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Generalized Solution of Transport Equation

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Abstract: This paper proves the existence and uniqueness of solution of irregular transport problem with variable speed and initial data in the Colombeau algebra $\mathcal{G}$, and some important proprieties of Colombeau algebra. The existence of distribution solutions to some classes of such equations is proven.

Key-Words: Transport equation, Colombeau algebra, Generalized solution, Distributions, Association.

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1 Introduction

The transport equation describes the movement of a scalar quantity over space. It is often used to transfer a scalar field (such as chemical concentration, material characteristics, or temperature) within an incompressible flow. The transport equation is also known as the convection-diffusion equation, which is a first-order PDE from a mathematical standpoint (partial differential equation). The most prevalent transportation models are based on the convection-diffusion equation.

In colombeau algebra, the concept of association is a true generalization of the equality of distributions, allowing us to explain findings in terms of distributions once again. Colombeau theory has found widespread use in a variety of natural sciences and technical sectors, particularly in domains involving products of distributions with coincident singularities [1][2].

The paper is organized as follows. After the introductory part,we give some basic preliminaries such as notations and definitions of the objects we shall work with, we also introduce different spaces of Colombeau algebra of generalized functions. In the third section we proved the existence and uniqueness of solution of transport equation with variable speed and initial data in the Colombeau algebra $\mathcal{G}$. Finally, in the fifth section we study the association (Application).

2 Preliminaries

2.1 Colombeau algebra

In this section, we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in[0;1]}$ of elements of a set X will be denoted by $X^{[0;1]}$, such sequences will also be called nets and simply written as u_ε.
Let $\mathcal{D}(\mathbb{R}^n)$ be the space of all smooth functions $\varphi: \quad \mathbb{R}^n \longrightarrow \mathbb{C}$ with compact support.
For $q \in \mathbb{N}$ we denote:

$$\mathcal{A}_q(\mathbb{R}^n) = \{\varphi \in \mathcal{D}(\mathbb{R}^n) \,/ \int \varphi(x)dx = 1 and$$

$$\int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \leq \alpha \leq q\}$$

The elements of the set $\mathcal{A}_q$ are called test functions.

It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \dots$. Colombeau in his books has proved that the sets $\mathcal{A}_k$ are non empty for all $k \in \mathbb{N}$.

For $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ and $\epsilon > 0$ it is denoted as $\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\check{\varphi}(x) = \varphi(-x)$.

We denote by:

$$\mathcal{E}(\mathbb{R}^n) = \{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/ \text{ with } u(\varphi, x) \text{ is } \mathcal{C}^\infty \text{ to the second variable } x\}$$

$$u(\varphi_\varepsilon, x) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1$$

$$\mathcal{E}_M(\mathbb{R}^n) = \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) \,/ \forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \sup_{x\in K} \|D^\alpha u_\varepsilon(x)\| = \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0\}$$

$$\mathcal{N}(\mathbb{R}^n) = \{(u_\varepsilon)_{\varepsilon>0} \in \mathcal{E}(\mathbb{R}^n) \,/ \forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \forall p \in \mathbb{N} \text{ such that } \sup_{x\in K} \|D^\alpha u_\varepsilon(x)\| = \mathcal{O}(\varepsilon^p) \text{ as } \varepsilon \to 0\}$$

The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M[\mathbb{R}^n]/\mathcal{N}[\mathbb{R}^n]$, where the elements of the set $\mathcal{E}_M(\mathbb{R}^n)$ are moderate while the elements of the set $\mathcal{N}(\mathbb{R}^n)$ are negligible.

The meaning of the term association in $\mathcal{G}(\mathbb{R})$ is given with the next two definitions.

Definition 1. *Generalized functions $f, g \in \mathcal{G}(\mathbb{R})$ are said to be associated, denoted $f \approx g$, if for each representative $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \|f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)\| \psi(x) dx = 0$$

Definition 2. *Generalized functions $f \in \mathcal{G}(\mathbb{R})$ is said to admit some as $u \in \mathcal{D}'(\mathbb{R})$ associated distribution, denoted $f \approx u$, if for each representative $f(\varphi_\varepsilon, x)$ of f and any $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x) \psi(x) dx = \langle u, \psi \rangle$$

Below is the statement of a problem called irregular transport problem on the domain $\Omega = \mathbb{R}^+ \times \mathbb{R}$.

$$\begin{cases} \partial_t u(t,x) + c(t,x)\partial_x u(t,x) = f(t,x)u(t,x) \\ +a(t,x), \quad (t,x) \in \mathbb{R}^{*,+} \times \mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1}$$

with c, f and a are discontinuous functions such that $c > 0$.

