	0.40674	1.5037	1.5037	1.5037	1.5037
25	0.42262	1.4981	1.4981	1.4981	1.4981
26	0.43837	1.4924	1.4924	1.4924	1.4924
27	0.45399	1.4864	1.4864	1.4864	1.4864
28	0.46947	1.4803	1.4803	1.4803	1.4803
29	0.48481	1.4740	1.4740	1.4740	1.4740
30	0.50000	1.4675	1.4675	1.4675	1.4675
31	0.51504	1.4608	1.4608	1.4608	1.4608
32	0.52992	1.4539	1.4539	1.4539	1.4539
33	0.54464	1.4469	1.4469	1.4469	1.4469
34	0.55919	1.4397	1.4397	1.4397	1.4397
35	0.57358	1.4323	1.4323	1.4323	1.4323
36	0.58779	1.4248	1.4248	1.4248	1.4248
37	0.60182	1.4171	1.4171	1.4171	1.4171
38	0.61566	1.4092	1.4093	1.4092	1.4092
39	0.62932	1.4013	1.4013	1.4013	1.4013
40	0.64279	1.3931	1.3932	1.3931	1.3931

Table 5. Values of the functions E (part two)

41	0.65606	1.7992	1.7992	1.7992	1.7992
42	0.66913	1.8122	1.8121	1.8122	1.8122
43	0.68200	1.8256	1.8256	1.8256	1.8256
44	0.69466	1.8396	1.8395	1.8396	1.8396
45	0.70711	1.8541	1.8540	1.8541	1.8541
46	0.71934	1.8691	1.8691	1.8691	1.8691
47	0.73135	1.8848	1.8847	1.8848	1.8848
48	0.74314	1.9011	1.9009	1.9011	1.9011
49	0.75471	1.9180	1.9178	1.9180	1.9180
50	0.76604	1.9356	1.9354	1.9356	1.9356
51	0.77715	1.9539	1.9536	1.9539	1.9539
52	0.78801	1.9729	1.9726	1.9729	1.9729
53	0.79864	1.9927	1.9923	1.9927	1.9927
54	0.80902	2.0133	2.0128	2.0133	2.0133
55	0.81915	2.0347	2.0341	2.0347	2.0347
56	0.82904	2.0571	2.0564	2.0571	2.0571
57	0.83867	2.0804	2.0795	2.0804	2.0804
58	0.84805	2.1047	2.1037	2.1047	2.1047
59	0.85717	2.1300	2.1288	2.1300	2.1300
60	0.86603	2.1565	2.1551	2.1565	2.1565
61	0.87462	2.1842	2.1825	2.1842	2.1842
62	0.88295	2.2132	2.2111	2.2132	2.2132
63	0.89101	2.2435	2.2410	2.2435	2.2435
64	0.89879	2.2754	2.2723	2.2754	2.2754
65	0.90631	2.3088	2.3051	2.3088	2.3088
66	0.91355	2.3439	2.3394	2.3439	2.3439
67	0.92050	2.3809	2.3754	2.3809	2.3809
68	0.92718	2.4198	2.4132	2.4198	2.4198
69	0.93358	2.4610	2.4530	2.4610	2.4610
70	0.93969	2.5046	2.4948	2.5045	2.5046
70.5	0.94264	2.5273	2.5165	2.5273	2.5273
71	0.94552	2.5507	2.5389	2.5507	2.5507
71.5	0.94832	2.5749		2.5749	2.5749
72	0.95106	2.5998		2.5998	2.5998
72.5	0.95372	2.6256		2.6255	2.6256
73	0.95630	2.6521		2.6521	2.6521
73.5	0.95882	2.6796		2.6796	2.6796
74	0.96126	2.7081		2.7081	2.7081
74.5	0.96363	2.7375		2.7375	2.7375
75	0.96593	2.7681		2.7680	2.7681
75.5	0.96815	2.7998		2.7997	2.7998
76	0.97030	2.8327		2.8326	2.8327
76.5	0.97237	2.8669		2.8669	2.8669
77	0.97437	2.9026		2.9025	2.9026
77.5	0.97630	2.9397		2.9397	2.9397
78	0.97815	2.9786		2.9785	2.9786
78.5	0.97992	3.0192		3.0191	3.0192
79	0.98163	3.0617		3.0616	3.0617
79.5	0.98325	3.1064		3.1063	3.1064
80	0.98481	3.1534		3.1533	3.1534
80.2	0.98541	3.1729		3.1727	3.1729
80.4	0.98600	3.1928		3.1927	3.1928
80.6	0.98657	3.2132		3.2130	3.2132
80.8	0.98714	3.2340		3.2338	3.2340

