# Formulas to approximate Legendre's Complete Elliptic Integrals using Peano's Law on Ellipse's Perimeter and a Recurrent-Iterative Scheme (Landen's Transform Included)

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**Abstract: Two sets of closed analytic functions are proposed for the approximate calculus of the complete elliptic integrals K(k) and E(k) in the normal form due to Legendre, their expressions having a remarkable simplicity and accuracy. The special usefulness of the newly proposed formulas consists in they allow performing the analytic study of variation of the functions in which they appear, using derivatives (they being expressed in terms of elementary functions only, without any special function; this would mean replacing one difficulty by another of the same kind). Comparative tables of so found approximate values with the exact ones, reproduced from special functions tables, are given (vs. the elliptic integrals' modulus k). The 1st set of formulas was suggested by Peano's law on ellipse's perimeter. The new functions and their derivatives coincide with the exact ones at the left domain's end only. As for their simplicity, the formulas in k / k' do not need mathematical tables (are purely algebraic). As for accuracy, the 2nd set, more intricate, gives more accurate values and extends itself more closely to the right domain's end. An original fast converging recurrent-iterative scheme to get sets of formulas with the desired accuracy is given in appendix.**

**Key-Words: analytic methods; Legendre complete elliptic integrals K(k) and E(k); elliptic integrals' moduli k, k′; tables of Legendre complete elliptic integrals; Peano's approximate law for ellipse's perimeter; recurrent-iterative scheme; Landen transformation**

## I. INTRODUCTION – ELLIPTIC INTEGRALS

There are many interesting domains in pure and applied mathematics where appear both (or, often, only one) complete elliptic integrals of the 1st and 2nd kind in the normal form due to Legendre. The arc length of a lemniscate, as well as the period of oscillations in a vacuum of the simple pendulum, in the dynamics of a constrained heavy particle, are given by a complete elliptic integral of the 1st kind. The perimeter of an ellipse, as well as the lift coefficient of a thin delta wing with subsonic leading edges, in supersonic aerodynamics (small perturbations theory), are given by a complete elliptic integral of the 2nd kind. In electromagnetic theory, the electric and magnetic fields from a circular coil can be expressed using the complete elliptic integrals. The relations below define the integrals of the 1st and 2nd kind, in *canonical* form, K(*k*) and E(*k*), resp.:

$$\mathrm{K}(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{-1/2}d\varphi=\int_0^1[(1-t^2)(1-k^2t^2)]^{-1/2}dt;$$

$$\mathrm{E}(k)=\int_0^{\pi/2}(1-k^2\sin^2\varphi)^{1/2}d\varphi=\int_0^1[(1-t^2)(1-k^2t^2)]^{1/2}dt;$$

$k=\sin\theta\geq 0$ is called *modulus*. K(*k*), E(*k*) are typical *elliptic* integrals. They do not admit primitive functions (cannot be expressed in terms of elementary functions), being calculated by expanding the integrands into series, integrating term-by-term, and presented vs. $k\in[0,1]$, or vs. $\theta\in[0,\pi/2]$, in some mathematical tables [1]–[6]. Modern mathematics defines an elliptic integral as any function $f$ which can be expressed in the form $f(x)=\int_c^x R[t, P(t)^{1/2}]\,dt$; $R$ is a rational function of its two arguments; $P$ is a polynomial of degree 3 or 4 with no repeated roots; $c$ is a constant. The values given in some special tables allow performing the calculus for a given case (point), but not the analytic study of variation of the functions in which these integrals appear, using the derivatives. In chapter 2 *two original sets* (subscripts 0; 1) of *closed analytic functions* are given for the approximate calculus of K(*k*) and E(*k*). We use *an original purely analytic method* (not some numerical, or sophisticated computer programs, like most authors). There also is a Legendre complete elliptic integral of the 3rd kind. With an appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the *three Legendre canonical forms* (of the 1st, 2nd & 3rd kind).

## II. THE TWO SETS OF NEWLY PROPOSED FORMULAS

The *complementary modulus* is $k'=(1-k^2)^{1/2}=\cos\theta$. The $\mathrm{E}_0(k)$ formula in the 1st set ($\mathrm{K}_0$, $\mathrm{E}_0$) is suggested by Peano's law.

