Domination, Independence and Fibonacci Numbers in Graphs Containing Disjoint Cycles

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Abstract: - Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics and its results have applications in many areas of the computing, social, and natural sciences. The fastest growing area within graph theory is the study of domination and Independence numbers. Domination number is the cardinality of a minimum dominating set of a graph. Independence number is the maximal cardinality of an independent set of vertices of a graph. The concept of Fibonacci numbers of graphs was first introduced by Prodinger and Tichy in 1982. The Fibonacci numbers of a graph is the number of independent vertex subsets. In this paper, introduce the identities of domination, independence and Fibonacci numbers of graphs containing vertex-disjoint cycles and edge-disjoint cycles.

Key-Words: - Fibonacci numbers, Independent numbers, Domination numbers, disjoint cycles, Graph.

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1 Introduction

Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics and its results have applications in many areas of the computing, social, and natural sciences. The fastest growing area within graph theory is the study of domination, the reason being its many and varied applications in such fields as social sciences, communications networks, algorithmic designs etc. Dominating and independent sets are among the most well-studied graph sets. Domination can be a useful tool for determining business network and making decisions.

The topic of domination was given formal Mathematical definition by C. Berge in 1958 and O. Ore [15] in 1962. Berge called the domination as external stability and domination number of coefficient of external stability. Ore introduced the world domination in his famous book [15]. This concept lived in hibernation until 1975 when a paper [8] published in 1977. This paper brought to light new ideas and potentiality of being applied in variety of areas. The research in domination theory has been broadly classified in [17], [18]. Domination can be a useful tool for determining business network and making decisions. Business would benefit from the use of the concept of domination to strategically plan the location of their stores in order to reach the maximum amount of areas with minimal stores locations.

Independent sets were introduced into the communication theory on noisy channels [9].

In the literature, especially in mathematics and physics, there are a lot of integer sequences, which are used in almost every field of modern sciences. Admittedly, the Fibonacci sequence is one of the most famous and curious numerical sequence in mathematics and have been widely studied from both algebraic and combinatorial prospective. Also, there is the Lucas sequence, which is as important as the Fibonacci sequence. Fibonacci numbers are a sequence of numbers in which each successive number is the sum of two previous numbers: 0,1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

Similarly Lucas numbers are 2,1, 3, 4, 7, 11, 18, 29,….

The Fibonacci sequence and Lucas Sequence [19] are defined by the recurrence relation:

$F_{n+1} = F_n + F_{n-1}$, where $n \geq 1$ with initial conditions $F_0 = 0, F_1 = 1$ [1.1]

$L_{n+1} = L_n + L_{n-1}$, where $n \geq 1$ with initial conditions $L_0 = 2, L_1 = 1$ [1.2]

In this paper, we present results on domination, independence, Fibonacci numbers of graphs containing vertex-disjoint cycles and edge-disjoint cycles.

2 Preliminaries and Notations

Let $G = (V, E)$ be a simple graph (i.e., undirected, without loops and multi edges). The number of vertices namely the cardinality of V is called the order of G and is denoted by $|G|$. The number of edges of a graph namely the cardinality of E is called the size of G and is denoted by $|E|$. We write $e = v_i v_j \in E(G)$ to mean the pair $v_i, v_j \in E(G)$ and if $e = v_i v_j \in E(G)$ we say that v_i and v_j are adjacent and e and v_i, e and v_j are incident.

The open neighbourhood $N(v)$ of the vertex v consists of the set of vertices adjacent to v. That is $N(v) = \{w \in V : vw \in E\}$. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighbourhood N(S) is defined by $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood $N[S]$ by $N[S] = N(S) \cup S$. A vertex $v \in S$ is called “an enclave of S”, if $N[S] \subset S$. A vertex $v \in S$ is called “an isolate of of S”, if $N(S) \subset V - S$.

The degree of a point v is denoted by $\deg(v)$ is defined as the number of edges incident with v. That is $\deg(v) = |N(v)|$. The maximum and minimum of the degree of vertices of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. If $\Delta(G) = \delta(G) = r$, then G is said to be a regular graph of degree r or simply r-regular.

Now explain domination, independence and Fibonacci numbers of graph.