81	0.98769	1.0338		1.0339	1.0338
81.2	0.98823	1.0326		1.0327	1.0326
81.4	0.98876	1.0314		1.0315	1.0314
81.6	0.98927	1.0302		1.0303	1.0302
81.8	0.98978	1.0290		1.0292	1.0290
82	0.99027	1.0278		1.0280	1.0278
82.2	0.99075	1.0267		1.0269	1.0267
82.4	0.99122	1.0256		1.0258	1.0256
82.6	0.99167	1.0245		1.0247	1.0245
82.8	0.99211	1.0234		1.0236	1.0234
83	0.99255	1.0223		1.0226	1.0223
83.2	0.99297	1.0213		1.0215	1.0213
83.4	0.99337	1.0202		1.0205	1.0202
83.6	0.99377	1.0192	false min.	1.0196	1.0192
83.8	0.99415	1.0182		1.0186	1.0182
84	0.99452	1.0172		1.0176	1.0172
84.2	0.99488	1.0163		1.0167	1.0163
84.4	0.99523	1.0153		1.0158	1.0153
84.6	0.99556	1.0144		1.0150	1.0144
84.8	0.99588	1.0135		1.0141	1.0135
85	0.99619	1.0127		1.0133	1.0127
85.2	0.99649	1.0118		1.0125	1.0118
85.4	0.99678	1.0110		1.0118	1.0110
85.6	0.99705	1.0102		1.0110	1.0102
85.8	0.99731	1.0094		1.0103	1.0094
86	0.99756	1.0086		1.0097	1.0086
86.2	0.99780	1.0079		1.0091	1.0079
86.4	0.99803	1.0072		1.0085	1.0072
86.6	0.99824	1.0065		1.0080	1.0065
86.8	0.99844	1.0059		1.0075	1.0059
87	0.99863	1.0053		1.0071	1.0053
87.2	0.99881	1.0047		1.0067	1.0047
87.4	0.99897	1.0041		1.0064	1.0041
87.6	0.99912	1.0036		1.0062	1.0036
87.8	0.99926	1.0031	false min.	1.0060	1.0031
88	0.99939	1.0026	for $E_1(k)$	1.0060	1.0026
88.2	0.99951	1.0021		1.0061	1.0021
88.4	0.99961	1.0017			1.0017
88.6	0.99970	1.0014			1.0014
88.8	0.99978	1.0010			1.0011
89	0.99985	1.0008			1.0008
89.1	0.99988	1.0006			1.0006
89.2	0.99990	1.0005			1.0005
89.3	0.99993	1.0004			1.0004
89.4	0.99995	1.0003			1.0003
89.5	0.99996	1.0002			1.0003
89.6	0.99998	1.0001	for $\theta_m \in (89.6, 89.7)°$		1.0002
89.7	0.99999	1.0001	$E_2(k)$ has a false min.		1.0002
89.8	0.99999	1.0000			1.0003
89.9	1.00000	1.0000			1.0007
90	1.00000	1.0000	1.1781	1.1781	1.1781

The values string in the last column is given by:

$$E_2(k) = (1+\sqrt{1-k^2})E_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right) - \sqrt{1-k^2}\,K_2(k) =$$

$$= (1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right) - k'\mathrm{K}_2(k) =$$
$$= (1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right) - \frac{2k'}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right), \text{with:}$$
$$\frac{1-k'}{1+k'} = k_1 \text{(descending Landen transformation), getting}$$
$$\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2}\sqrt{1+k_1'}\,\sqrt[4]{k_1'}\right] - k_1'\cdot \mathrm{K}_1(k_1),$$
$$\mathrm{K}_1(k_1) = \frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right) \text{ – given before table 5;}$$
$$\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2}\sqrt{(1+k_1')\sqrt{k_1'}}\right] -$$
$$- \frac{\pi k_1'\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right) =$$
$$= \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2(1+k_1')\sqrt{k_1'}} - \right.$$
$$\left. - \frac{k_1'\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(4-\frac{\sqrt[4]{2}\left(1+\sqrt{k_1'}\right)}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right)\right].$$

Expressing $k_1'(k')$: $k_1' = (1-k_1^2)^{1/2} = 2(k')^{1/2}/(1+k')$, (ascending Landen transformation), and replacing it:

$$\boxed{\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}\right)^2 - \sqrt{2\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}} - \frac{\sqrt{2}\cdot\frac{2\sqrt{k'}}{1+k'}}{\sqrt{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\left(4-\frac{\sqrt[4]{2}\left(1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}\right)}{\sqrt[4]{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\right)\right],}$$

and then: $\boxed{\mathrm{E}_2(k) = (1+k')\mathrm{E}_1(k_1) - k'\mathrm{K}_2(k)}$, with $\mathrm{K}_2(k)$ given just before table 5, getting the most accurate (seemingly intricate) formula, leading to a *new accurate expression for ellipse's perimeter* ($k \neq 1$; $k' \neq 0$). Similarly, the functions $\mathrm{K}_{0-2}(k)$ ($k, k' = 2^{-1/2}$) approximate the arc length of the entire Bernoulli's lemniscate. Concluding, the 3rd set of formulas is given by the recurrences:

$$\boxed{\mathrm{K}_2(k) = 2\mathrm{K}_1(k_1)/(1+k');\ \mathrm{E}_2(k) = (1+k')\mathrm{E}_1(k_1) - k'\mathrm{K}_2(k).}$$