2.2 Positive, negative and bounded generalized function

Definition 3. *[5]*
a- $U \in \mathcal{G}[\Omega]$ is said globally bound if it exists $c > 0$ and a representative $u \in \mathcal{E}_M[\Omega]$ of U and $N \in \mathbb{N}$ such as $\forall \phi \in \mathcal{A}_N$, we have

$$\sup_{y \in \Omega} \|u(\phi_\varepsilon, y)\| \leq c$$

when $\varepsilon \to 0$
b- $U \in \mathcal{G}[\Omega]$ is said to have local logarithmic growth if for any representative $u \in \mathcal{E}_M[\Omega]$ of U and for any compact K of Ω it exists $N \in \mathbb{N}$ such as $\forall \phi \in \mathcal{A}_N, \exists c > 0$, such as

$$\sup_{y \in \Omega} \|u(\phi_\varepsilon, y)\| \leq c \ln\left(\frac{1}{\varepsilon}\right)$$

when $\varepsilon \to 0$
c- $U \in \mathcal{G}[\Omega]$ is said to be strictly positive and we note $U > 0$ if for all compact K of Ω, there is a representative $u \in \mathcal{E}_M[\Omega]$ of $U, \exists N \in \mathbb{N}, c > 0, \forall \phi \in \mathcal{A}_N$, we have

$$u(\phi_\varepsilon, y) \geq c\varepsilon^N \quad \forall y \in K$$

when $\varepsilon \to 0$
d- $U \in \mathcal{G}[\Omega]$ is said to be strictly negative and we note $U < 0$ if for all compact K of Ω, there is a representative $u \in \mathcal{E}_M[\Omega]$ de $U, \exists N \in \mathbb{N}, c > 0, \forall \phi \in \mathcal{A}_N$, we have

$$u(\phi_\varepsilon, y) \leq -c\varepsilon^N \quad \forall y \in K$$

when $\varepsilon \to 0$

Proposition 1. *(c) and (d) of the previous definition do not depend on the chosen representative.*

Proof. c- Let U an element of $\mathcal{G}[\Omega]$ strictly positive. Be a compact K of Ω. So there is a representative $u \in \mathcal{E}_M[\Omega]$ of $U, \exists N \in \mathbb{N}, \exists c > 0, \forall \phi \in \mathcal{A}_N$ such as

$$u(\phi_\varepsilon, y) \geq c\varepsilon^N \quad y \in K$$

Let u_2 another representative of U. So

$$u - u_2 \in \mathcal{N}[\Omega]$$

i.e

$$u(\phi_\varepsilon, y) - u_2(\phi_\varepsilon, y) < \varepsilon^q \quad \forall q$$

so

$$\begin{aligned} u_2(\phi_\varepsilon, y) &> -\varepsilon^q + u(\phi_\varepsilon, y) \\ &> -\varepsilon^q + c\varepsilon^N \\ &> c\varepsilon^N \left(1 - \frac{\varepsilon^{q-N}}{c}\right) \end{aligned}$$

By crossing the limit $q \longrightarrow +\infty$, so

$$u_2(\phi_\varepsilon, y) > c\varepsilon^N, \quad y \in K$$

d- Let U an element of $\mathcal{G}[\Omega]$, suppose that U is strictly negative. Be a compact K of Ω, So there is a representative $u \in \mathcal{E}_M[\Omega]$ of $U, \exists N \in \mathbb{N}, \exists c > 0, \forall \phi \in \mathcal{A}_N$ such that

$$U(\phi_\varepsilon, y) < -c\varepsilon^N \quad y \in K$$

Let u_2 another representative of U. so

$$u - u_2 \in \mathcal{N}[\Omega]$$

thus

$$u_2(\phi_\epsilon, y) - u(\phi_\epsilon, y) < \epsilon^q$$

so

$$\begin{aligned} u_2\left(\phi_{\varepsilon}, y\right) &< \varepsilon^q + u\left(\phi_{\varepsilon}, y\right) \\ &< \varepsilon^q - c\varepsilon^N \\ &< -c\varepsilon^N\left(1 - \frac{\varepsilon^{q-N}}{c}\right) \end{aligned}$$