$$\mathrm{K}_0(k)=\frac{\pi}{\sqrt[4]{1-k^2}}\left(1-\frac{1}{2\sqrt{2}}\sqrt{\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}}\right)=\pi\left(\frac{1}{\sqrt{k'}}-\frac{1}{2\sqrt{2}}\frac{\sqrt{1+k'}}{k'^{3/4}}\right),$$

$$\mathrm{K}_0(\theta)=\frac{\pi}{\cos^{1/2}\theta}\left(1-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{1/4}\theta}\right)=\pi\left(\frac{1}{\cos^{1/2}\theta}-\frac{1}{2}\frac{\cos(\theta/2)}{\cos^{3/4}\theta}\right).$$

$$\mathrm{E}_0(k)=\frac{\pi}{4}\sqrt[4]{1-k^2}\left(\frac{3}{2}\frac{1+\sqrt{1-k^2}}{\sqrt[4]{1-k^2}}-1\right)=\frac{\pi}{4}\left[\frac{3}{2}(1+k')-\sqrt{k'}\right],$$

$$\mathrm{E}_0(\theta)=\frac{\pi}{4}\cos^{1/2}\theta\left(3\frac{\cos^2(\theta/2)}{\cos^{1/2}\theta}-1\right)=\frac{\pi}{4}\left(3\cos^2\frac{\theta}{2}-\sqrt{\cos\theta}\right).$$

Similarly, for the 2nd set ($\mathrm{K}_1$, $\mathrm{E}_1$), are proposed the formulas:

$$\mathrm{K}_1(k)=\frac{\pi\sqrt{2}}{\sqrt{(1+k')\sqrt{k'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k'}}{\sqrt[4]{(1+k')\sqrt{k'}}}\right),$$

$$\mathrm{K}_1(\theta)=\frac{\pi}{\cos(\theta/2)\cos^{1/4}\theta}\left(1-\frac{1}{4}\frac{1+\cos^{1/2}\theta}{\cos^{1/2}(\theta/2)\cos^{1/8}\theta}\right).$$

$$\mathrm{E}_1(k)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k'}\right)^2-\sqrt{2}\sqrt{1+k'}\sqrt[4]{k'}\right]-k'\cdot \mathrm{K}_1(k),$$

$$\mathrm{E}_1(\theta)=\frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{\cos\theta}\right)^2-2\cos\frac{\theta}{2}\sqrt[4]{\cos\theta}\right]-\cos\theta\cdot \mathrm{K}_1(\theta).$$

A 3[rd] set ($K_2$, $E_2$), even more accurate than previous two sets, can be built (a recurrent-iterative scheme) – see appendix.

Table 1. Values of the functions K (part one)