Noting: $k_1' = x$ and $[(1+x)\cdot x^{1/2}]^{1/2} = y$, one can write:

$$\boxed{\begin{aligned}&\mathrm{K}_2(k) = \pi(2/k')^{1/2}\cdot(x/y)[1-(2^{1/4}/4)(1+x^{1/2})/y^{1/2}];\\ &\mathrm{E}_2(k) = \pi(k')^{1/2}/(2x)\cdot\{(3/2)(1+x^{1/2})^2 - 2^{1/2}y -\\ &\qquad - 2^{1/2}(x/y)[4-2^{1/4}(1+x^{1/2})/y^{1/2}]\} - k'\mathrm{K}_2(k),\end{aligned}}$$

much simpler than previous ones (for calculation only). The validity of all approximate sets is limited to $k \in [0, k_{extr})$; $k_{extr} \leq 1$, "extr" ≡ extremum (max. for K, and min. for E; $k_{max} \neq k_{min}$) (see figs. 1 & 2 – the dashed black lines, and the solid red ones, resp.). The higher the "n" index is, the better the approximation is (the contact order at $k = 0$ is higher – hyperosculation, and the extrema are located closer to the range's right end, $k = 1$). We will cancel the recurrent-iterative scheme (stopping it to a specific "n" index value) when the maximum relative error (over the whole valid domain of variation $k \in [0, k_{extr})$) becomes lesser than the desired (required) accuracy. The first important application of the results obtained in chapter 4 consists in determining the locations of the extrema values k_{extr} (k_{max} for $\mathrm{K}_{n-1}(k)$ and k_{min} for $\mathrm{E}_{n-1}(k)$), corresponding to the annulment of their first derivatives with respect to k, using the relations:

$$\mathrm{K}_{n-1}'(k) = d\mathrm{K}_{n-1}(k)/dk = 0;\ \mathrm{E}_{n-1}'(k) = d\mathrm{E}_{n-1}(k)/dk = 0,$$

and adding the recurrent definitions for $\mathrm{K}_{n-1}(k)$ and $\mathrm{E}_{n-1}(k)$. The 1st ODE above gives the value k_{max} and the 2nd one gives the value k_{min}. Each of these ODEs has really two solutions. Besides the searched for one, both ODEs admit the solution $k = 0$, corresponding to a minimum for $\mathrm{K}_{n-1}(k)$ and to a maximum for $\mathrm{E}_{n-1}(k)$, both with the value $\pi/2$ (for both approximate and exact functions: $\mathrm{K}_{n-1}(0) = \mathrm{E}_{n-1}(0) = \mathrm{K}(0) = \mathrm{E}(0) = \pi/2$, with: $\mathrm{K}_{n-1}'(0) = \mathrm{E}_{n-1}'(0) = \mathrm{K}'(0) = \mathrm{E}'(0) = 0$), but with: $\mathrm{K}_{n-1}''(0) > 0$ and $\mathrm{K}''(0) > 0$ – a minimum, while: $\mathrm{E}_{n-1}''(0) < 0$ and $\mathrm{E}''(0) < 0$ – a maximum).

Thus one knows now the values k_{max} and k_{min} (the right ends of the validity domains of the approximate functions). Using the direct formulas here found for (K_2, E_2), the iterative scheme's steps can be bypassed. In order to evaluate the accuracy of the 3rd set (K_2, E_2), similarly as for the previous two sets, (K_0, E_0) and (K_1, E_1), we define the relative errors: $\varepsilon_{K2}(k) = \mathrm{K}_2(k)/\mathrm{K}(k) - 1$, and: $\varepsilon_{E2}(k) = \mathrm{E}_2(k)/\mathrm{E}(k) - 1$, for the approximate formulas of 1st & 2nd kind integrals. Their values, expressed in thousandths (‰) are given in table 6. These errors were calculated for the 3rd set (K_2, E_2) only, with an increment of 0°.2 in the field $\theta \in$ [84°, 89°], and of 0°.1 beyond 89°. To get table 6, in table 3 were suppressed the columns ε_{K0}(‰), ε_{E0}(‰) (the most inaccurate) and were inserted the columns ε_{K2}(‰), ε_{E2}(‰), keeping for comparison the columns "θ(°)", "$k = \sin\theta$", "ε_{K1}(‰)" and "ε_{E1}(‰)" (from table 3), only.