By crossing the limit $q \longrightarrow +\infty$, so

$$u_2\left(\phi_{\varepsilon}, y\right) < -c\varepsilon^N, \quad y \in K$$

□

Proposition 2. *(a) of the previous definition does not depend on the chosen representative.*

Proof. Let u_2 another representative of U. so

$$u - u_2 \in \mathcal{N}[\Omega]$$

then

$$\begin{gathered} \left\|u_2\left(\phi_{\varepsilon}, y\right) - u\left(\phi_{\varepsilon}, y\right)\right\| < \varepsilon^q \quad \forall y \in \Omega \quad \forall q \\ \left\|u_2\left(\phi_{\varepsilon}, y\right)\right\| - \left\|u\left(\phi_{\varepsilon}, y\right)\right\| < \varepsilon^q \\ \left\|u_2\left(\phi_{\varepsilon}, y\right)\right\| < \varepsilon^q + \left\|u\left(\phi_{\varepsilon}, y\right)\right\| \end{gathered}$$

$\left\|u_2\left(\phi_{\varepsilon}, y\right)\right\| < \varepsilon^q + c$ By crossing the limit $q \longrightarrow +\infty$, we find

$$\left\|u_2\left(\phi_{\varepsilon}, y\right)\right\| < c \quad \forall y \in \Omega$$

□

3 Existence and uniqueness of the generalized solution

we consider the following irregular transport problem on the domain $\Omega = \mathbb{R}^+ \times \mathbb{R}$:

$$\left\{\begin{array}{l} \partial_t u(t,x) + c(t,x)\partial_x u(t,x) = f(t,x)u(t,x) \\ +a(t,x), \quad (t,x) \in \mathbb{R}^{*,+} \times \mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{array}\right. \tag{2}$$

Now we will transform the problem (2) to the Colombeau algebra $\mathcal{G}$.

$$\left\{\begin{array}{l} \partial_t u_{\varepsilon}(t,x) + c_{\varepsilon}(t,x)\partial_x u_{\varepsilon}(t,x) = f_{\varepsilon}(t,x)u_{\varepsilon}(t,x) \\ +a_{\varepsilon}(x,t), (t,x) \in \mathbb{R}^{+,*} \times \mathbb{R} \\ u_{\varepsilon}(0,x) = u_{\varepsilon,0}(x), \quad x \in \mathbb{R} \end{array}\right. \tag{3}$$

Theorem 1. *We assume that c is globally bounded such that $c > 0$, $\partial_x c$ and f are with local logarithmic growth. So for an initial data $u_0 \in \mathcal{G}[\mathbb{R}]$ and a element of $\mathcal{G}[\Omega]$, the problem (3) admits a unique solution $u \in \mathcal{G}$.*

Proof. **Existence:**

The problem (3) admits a unique class solution C^{∞}.

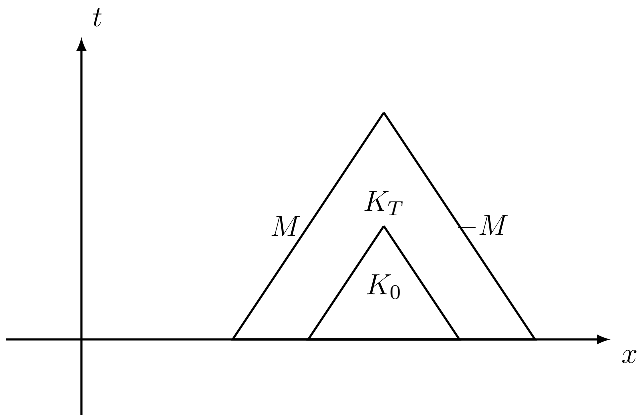
By hypothesis c is globally bounded, then $\exists M > 0$ such as :

$$\left\|c_{\varepsilon}(x,t)\right\| \leq M, \quad \forall(x,t) \in \Omega$$

so

$$\left\|\frac{d\lambda_{\varepsilon}(x,t,s)}{ds}\right\| \leq M, \quad \forall(x,t) \in \Omega$$

with $\lambda_{\varepsilon}(x,t,.)$ the characteristic curve corresponding to c_{ε} issue of the point (x,t), by drawing the lines passing through the point (x,t) and slope M and $-M$, we can determine a domain (compact of Ω) of determination of the solution that does not depend on ε.