| $\theta$ (°) | $k=\sin\theta$ | K($k$) | $K_0(k)$ | $K_1(k)$ |
|---|---|---|---|---|
| 0 | 0.00000 | 1.5708 | 1.5708 | 1.5708 |
| 1 | 0.01745 | 1.5709 | 1.5709 | 1.5709 |
| 2 | 0.03490 | 1.5713 | 1.5713 | 1.5713 |
| 3 | 0.05234 | 1.5719 | 1.5719 | 1.5719 |
| 4 | 0.06976 | 1.5727 | 1.5727 | 1.5727 |
| 5 | 0.08716 | 1.5738 | 1.5738 | 1.5738 |
| 6 | 0.10453 | 1.5751 | 1.5751 | 1.5751 |
| 7 | 0.12187 | 1.5767 | 1.5767 | 1.5767 |
| 8 | 0.13917 | 1.5785 | 1.5785 | 1.5785 |
| 9 | 0.15643 | 1.5805 | 1.5805 | 1.5805 |
| 10 | 0.17365 | 1.5828 | 1.5828 | 1.5828 |
| 11 | 0.19081 | 1.5854 | 1.5854 | 1.5854 |
| 12 | 0.20791 | 1.5882 | 1.5882 | 1.5882 |
| 13 | 0.22495 | 1.5913 | 1.5913 | 1.5913 |
| 14 | 0.24192 | 1.5946 | 1.5946 | 1.5946 |
| 15 | 0.25882 | 1.5981 | 1.5981 | 1.5981 |
| 16 | 0.27564 | 1.6020 | 1.6020 | 1.6020 |
| 17 | 0.29237 | 1.6061 | 1.6061 | 1.6061 |
| 18 | 0.30902 | 1.6105 | 1.6105 | 1.6105 |
| 19 | 0.32557 | 1.6151 | 1.6151 | 1.6151 |
| 20 | 0.34202 | 1.6200 | 1.6200 | 1.6200 |
| 21 | 0.35837 | 1.6252 | 1.6252 | 1.6252 |
| 22 | 0.37461 | 1.6307 | 1.6307 | 1.6307 |
| 23 | 0.39073 | 1.6365 | 1.6365 | 1.6365 |
| 24 | 0.40674 | 1.6426 | 1.6426 | 1.6426 |
| 25 | 0.42262 | 1.6490 | 1.6490 | 1.6490 |
| 26 | 0.43837 | 1.6557 | 1.6557 | 1.6557 |
| 27 | 0.45399 | 1.6627 | 1.6627 | 1.6627 |
| 28 | 0.46947 | 1.6701 | 1.6701 | 1.6701 |
| 29 | 0.48481 | 1.6777 | 1.6777 | 1.6777 |
| 30 | 0.50000 | 1.6858 | 1.6857 | 1.6858 |
| 31 | 0.51504 | 1.6941 | 1.6941 | 1.6941 |
| 32 | 0.52992 | 1.7028 | 1.7028 | 1.7028 |
| 33 | 0.54464 | 1.7119 | 1.7119 | 1.7119 |
| 34 | 0.55919 | 1.7214 | 1.7214 | 1.7214 |
| 35 | 0.57358 | 1.7312 | 1.7312 | 1.7312 |
| 36 | 0.58779 | 1.7415 | 1.7415 | 1.7415 |
| 37 | 0.60182 | 1.7522 | 1.7522 | 1.7522 |
| 38 | 0.61566 | 1.7633 | 1.7632 | 1.7633 |
| 39 | 0.62932 | 1.7748 | 1.7748 | 1.7748 |
| 40 | 0.64279 | 1.7868 | 1.7867 | 1.7868 |
| 41 | 0.65606 | 1.7992 | 1.7992 | 1.7992 |
| 42 | 0.66913 | 1.8122 | 1.8121 | 1.8122 |
| 43 | 0.68200 | 1.8256 | 1.8256 | 1.8256 |
| 44 | 0.69466 | 1.8396 | 1.8395 | 1.8396 |
| 45 | 0.70711 | 1.8541 | 1.8540 | 1.8541 |
| 46 | 0.71934 | 1.8691 | 1.8691 | 1.8691 |
| 47 | 0.73135 | 1.8848 | 1.8847 | 1.8848 |
| 48 | 0.74314 | 1.9011 | 1.9009 | 1.9011 |
| 49 | 0.75471 | 1.9180 | 1.9178 | 1.9180 |
| 50 | 0.76604 | 1.9356 | 1.9354 | 1.9356 |
| 51 | 0.77715 | 1.9539 | 1.9536 | 1.9539 |
| 52 | 0.78801 | 1.9729 | 1.9726 | 1.9729 |
| 53 | 0.79864 | 1.9927 | 1.9923 | 1.9927 |
| 54 | 0.80902 | 2.0133 | 2.0128 | 2.0133 |
| 55 | 0.81915 | 2.0347 | 2.0341 | 2.0347 |
| 56 | 0.82904 | 2.0571 | 2.0564 | 2.0571 |
| 57 | 0.83867 | 2.0804 | 2.0795 | 2.0804 |
| 58 | 0.84805 | 2.1047 | 2.1037 | 2.1047 |
| 59 | 0.85717 | 2.1300 | 2.1288 | 2.1300 |
| 60 | 0.86603 | 2.1565 | 2.1551 | 2.1565 |
| 61 | 0.87462 | 2.1842 | 2.1825 | 2.1842 |
| 62 | 0.88295 | 2.2132 | 2.2111 | 2.2132 |
| 63 | 0.89101 | 2.2435 | 2.2410 | 2.2435 |
| 64 | 0.89879 | 2.2754 | 2.2723 | 2.2754 |
| 65 | 0.90631 | 2.3088 | 2.3051 | 2.3088 |
| 66 | 0.91355 | 2.3439 | 2.3394 | 2.3439 |
| 67 | 0.92050 | 2.3809 | 2.3754 | 2.3809 |
| 68 | 0.92718 | 2.4198 | 2.4132 | 2.4198 |
| 69 | 0.93358 | 2.4610 | 2.4530 | 2.4610 |
| 70 | 0.93969 | 2.5046 | 2.4948 | 2.5045 |
| 70.5 | 0.94264 | 2.5273 | 2.5165 | 2.5273 |
| 71 | 0.94552 | 2.5507 | 2.5389 | 2.5507 |
| 71.5 | 0.94832 | 2.5749 | | 2.5749 |
| 72 | 0.95106 | 2.5998 | | 2.5998 |
| 72.5 | 0.95372 | 2.6256 | | 2.6255 |
| 73 | 0.95630 | 2.6521 | | 2.6521 |
| 73.5 | 0.95882 | 2.6796 | | 2.6796 |
| 74 | 0.96126 | 2.7081 | | 2.7081 |
| 74.5 | 0.96363 | 2.7375 | | 2.7375 |
| 75 | 0.96593 | 2.7681 | | 2.7680 |
| 75.5 | 0.96815 | 2.7998 | | 2.7997 |
| 76 | 0.97030 | 2.8327 | | 2.8326 |
| 76.5 | 0.97237 | 2.8669 | | 2.8669 |
| 77 | 0.97437 | 2.9026 | | 2.9025 |
| 77.5 | 0.97630 | 2.9397 | | 2.9397 |
| 78 | 0.97815 | 2.9786 | | 2.9785 |
| 78.5 | 0.97992 | 3.0192 | | 3.0191 |
| 79 | 0.98163 | 3.0617 | | 3.0616 |
| 79.5 | 0.98325 | 3.1064 | | 3.1063 |
| 80 | 0.98481 | 3.1534 | | 3.1533 |
| 80.2 | 0.98541 | 3.1729 | | 3.1727 |
| 80.4 | 0.98600 | 3.1928 | | 3.1927 |
| 80.6 | 0.98657 | 3.2132 | | 3.2130 |
| 80.8 | 0.98714 | 3.2340 | | 3.2338 |
| 81 | 0.98769 | 3.2553 | | 3.2551 |