Table 6. Relative errors ε distribution (part one) (this table completes and replaces table 3; $\theta \geq 84.8°$)

θ(°)	$k = \sin\theta$	ε_{K1}(‰)	ε_{K2}(‰)	ε_{E1}(‰)	ε_{E2}(‰)
84.8	0.99588	– 0.369	0	+ 0.607	0
85	0.99619	– 0.396	0	+ 0.592	0
85.2	0.99649	– 0.451	0	+ 0.705	0
85.4	0.99678	– 0.500	0	+ 0.748	0
85.6	0.99705	– 0.582	0	+ 0.823	0
85.8	0.99731	– 0.652	0	+ 0.932	0
86	0.99756	– 0.737	0	+ 1.076	0
86.2	0.99780	– 0.832	0	+ 1.160	0
86.4	0.99803	– 0.945	0	+ 1.284	0
86.6	0.99824	– 1.077	0	+ 1.453	0
86.8	0.99844	– 1.214	0	+ 1.571	0
87	0.99863	– 1.421	0	+ 1.743	0
87.2	0.99881	– 1.626	0	+ 1.976	0
87.4	0.99897	– 1.894	0	+ 2.275	0
87.6	0.99912	– 2.234	0	+ 2.553	0
87.8	0.99926	– 2.655	0	+ 2.922	0
88	0.99939	– 3.156	0	+ 3.397	0
88.2	0.99951	– 3.808	0	+ 4.004	0

Table 6. Relative errors ε distribution (part two)

88.4	0.99961	-	0	-	0
88.6	0.99970	-	0	-	0
88.8	0.99978	-	0	-	0
89	0.99985	-	0	-	0
89.1	0.99988	-	0	-	0
89.2	0.99990	-	0	-	0
89.3	0.99993	-	0	-	0
89.4	0.99995	-	0	-	0
89.5	0.99996	-	– 0.033	-	+ 0.1
89.6	0.99998	-	– 0.047	-	+ 0.1
89.7	0.99999	-	– 0.075	-	+ 0.1
89.8	0.99999	-	– 0.170	-	+ 0.3
89.9	1.00000	-	– 0.452	-	+ 0.7
90	1.00000	– 2000	– 2000	178.097	178.097

The errors strings are stopped if their modulus is ≥ 4 ‰. From the tables 3 and 6 one can see that, for any n^{th} set of approximation and at any k value, $\varepsilon_K < 0$ ($K_n < K$) and $\varepsilon_E > 0$ ($E_n > E$), i.e. K is approximated by lack, while E – by excess. The "0" $\varepsilon_{K,E}$ values mean "the first 4 decimal digits identical to those in tables [6]". One can also build the 3^{rd} set [$K_2(\theta)$, $E_2(\theta)$], expressed in trigonometric functions, replacing k' in [$K_2(k)$, $E_2(k)$] set by $\cos\theta$ and applying usual trigonometric identities. The comparative series representations and the graphic comparison are superfluous, due to the great accuracy of the approximate values given by the 3^{rd} set (practically identical to the exact ones, which could be already noticed from the analysis of the 2^{nd} set, this showing the fast converging character of this recurrent-iterative scheme). Except for the domain's right end (k = 1), the 3^{rd} set of approximation (K_2, E_2), even more accurate than the 2^{nd} one (K_1, E_1), may be considered and successfully used instead of the exact values of $K(k)$ and $E(k)$ from mathematical tables. A false minimum takes place for all $E_n(k)$: for $E_2(k)$, at $\theta = 89°.7$ (k = 0.99999); for $E_1(k)$, at $\theta = 88°$ (k = 0.99939), and for $E_0(k)$, at $\theta = 83°.62$ (k = 0.99381). The graphs of all $E_n(k)$ pass through the point (1, 3π/8 = 1.178097); for k tending to unity, the graphs of all $K_n(k)$ go toward (– ∞) – singularity; the higher n^{th} sets (n ≥ 4) give better accuracy). Unlike the mathematical tables (and in addition to them), all approximation sets (the 1^{st}, 2^{nd}, 3^{rd} and the higher n^{th} (n ≥ 4) ones) allow performing the *analytic study of variation* of the functions in which $K(k)$ and / or $E(k)$ appear /s, *using the derivatives of the 1^{st} and 2^{nd} order* (with respect to k).