In this case, for all $(x,t) \in K_T \subset \Omega$,, the characteristic curves resulting from this point remain in K_T, so the integral solution of the equation (3) is:

$$\begin{aligned} u_{i,\varepsilon}(x,t) = {} & u_{0,\varepsilon}\left(\lambda_{\varepsilon}(x,t,0)\right) \\ & + \int_0^t f_{\varepsilon}\left(\lambda_{\varepsilon}(x,t,s), s\right) u_{\varepsilon}\left(\lambda_{\varepsilon}(x,t,s), s\right) ds \\ & + \int_0^t a_{i,\varepsilon}\left(\lambda_{\varepsilon}(x,t,s), s\right) ds \end{aligned}$$

$\|u_{\varepsilon}(x,t)\| \leq \sup_{x \in K_0}\|u_{0,\varepsilon}(x)\|$
$+ \int_0^T \sup_{(t,x)\in K_T}\|f_{\varepsilon}(t,x)\| \sup_{K_T}\|u_{\varepsilon}(t,x)\| ds$
$+ \int_0^T \sup_{(t,x)\in K_T}\|a_{\varepsilon}(t,x)\| ds$
$\sup_{(t,x)\in K_T}\|u_{\varepsilon}(t,x)\| \leq \sup_{x\in K_0}\|u_{0,\varepsilon}(x)\| + T\sup_{(t,x)\in K_T}\|a_{\varepsilon}(t,x)\|$
$+ \int_0^t \sup_{(t,x)\in K_T}\|f_{\varepsilon}(t,x)\| \sup_{(t,x)\in K_T}\|u_{\varepsilon}(t,x)\| ds$

Apply Gronwall's lemma to the function $u_{\varepsilon} \rightarrow \sup_{(t,x)\in K_T}\|u_{\varepsilon}(t,x)\|$

$$\begin{aligned} & \sup_{(t,x)\in K_T}\|u_{\varepsilon}(t,x)\| \leq \\ & \quad \left[\sup_{x\in K_0}\|u_{0,\varepsilon}(x)\| + T\sup_{(t,x)\in K_T}\|a_{\varepsilon}(t,x)\|\right] \\ & \quad \times \exp\left(T\sup_{(t,x)\in K_T}\|f_{\varepsilon}(t,x)\|\right) \end{aligned}$$

As f is local logarithmic growth, then:

$$\sup_{(t,x)\in K_T} \|u_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^{-N}\right), \quad N \in \mathbb{N}$$

Now let's apply the operator ∂_x on equation (3)

$$\begin{cases} \partial_t(\partial_x u_\varepsilon(t,x)) + c_\varepsilon(t,x)\partial_x\partial_x u_\varepsilon(t,x) = \\ f_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) - \partial_x c_\varepsilon(t,x)u_\varepsilon(t,x) \\ +\partial_x f_\varepsilon(t,x)u_\varepsilon(t,x) + \partial_x a_\varepsilon(x,t), (t,x) \in \mathbb{R}^{+,*} \times \mathbb{R} \\ \partial_x u_\varepsilon(0,x) = u_0'(x), x \in \mathbb{R} \end{cases} \tag{4}$$

Then, the integral solution is:
$\partial_x u_\varepsilon(t,x) = u'_{0,\varepsilon}\left(\lambda_\varepsilon(x,t,0)\right)$
$+ \int_0^t \partial_x f_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) u_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) ds$
$+ \int_0^t f_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) \partial_x u_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) ds$
$- \int_0^t \partial_x c_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) u_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right) ds$
$+ \int_0^t \partial_x a_{i,\varepsilon}\left(\lambda_\varepsilon(x,t,s),s\right) ds$