Table 1. Values of the functions K (part two)

| | | | | |
|---|---|---|---|---|
| 81.2 | 0.98823 | 3.2771 | | 3.2769 |
| 81.4 | 0.98876 | 3.2995 | | 3.2992 |
| 81.6 | 0.98927 | 3.3223 | | 3.3221 |
| 81.8 | 0.98978 | 3.3458 | | 3.3455 |
| 82 | 0.99027 | 3.3699 | | 3.3696 |
| 82.2 | 0.99075 | 3.3946 | | 3.3942 |
| 82.4 | 0.99122 | 3.4199 | | 3.4196 |
| 82.6 | 0.99167 | 3.4460 | | 3.4456 |
| 82.8 | 0.99211 | 3.4728 | | 3.4724 |
| 83 | 0.99255 | 3.5004 | | 3.4999 |
| 83.2 | 0.99297 | 3.5288 | | 3.5283 |
| 83.4 | 0.99337 | 3.5581 | | 3.5575 |
| 83.6 | 0.99377 | 3.5884 | | 3.5877 |
| 83.8 | 0.99415 | 3.6196 | | 3.6188 |
| 84 | 0.99452 | 3.6519 | | 3.6510 |
| 84.2 | 0.99488 | 3.6852 | | 3.6843 |
| 84.4 | 0.99523 | 3.7198 | | 3.7187 |
| 84.6 | 0.99556 | 3.7557 | | 3.7545 |
| 84.8 | 0.99588 | 3.7930 | | 3.7916 |
| 85 | 0.99619 | 3.8317 | | 3.8302 |
| 85.2 | 0.99649 | 3.8721 | | 3.8704 |
| 85.4 | 0.99678 | 3.9142 | | 3.9122 |
| 85.6 | 0.99705 | 3.9583 | | 3.9560 |
| 85.8 | 0.99731 | 4.0044 | | 4.0018 |
| 86 | 0.99756 | 4.0528 | | 4.0498 |
| 86.2 | 0.99780 | 4.1037 | | 4.1003 |
| 86.4 | 0.99803 | 4.1574 | | 4.1535 |
| 86.6 | 0.99824 | 4.2142 | | 4.2097 |
| 86.8 | 0.99844 | 4.2744 | | 4.2692 |
| 87 | 0.99863 | 4.3387 | | 4.3325 |
| 87.2 | 0.99881 | 4.4073 | | 4.4001 |
| 87.4 | 0.99897 | 4.4811 | | 4.4726 |
| 87.6 | 0.99912 | 4.5609 | | 4.5507 |
| 87.8 | 0.99926 | 4.6477 | | 4.6354 |
| 88 | 0.99939 | 4.7427 | | 4.7277 |
| 88.2 | 0.99951 | 4.8478 | | 4.8293 |
| 88.4 | 0.99961 | 4.9654 | | |
| 88.6 | 0.99970 | 5.0988 | | |
| 88.8 | 0.99978 | 5.2527 | | |
| 89 | 0.99985 | 5.4349 | | |
| 89.1 | 0.99988 | 5.5402 | | |
| 89.2 | 0.99990 | 5.6579 | | |
| 89.3 | 0.99993 | 5.7914 | | |
| 89.4 | 0.99995 | 5.9455 | | |
| 89.5 | 0.99996 | 6.1278 | | |
| 89.6 | 0.99998 | 6.3509 | | |
| 89.7 | 0.99999 | 6.6385 | | |
| 89.8 | 0.99999 | 7.0440 | | |
| 89.9 | 1.00000 | 7.7371 | | |
| 90 | 1.00000 | $\infty$ | $-\infty$ | $-\infty$ |

The values strings in the last two columns of table 1 were canceled when each of the two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 1st kind K(*k*) gives too great relative errors (≥ 4 ‰ – also see chapter 3) for being still accepted in the usual mathematical / technical calculus. The same procedure will be applied in case of the next table (no. 2), for the same reason, concerning the accuracy of the values given by each of the other two closed analytic formulas proposed for the approximation of the Legendre complete elliptic integral of the 2nd kind E(*k*). The accuracy analysis of the two sets of formulas will be performed in the next chapter (no. 3). In chapter 4 some series representations for the exact functions and for both sets of approximation, as well as for their first order derivatives, will be given. For ($K_{0,1}$, $E_{0,1}$) behaviour in the right domain's side see appendix.