Remarks: 1. As a first step in applying the new recurrent-iterative scheme, even the obtaining of the 2^{nd} set (K_1, E_1) as a function of the 1^{st} one (K_0, E_0) (in ch. 2) may be considered, i.e. *this scheme starts really at the 2^{nd} set.* It is to be highlighted the used method is a *purely analytic* one (neither numerical methods nor sophisticated software, *at most* using MatLab's (software package for engineers) "Symbolic Math" toolbox, for analytically solving the more intricate algebraic equations encountered). Its *simplicity, accuracy* and *fast convergence*, as well as its *limitations* depend *exclusively* on the *correct choice of its starting point* (approximation set) (K_0, E_0). It must be quite precise, and especially, *as simple as possible.* The starting approximate formula-definition giving $E_0(k)$ was suggested to the author by an old approximate formula (Peano, [20], [21]) for the perimeter L of an ellipse of semiaxes a and b (≤ a): $L \approx \pi[1.5(a + b) - (ab)^{1/2}]$ – a good (& *simple*) approx. with the best accuracy for $b = a$ (circle): $L = 2\pi a$, and the worst one for $b = 0$ (Ox' segment): $L = 1.5\pi a$, instead of $L = 4a$ (ε ≈ 17.81 %), or by Peano's optimized law: $L_1 \approx \pi[1.32(a + b) - 0.64(ab)^{1/2}]$, *with the smallest overall error* [22] (about 7 times smaller than that of the original law); for $b = a$: $L_1 = L = 2\pi a$, and for $b = 0$: $L_1 = 1.32\pi a$, much closer to the exact value $L = 4a$ (ε ≈ 3.67 %). For its behaviour at low b/a ratios (is not tangent at k = 0 to the exact curve, cutting it), this formula is not found on the list of the very accurate (but not simple) approximations [22] (Padé, Jacobsen, Ramanujan (2 expressions), Rackauckas), all expressed in terms of a particular ratio: $h = [(a - b)/(a + b)]^2 \equiv k_1^2$. Thus a reliable approximate (by excess) formula-definition was obtained (see chapter 2) for the Legendre complete elliptic integral of the 2^{nd} kind (in the 1^{st} set of approximation): $E_0(k) = (\pi/4)[1.5(1 + k') - (k')^{0.5}]$; $k' = (1 - k^2)^{0.5} = b/a$; or: $E_{01}(k) = (\pi/4)[1.32(1 + k') - 0.64(k')^{0.5}]$ (Peano's optimized). If in the power series we stop at the 'rank 5' term (see ch. 4), the error is $(3/2^{14})k^8 = (3/16384)k^8$, i.e. small enough (asymptotic expansion). As for the pair approximate formula-definition giving $K_0(k)$, this was obtained using the previous one for $E_0(k)$ and applying the definition of the first derivative of $E(k)$ with respect to k: $dE(k)/dk = [E(k) - K(k)]/k$ (see chapter 4), thus getting: $K(k) = E(k) - k[dE(k)/dk]$; replacing $K(k)$ and $E(k)$ by their 1^{st} approximations: $K_0(k)$ and the previously given $E_0(k)$, one gets: $K_0(k) = (\pi/8)[3/2(1 + 1/k') - (k')^{0.5}(1 + 1/(k')^2]$, of a lesser accuracy (esp. for $\theta > \pi/3$) than $E_0(k)$. To improve this, one uses a descending Landen transformation: $K(k) = (1 + k_1)K(k_1)$ with $k_1 = (1 - k')/(1 + k') \le k$, and replacing in $K(k)$, one gets: $K_0(k) = \pi[1/(k')^{0.5} - (1/2^{1.5})(1 + k')^{0.5}/(k')^{0.75}] \ge K_0(k)$ (see ch. 2), of an accuracy (in modulus) much closer to that of its pair $E_0(k)$. Being practically generated by the same mathematical source, $K_0(k)$ and $E_0(k)$ vary (ordinates, slopes, asymptote, extrema, concavities, convexities, inflections) in perfectly correlated way. So, at the value k_{extr} corresponding to a false minimum for $E_0(k)$, $K_0(k)$ must equate $E_0(k)$, to satisfy the annulment of $dE_0(k)/dk$. To prepare this, $K_0(k)$ must stop its vertiginous ascension to ∞, making a false inflection, and then a false max. at $k_{Extr} < k_{extr}$, and a vertiginous (k = 1 – vertical asymptote) fall toward (– ∞); so $K_0 = E_0$ at $k = 0$ and $k = k_{extr}$. But, due to its additional step (to have $|\varepsilon_{K0}| \approx |\varepsilon_{E0}|$), K_0 is not generated by the same mathematical source as E_0. To minimise the unwished events, limiting them to a getting thinner region for rising n (already 1/300 of the field [0, π/2] at n = 2) near k = 1, one applies the descending Landen transformation, passing from k to $k_1 \le k$, where all goes well, also keeping all advantages of the asymptotic behaviour of the new functions (K_n, E_n), i.e. applying a higher n^{th} (n ≥ 2) set (repeating this scheme until the desired accuracy for (K_n, E_n) is obtained; fortunately, this scheme is fast converging); though it keeps the limitation at k = 1, Peano's optimized law accelerates the scheme.