$\|\partial_x u_\varepsilon(x,t)\| \leq \sup_{x\in K_0} \|u'_{0,\varepsilon}(x)\| + \int_0^T \sup_{(t,x)\in K_T} \|\partial_x f_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x)\| ds$
$+\int_0^T \sup_{(t,x)\in K_T} \|f_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|\partial_x u_\varepsilon(t,x)\| ds$
$+\int_0^T \sup_{(t,x)\in K_T} \|\partial_x c_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x)\| ds$
$+ \int_0^T \sup_{(t,x)\in K_T} \|\partial_x a_\varepsilon(t,x)\| ds$

Apply Gronwall's lemma to the function $u\varepsilon \to \sup_{(t,x)\in K_s} \|\partial_x u_\varepsilon(t,x)\|$, then:

$\sup_{K_T} \|\partial_x u_\varepsilon(t,x)\| \leq [\sup_{x\in K_0} \|u'_{0,\varepsilon}(x)\| + T \sup_{(t,x)\in K_T} \|\partial_x f_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x)\|$
$+ T \sup_{K_T} \|\partial_x c_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x)\| + T \sup_{(t,x)\in K_T} \|\partial_x a_\varepsilon(t,x)\|]$
$\times \exp(T \sup_{(t,x)\in K_T} \|f_\varepsilon(t,x)\|)$

As f is local logarithmic growth and (u_ε) is moderate, so

$$\sup_{(t,x)\in K_T} \|\partial_x u_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^{-N}\right), \quad N \in \mathbb{N}$$

By doing the same reasoning, we find that for all $m \in \mathbb{N}$ and $m \geq 2$, we have:

$$\sup_{(t,x)\in K_T} \|\partial_x^m u_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^{-N}\right), \quad N \in \mathbb{N}$$

on the other hand, we have

$\partial_t u_\varepsilon(t,x) = -c_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) + f_\varepsilon(t,x)u_\varepsilon(t,x) + a_\varepsilon(x,t)$

as f has local logarithmic growth and $(\partial_x u_\varepsilon)$ is moderate, then:

$$\sup_{(t,x)\in K_T} \|\partial_t u_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^{-N}\right), \quad N \in \mathbb{N}$$

We also have

$\partial_t\partial_x u_\varepsilon(t,x) = -\partial_x c_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) - c_\varepsilon(t,x)\partial_x^2 u_\varepsilon(t,x) + \partial_x f_\varepsilon(t,x)u_\varepsilon(t,x) + f_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) + \partial_x a_\varepsilon(x,t)$

$\partial_t^2 u_\varepsilon(t,x) = -\partial_t c_\varepsilon(t,x)\partial_x u_\varepsilon(t,x) - c_\varepsilon(t,x)\partial_t\partial_x u_\varepsilon(t,x) + \partial_t f_\varepsilon(t,x)u_\varepsilon(t,x) + f_\varepsilon(t,x)\partial_t u_\varepsilon(t,x) + \partial_t a_\varepsilon(x,t)$

etc. ....
Hence, for any derivative operator $\partial_t^n\partial_x^m, \exists N \in \mathbb{N}$

$$\sup_{(t,x)\in K_T} \|\partial_t^n\partial_x^m u_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^{-N}\right)$$

then

$$u \in \mathcal{G}(\Omega)$$

Uniqueness:
Suppose that problem (3) admits two solutions $u_\varepsilon, v_\varepsilon \in \mathcal{G}[\Omega]$, so $\exists d_{0,\varepsilon} \in \mathcal{N}[\mathbb{R}]$ such as:

$$\begin{cases} \partial_t(u_\varepsilon(t,x) - v_\varepsilon(t,x)) + \lambda_{i\varepsilon}(x,t)\partial_x(u_\varepsilon(t,x) \\ -v_\varepsilon(t,x)) = f_\varepsilon(x,t)(u_\varepsilon(x,t) - v_\varepsilon(x,t)) \\ u_\varepsilon(x,0) - v_\varepsilon(x,0) = d_{0,\varepsilon}(x) \end{cases}$$

The integral solution is:

$u_\varepsilon(t,x) - v_\varepsilon(t,x) = d_{0,\varepsilon}\left(\lambda_\varepsilon(x,t,0)\right) + \int_0^t f_\varepsilon\left(\lambda_\varepsilon(x,t,s),s\right)(u_\varepsilon(\lambda_\varepsilon(x,t,s),s) - v_\varepsilon(\lambda_\varepsilon(,,s),s))ds$