Table 2. Values of the functions E (part one)

| $\theta$ (°) | $k = \sin\theta$ | E(*k*) | $E_0(k)$ | $E_1(k)$ |
|---|---|---|---|---|
| 0 | 0.00000 | 1.5708 | 1.5708 | 1.5708 |
| 1 | 0.01745 | 1.5707 | 1.5707 | 1.5707 |
| 2 | 0.03490 | 1.5703 | 1.5703 | 1.5703 |
| 3 | 0.05234 | 1.5697 | 1.5697 | 1.5697 |
| 4 | 0.06976 | 1.5689 | 1.5689 | 1.5689 |
| 5 | 0.08716 | 1.5678 | 1.5678 | 1.5678 |
| 6 | 0.10453 | 1.5665 | 1.5665 | 1.5665 |
| 7 | 0.12187 | 1.5649 | 1.5649 | 1.5649 |
| 8 | 0.13917 | 1.5632 | 1.5632 | 1.5632 |
| 9 | 0.15643 | 1.5611 | 1.5611 | 1.5611 |
| 10 | 0.17365 | 1.5589 | 1.5589 | 1.5589 |
| 11 | 0.19081 | 1.5564 | 1.5564 | 1.5564 |
| 12 | 0.20791 | 1.5537 | 1.5537 | 1.5537 |
| 13 | 0.22495 | 1.5507 | 1.5507 | 1.5507 |
| 14 | 0.24192 | 1.5476 | 1.5476 | 1.5476 |
| 15 | 0.25882 | 1.5442 | 1.5442 | 1.5442 |
| 16 | 0.27564 | 1.5405 | 1.5405 | 1.5405 |
| 17 | 0.29237 | 1.5367 | 1.5367 | 1.5367 |
| 18 | 0.30902 | 1.5326 | 1.5326 | 1.5326 |
| 19 | 0.32557 | 1.5283 | 1.5283 | 1.5283 |
| 20 | 0.34202 | 1.5238 | 1.5238 | 1.5238 |
| 21 | 0.35837 | 1.5191 | 1.5191 | 1.5191 |
| 22 | 0.37461 | 1.5141 | 1.5141 | 1.5141 |
| 23 | 0.39073 | 1.5090 | 1.5090 | 1.5090 |
| 24 | 0.40674 | 1.5037 | 1.5037 | 1.5037 |
| 25 | 0.42262 | 1.4981 | 1.4981 | 1.4981 |
| 26 | 0.43837 | 1.4924 | 1.4924 | 1.4924 |
| 27 | 0.45399 | 1.4864 | 1.4864 | 1.4864 |
| 28 | 0.46947 | 1.4803 | 1.4803 | 1.4803 |
| 29 | 0.48481 | 1.4740 | 1.4740 | 1.4740 |
| 30 | 0.50000 | 1.4675 | 1.4675 | 1.4675 |
| 31 | 0.51504 | 1.4608 | 1.4608 | 1.4608 |
| 32 | 0.52992 | 1.4539 | 1.4539 | 1.4539 |
| 33 | 0.54464 | 1.4469 | 1.4469 | 1.4469 |
| 34 | 0.55919 | 1.4397 | 1.4397 | 1.4397 |
| 35 | 0.57358 | 1.4323 | 1.4323 | 1.4323 |
| 36 | 0.58779 | 1.4248 | 1.4248 | 1.4248 |
| 37 | 0.60182 | 1.4171 | 1.4171 | 1.4171 |
| 38 | 0.61566 | 1.4092 | 1.4093 | 1.4092 |
| 39 | 0.62932 | 1.4013 | 1.4013 | 1.4013 |
| 40 | 0.64279 | 1.3931 | 1.3932 | 1.3931 |
| 41 | 0.65606 | 1.3849 | 1.3849 | 1.3849 |

Table 2. Values of the functions E (part two)