2. Besides the formulas for transforming the modulus using the descending Landen transformation, there are formulas using the ascending Landen transformation (not of interest here).

Appendix' 1 conclusions

Some authors (e.g.: Bagis – see [14]) *choose to start from more precise formulas for the perimeter of an ellipse* (similar to Ramanujan's "type π formulas" (1914) – see [23]): $L_{\mathrm{I}} = \pi\{3(a + b) - [(a + 3b)(3a + b)]^{1/2}\} = \pi\{3(a + b) - [10ab + 3(a^2 + b^2)]^{1/2}\}$ – Ramanujan 1st approximation; $L_{\mathrm{II}} = \pi(a+b)\{1+3h/[10+(4-3h)^{1/2}]\}$; $h = [(a-b)/(a+b)]^2 \equiv k_1^2$ – the more famous Ramanujan 2nd approximation; the errors in these empirical relations are of order h^3 and h^5 (both are very accurate, but not as simple as possible), *in order to get approximate formulas as accurate as possible for Legendre's complete elliptic integrals*. In [14], instead of 'π' from Ramanujan's formulas, appears the constant $\Gamma^2(1/4)/\pi^{3/2}$ (the length of the entire Bernoulli's lemniscate is: $L_B = 2^{3/2}a\mathrm{K}(2^{-1/2}) = [\Gamma^2(1/4)/(2\pi)^{1/2}]a$; $a = 2^{1/2}c$ – the half-width; c – focal coord.). We cite from [22]: "What makes Ramanujan's first formula interesting to this Author is the fact that, like the first form of Peano's approximation, it can be interpreted as a combination of the arithmetic mean with another one, denoted as $\mathrm{R}(a, b, w)$ and defined by: $\mathrm{R}(a, b, w) = [(a + wb)(b + wa)]^{1/2}/(1 + w)$. In Ramanujan's formula we have $w = 3$ and the two means are combined linearly with the relative weights + 3 and – 2, resp." Although it seems to conflict with the beginning of its remark 1, this appendix demonstrates that *even choosing as a starting point a "not so precise"* (with big problems at the domain's right end, $k = 1$), *but especially simple formula* (like Peano's one, or better, Peano's optimized one), *and applying the fast converging recurrent-iterative scheme* (including the descending Landen transformation, to solve the unwished behaviour of $\mathrm{E}_n(k)$ appeared near $k = 1$, due to any of Peano's approximate laws – method's major limitation (see the 2nd part of remark 1)), *similar results (from the viewpoint of their accuracy) for Legendre's complete elliptic integrals* [K(k), E(k)] (with very small values of the relative errors (ε_K, ε_E) – practically zero) *can be obtained* already beginning even with the 3rd set of approximation (K_2, E_2) – see tables 4 – 6. As regards the relations describing the recurrence, they are: $\mathrm{K}_n(k) = [2/(1 + k')]\mathrm{K}_{n-1}(k_1)$, and:
$\mathrm{E}_n(k) = (1 + k')\mathrm{E}_{n-1}(k_1) - [2k'/(1 + k')]\mathrm{K}_{n-1}(k_1)$, resp., with $k_1 = (1 - k')/(1 + k') \leq k$, this representing even the source of the descending Landen transformation; they express the values of the $(n + 1)^{\mathrm{th}}$ set in function of those of the n^{th} one. The recurrent-iterative scheme has two advantages over the interpolation, regression and spline methods: 1. does not require the points' coordinates; 2. its accuracy can be improved no matter how much. To bypass it, the here found direct formulas for (K_2, E_2) can be applied; (for practical applications, the 3rd set is accurate enough; it can be used until $\theta = 89°.7$; $k = 0.99999$; see tables 4 – 6). $\mathrm{E}_{1,2}(k)$ *lead also to better expressions for ellipse's perimeter than* $\mathrm{E}_0(k)$ (the original Peano's one).

Appendix' 1 references:

[16] Landen, J., *XXXVI. A Disquisition Concerning Certain Fluents, which are Assignable by the Arcs of the Conic Sections; Wherein are Investigated Some New and Useful Theorems for Computing Such Fluents*, Philosophical Transactions of the Royal Society of London, vol. 61, 1771, pp. 298 – 309; https://doi.org/doi:10.1098/rstl.1771.0037

[17] Landen, J., *XXVI. An Investigation of a General Theorem for Finding the Length of Any Arc of Any Conic Hyperbola, by means of Two Elliptic Arcs, with Some Other New and Useful Theorems Deduced Therefrom*, Philosophical Transactions of the Royal Society of London, vol. 65, 1775, pp. 283 – 289; https://doi.org/doi:10.1098/rstl.1775.0028.