2.1 Domination Number

The concept of domination in graphs is now well studied in graph theory. A subset $S \subseteq V(G)$ is called a dominating set, if every vertices of $V - S$ is adjacent to a member of S. A dominating set of G with minimum cardinality is called a minimum dominating set and the cardinality of a minimum dominating set is called the domination number and denoted by $\gamma(G)$. The upper domination number of a graph G denoted by $\Gamma(G)$ is defined as the maximum cardinality of a minimum dominating set of G.

2.2 Independence Number

A sub set $S \subseteq V(G)$ is an independent set, if no two vertices of S are adjacent. Moreover, the subset containing only one vertex and the empty set also are independent. The number of all independent sets in G is denoted by $NI(G)$. For a graph G on $V(G) = \phi$, we put $NI(G) = 1$. Independence number $\beta(G)$ of graph G is the maximal cardinality of an independent set of vertices.

2.3 Independent Domination Number

A set S of vertices in a graph G is called an independent dominating set of G if S is both an independent and a dominating set of G. This set is also called a Stable set or a Kernel of the graph. Independent dominating sets were introduced into the theory of games by Neumann and Margenstern in 1944 [14]. The independent domination number $i(G)$ is the cardinality of the smallest independent domination set.

2.4 Fibonacci Numbers of Graphs

The concept of Fibonacci numbers of graphs was first introduced by Prodinger and Tichy [13]. The Fibonacci number of a graph G is defined as the number of independent vertex subsets of G, where a set of vertices is said to be independent if it contains no pair of connected vertices. Prodinger and Tichy [13] defined

$NI(P_n) = F_{n+1}$, where F_n is nth Fibonacci number. [1.3]

$NI(C_n) = L_n$, $n \geq 3$ where L_n is nth Lucas number [1.4]

3 Graph Containing Vertex-Disjoint Cycles

Theorem 3.1 Let G be a graph of order n containing two vertex-disjoint cycles. Then $NI(G) = 5F_{n-3}$, $n \geq 6$

Proof Let G be a graph of order n and having two disjoint cycles. If n=6, 7 and 8 then $NI(G) = 5F_3 = 15$, $NI(G) = 5F_4 = 25$ and $NI(G) = 5F_5 = 40.$

For larger values of n, consider two fixed vertices v_i and v_j. If delete one vertex v_i from graph, the total number of independent set is $NI(G) = 5F_{n-4}$, If delete another vertex v_j from graph, the total number of independent set is $NI(G) = 5F_{n-5}$, Independent sets including vertices v_i and v_j must determine by combining both cases, Thus $NI(G) = 5F_{n-4} + 5F_{n-5} = 5F_{n-3}$.

Theorem 3.2 Let G be a graph of order n containing three vertex-disjoint cycles. Then $NI(G) = 7F_{n-4}$, $n \geq 9$

The proof can be given same as theorem 3.1.

Theorem3.3. Let G be a graph of order n containing four vertex-disjoint cycles. Then $NI(G) = 4F_{n-3}$, $n \geq 12$

The proof can be given same as theorem 3.1.

Theorem 3.3 Let G be a graph of order n containing vertex-disjoint cycles. Then

$\gamma(G) = i(G)$, $n \geq 6$

4 Graph Containing Edge-Disjoint Cycles

Theorem 4.1. Let G be a graph of order n containing two edge-disjoint cycles. Then $NI(G) = 3F_{n-2} + (n-4)$, $n \geq 5$

Proof. Let G be a graph of order n and having two edge-disjoint cycles. If n=5, 6 and 7 then $NI(G) = 3F_3 + 1 = 10$, $NI(G) = 3F_4 + 2 = 17$ and $NI(G) = 3F_5 + 3 = 27.$

By inspection, it can be derived and verified.

Theorem 4.2 Let G be a graph of order n containing three edge-disjoint cycles. Then $NI(G) \geq 3F_{n-2}$, $n \geq 7.$

The proof can be given same as theorem 4.1.

Theorem 4.3 Let G be a graph of order n containing edge-disjoint cycles. Then

$\gamma(G) = i(G)$, $n \geq 6$

5 Conclusions

Identities of Domination, independence, Fibonacci numbers of graphs containing vertex-disjoint cycles and edge-disjoint cycles are described.

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