| | | | | |
|---|---|---|---|---|
| 42 | 0.66913 | 1.3765 | 1.3765 | 1.3765 |
| 43 | 0.68200 | 1.3680 | 1.3680 | 1.3680 |
| 44 | 0.69466 | 1.3594 | 1.3594 | 1.3594 |
| 45 | 0.70711 | 1.3506 | 1.3507 | 1.3506 |
| 46 | 0.71934 | 1.3418 | 1.3419 | 1.3418 |
| 47 | 0.73135 | 1.3329 | 1.3330 | 1.3329 |
| 48 | 0.74314 | 1.3238 | 1.3239 | 1.3238 |
| 49 | 0.75471 | 1.3147 | 1.3148 | 1.3147 |
| 50 | 0.76604 | 1.3055 | 1.3057 | 1.3055 |
| 51 | 0.77715 | 1.2963 | 1.2964 | 1.2963 |
| 52 | 0.78801 | 1.2870 | 1.2872 | 1.2870 |
| 53 | 0.79864 | 1.2776 | 1.2778 | 1.2776 |
| 54 | 0.80902 | 1.2681 | 1.2684 | 1.2681 |
| 55 | 0.81915 | 1.2587 | 1.2590 | 1.2587 |
| 56 | 0.82904 | 1.2492 | 1.2496 | 1.2492 |
| 57 | 0.83867 | 1.2397 | 1.2401 | 1.2397 |
| 58 | 0.84805 | 1.2301 | 1.2307 | 1.2301 |
| 59 | 0.85717 | 1.2206 | 1.2212 | 1.2206 |
| 60 | 0.86603 | 1.2111 | 1.2118 | 1.2111 |
| 61 | 0.87462 | 1.2015 | 1.2024 | 1.2015 |
| 62 | 0.88295 | 1.1920 | 1.1930 | 1.1920 |
| 63 | 0.89101 | 1.1826 | 1.1838 | 1.1826 |
| 64 | 0.89879 | 1.1732 | 1.1745 | 1.1732 |
| 65 | 0.90631 | 1.1638 | 1.1654 | 1.1638 |
| 66 | 0.91355 | 1.1545 | 1.1564 | 1.1545 |
| 67 | 0.92050 | 1.1453 | 1.1475 | 1.1453 |
| 68 | 0.92718 | 1.1362 | 1.1387 | 1.1362 |
| 69 | 0.93358 | 1.1272 | 1.1301 | 1.1273 |
| 70 | 0.93969 | 1.1184 | 1.1217 | 1.1184 |
| 70.5 | 0.94264 | 1.1140 | 1.1176 | 1.1140 |
| 71 | 0.94552 | 1.1096 | 1.1135 | 1.1096 |
| 71.5 | 0.94832 | 1.1053 | | 1.1053 |
| 72 | 0.95106 | 1.1011 | | 1.1011 |
| 72.5 | 0.95372 | 1.0968 | | 1.0968 |
| 73 | 0.95630 | 1.0927 | | 1.0927 |
| 73.5 | 0.95882 | 1.0885 | | 1.0885 |
| 74 | 0.96126 | 1.0844 | | 1.0844 |
| 74.5 | 0.96363 | 1.0804 | | 1.0804 |
| 75 | 0.96593 | 1.0764 | | 1.0764 |
| 75.5 | 0.96815 | 1.0725 | | 1.0725 |
| 76 | 0.97030 | 1.0686 | | 1.0686 |
| 76.5 | 0.97237 | 1.0648 | | 1.0648 |
| 77 | 0.97437 | 1.0611 | | 1.0611 |
| 77.5 | 0.97630 | 1.0574 | | 1.0574 |
| 78 | 0.97815 | 1.0538 | | 1.0538 |
| 78.5 | 0.97992 | 1.0502 | | 1.0503 |
| 79 | 0.98163 | 1.0468 | | 1.0468 |
| 79.5 | 0.98325 | 1.0434 | | 1.0435 |
| 80 | 0.98481 | 1.0401 | | 1.0402 |
| 80.2 | 0.98541 | 1.0388 | | 1.0389 |
| 80.4 | 0.98600 | 1.0375 | | 1.0376 |
| 80.6 | 0.98657 | 1.0363 | | 1.0364 |
| 80.8 | 0.98714 | 1.0350 | | 1.0351 |
| 81 | 0.98769 | 1.0338 | | 1.0339 |
| 81.2 | 0.98823 | 1.0326 | | 1.0327 |
| 81.4 | 0.98876 | 1.0314 | | 1.0315 |
| 81.6 | 0.98927 | 1.0302 | | 1.0303 |
| 81.8 | 0.98978 | 1.0290 | | 1.0292 |
| 82 | 0.99027 | 1.0278 | | 1.0280 |
| 82.2 | 0.99075 | 1.0267 | | 1.0269 |
| 82.4 | 0.99122 | 1.0256 | | 1.0258 |
| 82.6 | 0.99167 | 1.0245 | | 1.0247 |
| 82.8 | 0.99211 | 1.0234 | | 1.0236 |
| 83 | 0.99255 | 1.0223 | | 1.0226 |
| 83.2 | 0.99297 | 1.0213 | | 1.0215 |
| 83.4 | 0.99337 | 1.0202 | | 1.0205 |
| 83.6 | 0.99377 | 1.0192 | false min. | 1.0196 |
| 83.8 | 0.99415 | 1.0182 | | 1.0186 |
| 84 | 0.99452 | 1.0172 | | 1.0176 |
| 84.2 | 0.99488 | 1.0163 | | 1.0167 |
| 84.4 | 0.99523 | 1.0153 | | 1.0158 |
| 84.6 | 0.99556 | 1.0144 | | 1.0150 |
| 84.8 | 0.99588 | 1.0135 | | 1.0141 |
| 85 | 0.99619 | 1.0127 | | 1.0133 |
| 85.2 | 0.99649 | 1.0118 | | 1.0125 |
| 85.4 | 0.99678 | 1.0110 | | 1.0118 |
| 85.6 | 0.99705 | 1.0102 | | 1.0110 |
| 85.8 | 0.99731 | 1.0094 | | 1.0103 |
| 86 | 0.99756 | 1.0086 | | 1.0097 |
| 86.2 | 0.99780 | 1.0079 | | 1.0091 |
| 86.4 | 0.99803 | 1.0072 | | 1.0085 |
| 86.6 | 0.99824 | 1.0065 | | 1.0080 |
| 86.8 | 0.99844 | 1.0059 | | 1.0075 |
| 87 | 0.99863 | 1.0053 | | 1.0071 |
| 87.2 | 0.99881 | 1.0047 | | 1.0067 |
| 87.4 | 0.99897 | 1.0041 | | 1.0064 |
| 87.6 | 0.99912 | 1.0036 | | 1.0062 |
| 87.8 | 0.99926 | 1.0031 | | 1.0060 |
| 88 | 0.99939 | 1.0026 | | 1.0060 |
| 88.2 | 0.99951 | 1.0021 | | 1.0061 |
| 88.4 | 0.99961 | 1.0017 | | |
| 88.6 | 0.99970 | 1.0014 | | |
| 88.8 | 0.99978 | 1.0010 | | |
| 89 | 0.99985 | 1.0008 | | |
| 89.1 | 0.99988 | 1.0006 | | |
| 89.2 | 0.99990 | 1.0005 | | |
| 89.3 | 0.99993 | 1.0004 | | |
| 89.4 | 0.99995 | 1.0003 | | |
| 89.5 | 0.99996 | 1.0002 | | |
| 89.6 | 0.99998 | 1.0001 | | |
| 89.7 | 0.99999 | 1.0001 | | |
| 89.8 | 0.99999 | 1.0000 | | |
| 89.9 | 1.00000 | 1.0000 | | |
| 90 | 1.00000 | 1.0000 | 1.1781 | 1.1781 |