[18] Selescu, R., *Simple Closed Analytic Formulas to approximate the First Two Legendre's Complete Elliptic Integrals by a Fast Converging Recurrent-Iterative Scheme*, ***WSEAS Transactions on Computer Research***, Vol. 9, pp. 55 – 67, 6 July 2021, DOI: 10.37394/232018.2021.9.7; https://wseas.com/journals/cr/2021.php.

[19] Selescu, R., *Formulas to approximate Legendre's Complete Elliptic Integrals using Peano's Law on Ellipse's Perimeter and a Recurrent-Iterative Scheme (Landen's Transform Included)*, ***International Journal of Computational and Applied Mathematics & Computer Science***, Vol. 1, pp. 46 – 58, 2021, ISSN 2769 – 2477; http://www.icamcs.co/papers/2021/icamcs(2021)-007.pdf.

[20] Peano, G., *Applicazioni geometriche del calcolo infinitesimale*, Fratelli Bocca Editori, Torino, 1887; (in Italian), p. 233; the approximate formula for the ellipse perimeter was: $L \approx \pi(a + b) + (\pi/2)(a^{1/2} - b^{1/2})^2$.

[21] Peano, G., *VIII An approximation formula for the perimeter of the ellipse*, 1889, pp. 135 – 136 in *Selected Works of Giuseppe Peano*, Translated and edited, with a biographical sketch and bibliography, by Hubert C. Kennedy; Series: Heritage; Copyright Date: 1973; Published by: University of Toronto Press; Pages: 262; the approximate formula for ellipse perimeter (due to J. Boussinesq in the *Comptes rendus,* Académie des Sciences, Paris, 1889, p. 695) was given in the well-known equivalent form: $L \approx \pi[3(a + b)/2 - (ab)^{1/2}]$; https://www.jstor.org/stable/10.3138/j.ctt1vxmd8x.

[22] Sýkora, St., *Approximations of Ellipse Perimeters and of the Complete Elliptic Integral* E(x). Review of known formulae, Review by Stanislav Sýkora, Extra Byte, Ed. S. Sýkora, Vol.I; First release: December 27, 2005. Permalink via DOI: 10.3247/SL1Math05.004; http://www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html.

[23] Ramanujan, S., *Modular Equations and Approximations to* π, § 16, ***Quart. J. Pure App. Math.***, vol. 45, pp. 350 – 372, 1914, ISBN 9780821820766.

Appendix 2 – Approximating the integral Π(*n*, *k*)

Using the previously given theoretical results (consisting of high-accuracy simple closed purely algebraic functions found by applying our scheme of approximation as desired for the integrals K(*k*) and E(*k*)) and ***the expressions established for the partial derivatives*** of the complete elliptic integral of the 3rd kind, Π(*n*, *k*) (introduced by Legendre, in canonical form: $\Pi(n,k) = \int_0^{\pi/2} d\varphi / [(1 - n\sin^2\varphi)(1 - k^2\sin^2\varphi)^{1/2}]$, being sometimes defined with an inverse sign for *n*: $\Pi'(n,k) = \int_0^{\pi/2} d\varphi / [(1 + n\sin^2\varphi)(1 - k^2\sin^2\varphi)^{1/2}]$), ***with respect to*** *n* (*characteristic*; *n* can take on any value), ***and to*** *k* (*modulus*; $k \in [0, 1]$) with $k^2 = m$ (*parameter*), ***resp.*** – a system of two linear PDEs of the first order (https://en.wikipedia.org/wiki/Elliptic_integral#Partial_derivatives):

$$\begin{cases} \dfrac{\partial\Pi(n,k)}{\partial n} = \dfrac{1}{2(k^2-n)(n-1)}\left[\mathrm{E}(k) + \dfrac{1}{n}(k^2-n)\mathrm{K}(k) + \dfrac{1}{n}(n^2-k^2)\Pi(n,k)\right]; \\ \dfrac{\partial\Pi(n,k)}{\partial k} = \dfrac{k}{n-k^2}\left[\dfrac{1}{k^2-1}\mathrm{E}(k) + \Pi(n,k)\right] \end{cases}$$