$\|u_\varepsilon(t,x) - v_\varepsilon(t,x)\| \leq \sup_{x\in K_0} \|d_{0,\varepsilon}(x)\|$
$+ \int_0^T \sup_{(t,x)\in K_T} \|f_\varepsilon(t,x)\| \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x) - v_\varepsilon(t,x)\| ds$

Apply Gronwall's lemma to the function $u_\varepsilon \to \sup_{(t,x)\in K_T} \|u_\varepsilon(t,x) - v_\varepsilon(t,x)\|$

$$\sup_{(t,x)\in K_T} \|u_\varepsilon(t,x) - v_\varepsilon(t,x)\| \leq (\sup_{x\in K_0} \|d_{0,\varepsilon}(x)\|) \times \exp\left(T \sup_{(t,x)\in K_T} \|f_\varepsilon(t,x)\|\right)$$

as f is local logarithmic growth, then

$$\sup_{(t,x)\in K_T} \|u_\varepsilon(t,x) - v_\varepsilon(t,x)\| = \mathcal{O}\left(\varepsilon^q\right), \quad \forall q \in \mathbb{N}$$

For the other derivatives, it is the same as the first part of the proof of the theorem.
Consequently the problem (3) admits a unique solution $u \in \mathcal{G}[\Omega]$. $\square$

4 Application

We consider the following problem which presents the propagation of a wave in a discontinuous medium:

$$\left\{\begin{array}{l}\partial_t u(t,x)+c(t,x)\partial_x u(t,x)=0,(t,x)\in\mathbb{R}_{+,*}\times\mathbb{R}\\ u(0,x)=u_0(x),\quad x\in\mathbb{R}\end{array}\right. \tag{5}$$

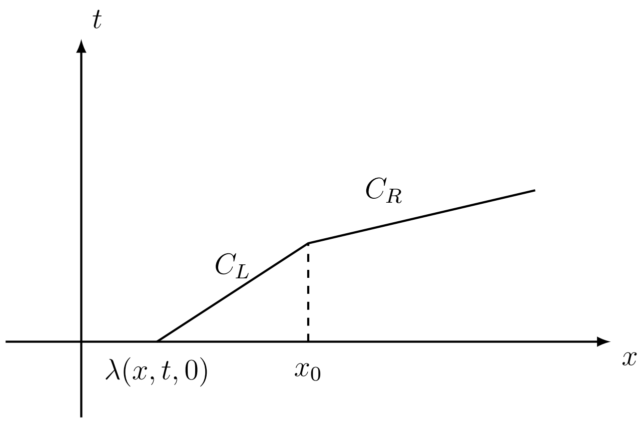
with

$$c(t,x)=\left\{\begin{array}{ll}c_L, & x\leq x_0\\ c_R, & x>x_0\end{array}\right.$$

and u_0 is a continuous function almost everywhere, and zero in the neighborhood of 0.
If we set a condition of transition to x_0 (continuity of u on point x_0), then the solution of the problem (5) is given by:

$$u(t,x)=u_0(\lambda(t,x,0))$$

with λ the characteristic curve resulting from the point (t,x) and sloping c (figure(2)).



$c\in L^\infty(\Omega)$, then there is $C\in\mathcal{G}[\Omega]$ such as $C\approx c$, c is globally bounded and $\partial_x C\in\mathcal{N}[\Omega]$ local logarithmic growth, and according to Theorem 1 the problem (5) admits a unique solution $U=[(u_\varepsilon)]\in\mathcal{G}[\Omega]$.

Now we will transform the problem (5) to the Colombeau algebra, we have:

$$\left\{\begin{array}{l}\partial_t u_\varepsilon(t,x)+c_\varepsilon(x)\partial_x u_\varepsilon(t,x)=0,(t,x)\in\mathbb{R}^{*+}\times\mathbb{R}\\ u_\varepsilon(0,x)=u_{0,\varepsilon}(x),\end{array}\right. \quad x\in\mathbb{R}$$

Take:

$$\lambda_\varepsilon=\lambda*\phi_{\eta_\varepsilon}$$

with $\phi\in\mathcal{D}(\mathbb{R}_+)$ such as :

$\int_{\mathbb{R}_+}\phi(x)dx=1 \quad \operatorname{supp}(\phi_{\eta_\varepsilon})\subset \quad]x_0-\eta_\varepsilon,x_0+\eta_\varepsilon[\quad \eta_\varepsilon=|\log_\varepsilon|^{-1}$