At $\theta = \cos^{-1}(1/9) = 83.62063°$, $E_0(\theta) = \pi/3 = 1.0472$ – false min. In the comparative tables 1 and 2, the 4*D* (four decimal digit) exact values of both Legendre complete elliptic integrals reproduced from special functions tables [6] (tab. 29, p. 117), as well as their 4*D* approximate values obtained by applying the two sets of closed analytic formulas were given (all versus the respective elliptic integrals modulus $k = \sin\theta$). It is to be noticed that both sets of approximate formulas are not given by spline or regression functions, but by asymptotic expressions, these ones having a remarkable simplicity (see, e.g.: the 2nd form of $E_0(k)$, suggested by Peano's law on ellipse's perimeter; *all newly found formulas in k / k′ do not need any mathematical table*, being purely algebraic) and accuracy (see table 3). The identity with the exact functions is satisfied for the left domain's end $k = 0$ ($\theta = 0°$). The 2nd set ($K_1$, $E_1$), although a bit more intricate, gives more accurate values than the 1st one ($K_0$, $E_0$) and arrives more closely to the right domain's end $k = 1$ ($\theta = 90°$).

## III. THE ACCURACY OF THE TWO SETS OF FORMULAS

Let us define the following relative error functions:

$\varepsilon_{K0}(k) = K_0(k)/K(k) - 1$; $\varepsilon_{K1}(k) = K_1(k)/K(k) - 1$,

$\varepsilon_{E0}(k) = E_0(k)/E(k) - 1$; $\varepsilon_{E1}(k) = E_1(k)/E(k) - 1$,

for both sets of approximation of the 1st and 2nd kind integrals, resp. Their values are given in table 3, expressed in thousandths (‰). These errors were calculated for the 1st set ($K_0$, $E_0$) only in the field $\theta \in [54°, 71°]$ of the domain, with an increment of 1°, while for the 2nd set ($K_1$, $E_1$) only in the field $\theta \in [84°.8, 88°.2]$, with an increment of 0°.2, like in tables 1 & 2.