(so that: $d\Pi(n, k) = [\partial\Pi(n, k)/\partial n]dn + [\partial\Pi(n, k)/\partial k]dk$) in canonical form, *both expressed through* K(*k*), E(*k*) *and* Π(*n*, *k*), with the unknown function Π(*n*, *k*). (The above definition of Π(*n*, *k*) has two remarkable particular cases: $\Pi(0, k) \equiv \mathrm{K}(k)$, and $\Pi(k^2, k) \equiv \mathrm{E}(k)/(1 - k^2)$ – see [7]. In the 1st PDE K(*k*) and E(*k*) are considered constants. Just like the other two complete elliptic integrals, (K(*k*) and E(*k*)), Π(*n*, *k*) can be computed very efficiently using the arithmetic-geometric mean AGM – see [24] – [27]); for the 2nd equation of the system above also see [24], § 19.4(i) Derivatives, Eq. 19.4.4 (https://dlmf.nist.gov/19). *One can solve (integrate, preferably analytically) this system, expecting to find for* Π(*n*, *k*) *similar closed analytic approximate functions*, of a similar order of precision to the purely algebraic (differentiable and integrable) ones found for K(*k*) and E(*k*). In the first solving step K and E are kept in symbolic form (not expressed wrt *k*), then our approximations are used. The indefinite integral of Π(*n*, *k*) wrt *k* (a "partial" integral) can also be expressed through K(*k*), E(*k*) and Π(*n*, *k*): $\int\Pi(n,k)dk = 2[\mathrm{E}(k) - \mathrm{K}(k) + (k-n)\Pi(n,k)]$ (https://functions.wolfram.com/EllipticIntegrals/EllipticPi/introductions/CompleteEllipticIntegrals/ShowAll.html; here is also the relationship of Π(*n*, *k*) with F(*φ*, *k*) and E(*φ*, *k*) – the incomplete elliptic integrals of the 1st and 2nd kind; see sect. "Connections within the group of complete elliptic integrals and with other function groups", sub-s "Representations through more general functions"). Besides the here given Legendre normal form, the elliptic integrals can also be expressed in Carlson symmetric form. For other computation method and for tables see [28] – [30]. In ([29], [30]) all three elliptic integrals K, E, Π are computed. The effective determining of the approximate formulas for Π(*n*, *k*) (Π_{0-2}) will form the object of a future research work.

Appendix' 2 references

[24] Carlson, B. C. (2010), *Elliptic Integrals*, ch. 19 in Olver, F. W. J., Lozier, D. M., Boisvert, R. F., Clark, C. W. (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, ISBN 978-0-521-19225-5, MR 2723248; version 1.1.7 – released on Oct. 15, 2022.
[25] Carlson, B. C., *A Table of Elliptic Integrals of the Third Kind*, ***Math. Comp.***, Vol. 51, No. 183 (1988), pp. 267 – 280, MR **89k**: 33003.
[26] Gray, N., *Automatic Reduction of Elliptic Integrals Using Carlson's Relation*, ***Math. Comp.***, Vol. 71, No. 237 (Jan. 2002), pp. 311 – 318, AMS; https://ams.org/journals/mcom/2002-71-237/S0025-5718-01-01333-3/S0025-5718-01-01333-3.pdf.
[27] Carlson, B. C., *Three Improvements in Reduction and Computation of Elliptic Integrals*, ***J Res Natl Inst Stand Technol***, 2002, Sep-Oct; 107(5), pp. 413 – 418, Published online 2002 Oct 1; doi: 10.6028/res.107.034.
[28] Fettis, H. E., *Calculation of Elliptic Integrals of the Third Kind by Means of Gauss' Transformation*, ***Math. Comp.***, Vol. 19, No. 89 (1965), pp. 97 – 104.
[29] Fettis, H. E., Caslin, J. C., *Tables of Elliptic Integrals of the First, Second and Third Kind*. Technical report Technical Report ARL 64-232, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, 1964.
[30] Horig, P., *Elliptic Integrals; the Landen Transformation and Carlson Duplication*, PhD thesis, 132 pp., 2014 (https://www.academia.edu/11191001/Elliptic_Integrals_the_Landen_Transformation_and_Carlson_Duplication).

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Final note (work history)

Without appendix' 1 conclusions and no acknowledgement, with a reduced remark 1, the work's 1st version appeared in unitary form (the sets (0; 1) of formulas + the appendix 1) as [18]. The 2nd one (revision) was needed (published as [19]). In this 3rd version the title was limited to ten words and an explanation in remark 1 was corrected (improved). The MSC2020 (Mathematics Subject Classification, namely: elliptic functions and integrals; systems of linear first-order PDEs; initial value problems for systems of linear first-order PDEs; boundary value problems for systems of linear first-order PDEs; initial-boundary value problems for systems of linear first-order PDEs; approximation by polynomials; approximation by rational functions; Padé approximation; asymptotic approximations, asymptotic expansions) was introduced and the appendix 2 and the acknowledgements were added.