We pose:

$$u_\varepsilon(t,x)=u_{0,\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right)$$

To proof that $U\approx u$, just prove that :

$$\lim_{\varepsilon\to0}\int_\Omega(u_{0,\varepsilon}(\lambda_\varepsilon(t,x,,0))-u_0(\lambda(t,x,0)))\psi(t,x)dtdx=0$$

for all $\psi\in\mathcal{D}(\Omega)$
We have

$$\begin{array}{l}\int_\Omega\left(u_{0,\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right)-u_0(\lambda(t,x,0))\right)\psi(t,x)dtdx=\\ \int_\Omega\left(u_{0,\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right)-u_0\left(\lambda_\varepsilon(t,x,0)\right)\right)\psi(t,x)dtdx\\ +\int_\Omega\left(u_0\left(\lambda_\varepsilon(t,x,0)\right)-u_0(\lambda(t,x,0))\right)\psi(t,x)dtdx\end{array}$$

But

$$\begin{array}{c}\int_\Omega\left(u_{0,\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right)-u_0\left(\lambda_\varepsilon(t,x,0)\right)\right)\psi(t,x)dtdx=\\ \int_\Omega\left(u_{0,\varepsilon}-u_0\right)\left(\lambda_\varepsilon(t,x,0)\right)\psi(t,x)dtdx\\ \leq\sup_{x\in\mathbb{R}}\|u_0*\phi_\varepsilon-u_0\|\|\int_{\sup p(\phi)}\psi(t,x)dtdx\|\end{array}$$

so

$$\lim_{\varepsilon\to0}\int_\Omega\left(u_{0,\varepsilon}\left(\lambda_\varepsilon(t,x,0)\right)-u_0(\lambda(t,x,0))\right)\psi(t,x)dtdx=0$$

To prove that

$$\lim_{\varepsilon\to0}\int_\Omega\left(u_0\left(\lambda_\varepsilon(t,x,0)\right)-u_0(\lambda(t,x,0))\right)\psi(t,x)dtdx=0$$

We must prove:

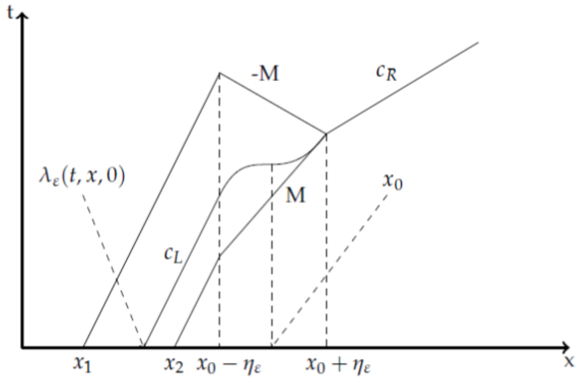
$$\lim_{\varepsilon\to0}\left(\lambda_\varepsilon(t,x,0)-\lambda(t,x,0)\right)=0$$

We know that c is globally bounded.
So there is $M>0$ such as

$$\sup_{(t,x)\in\Omega}\|c_\varepsilon(t,x)\|<M$$

So, we can frame the curve λ_ε between two broken curves, and we take the intersection of these two curves with the axis (ox), given by:

$$\begin{array}{c}x_1=\frac{1}{c_L}\left[-2M\eta_\varepsilon+c_R\left(x-x_0-\eta_\varepsilon\right)\right]-\frac{t}{c_L}-\eta_\varepsilon+x_0\\ x_2=\frac{1}{c_L}\left[2M\eta_\varepsilon+c_R\left(x-x_0-\eta_\varepsilon\right)\right]-\frac{t}{c_L}-\eta_\varepsilon+x_0\end{array}$$



such that

$$x_1 \leq \lambda_\varepsilon(t, x, 0) \leq x_2$$

Since

$$\begin{aligned}\lim_{\varepsilon\to 0} \lambda_\varepsilon(t, x, 0) &= \frac{c_R}{c_L}(x - x_0) - \frac{t}{c_L} + x_0 \\ &= \lambda(x, t, 0)\end{aligned}$$

so

$$U \approx u$$

□

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