Table 3. Relative errors ε distribution

| $\theta$(°) | $k = \sin\theta$ | $\varepsilon_{K0}$(‰) | $\varepsilon_{K1}$(‰) | $\varepsilon_{E0}$(‰) | $\varepsilon_{E1}$(‰) |
|---|---|---|---|---|---|
| 54 | 0.80902 | − 0.250 | | + 0.255 | |
| 55 | 0.81915 | − 0.272 | | + 0.243 | |
| 56 | 0.82904 | − 0.353 | | + 0.293 | |
| 57 | 0.83867 | − 0.420 | | + 0.334 | |
| 58 | 0.84805 | − 0.497 | | + 0.454 | |
| 59 | 0.85717 | − 0.558 | | + 0.502 | |
| 60 | 0.86603 | − 0.669 | | + 0.566 | |
| 61 | 0.87462 | − 0.799 | | + 0.742 | |
| 62 | 0.88295 | − 0.961 | | + 0.874 | |
| 63 | 0.89101 | − 1.118 | | + 0.973 | |
| 64 | 0.89879 | − 1.366 | | + 1.135 | |
| 65 | 0.90631 | − 1.619 | | + 1.377 | |
| 66 | 0.91355 | − 1.918 | | + 1.627 | |
| 67 | 0.92050 | − 2.299 | | + 1.900 | |
| 68 | 0.92718 | − 2.709 | | + 2.215 | |
| 69 | 0.93358 | − 3.253 | | + 2.573 | |
| 70 | 0.93969 | − 3.907 | | + 2.959 | |
| 71 | 0.94552 | − 4.642 | | + 3.525 | |
| | | - | | - | |
| 84.8 | 0.99588 | - | − 0.369 | - | + 0.607 |
| 85 | 0.99619 | - | − 0.396 | - | + 0.592 |
| 85.2 | 0.99649 | - | − 0.451 | - | + 0.705 |
| 85.4 | 0.99678 | - | − 0.500 | - | + 0.748 |
| 85.6 | 0.99705 | - | − 0.582 | - | + 0.823 |
| 85.8 | 0.99731 | - | − 0.652 | - | + 0.932 |
| 86 | 0.99756 | - | − 0.737 | - | + 1.076 |
| 86.2 | 0.99780 | - | − 0.832 | - | + 1.160 |
| 86.4 | 0.99803 | - | − 0.945 | - | + 1.284 |
| 86.6 | 0.99824 | - | − 1.077 | - | + 1.453 |
| 86.8 | 0.99844 | - | − 1.214 | - | + 1.571 |
| 87 | 0.99863 | - | − 1.421 | - | + 1.743 |
| 87.2 | 0.99881 | - | − 1.626 | - | + 1.976 |
| 87.4 | 0.99897 | - | − 1.894 | - | + 2.275 |
| 87.6 | 0.99912 | - | − 2.234 | - | + 2.553 |
| 87.8 | 0.99926 | - | − 2.655 | - | + 2.922 |
| 88 | 0.99939 | - | − 3.156 | - | + 3.397 |
| 88.2 | 0.99951 | - | − 3.808 | - | + 4.004 |

The relative errors strings are stopped for values ≥ 4 ‰. One can see that both sets given in chapter 2 have a much lesser relative error for $K(k)$ than the well-known asymptotic expression: $K(k) \approx \pi/2 + (\pi/8)[k^2/(1 - k^2)] - (\pi/16)[k^4/(1 - k^4)]$, with a relative precision of $3 \cdot 10^{-4}$ for $k < 0.5$ ($\theta < 30°$), only.

## IV. COMPARATIVE SERIES REPRESENTATIONS; LEGENDRE'S FUNCTIONAL RELATION

Expanding into power series, one obtains for the complete elliptic integrals the set of representations below ([5] – [7]):

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409225}{1073741824}k^{16} + \ldots\right) =$$

$$\frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 k^{2n}\right\} = \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}\right\};$$

$$E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760615}{1073741824}k^{16} - \ldots\right) =$$

$$\frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 \frac{k^{2n}}{2n-1}\right\} = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}\right\}.$$

At $k = 0$: $K(0) = E(0) = \pi/2$; at $k = 1$: $K(1) \uparrow \infty$; $E(1) = 1$. Proceeding in the same manner, we get for the 1st set (the most inaccurate) of approximate functions the expansions

$$K_0(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1222}{16384}k^8 + \ldots\right);$$

$$E_0(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{172}{16384}k^8 - \ldots\right),$$

for the 2nd set being *practically identical with the exact ones*

$$K_1(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + \frac{184041}{4194304}k^{14} + \frac{41409222}{1073741824}k^{16} + \ldots\right);$$

$$E_1(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{16384}k^8 - \frac{441}{65536}k^{10} - \frac{4851}{1048576}k^{12} - \frac{14157}{4194304}k^{14} - \frac{2760606}{1073741824}k^{16} - \ldots\right).$$

The difference with respect to the expansions of the exact functions (K, E) begins at the terms in $k^8$ for the 1st set of approximation ($K_0$, $E_0$), and at the terms in $k^{16}$ for the 2nd one ($K_1$, $E_1$). For the 1st derivatives of K, E we get

$$\frac{d\mathrm{K}(k)}{dk}=\frac{\mathrm{E}(k)}{k(1-k^2)}-\frac{\mathrm{K}(k)}{k}=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6+\frac{19845}{16384}k^8+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409225}{33554432}k^{14}+\right)$$

$$=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2 nk^{2n-1}=\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2 nk^{2n-1};$$

$$\frac{d\mathrm{E}(k)}{dk}=\frac{\mathrm{E}(k)-\mathrm{K}(k)}{k}=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\frac{2205}{16384}k^8+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{2760615}{33554432}k^{14}+\ldots\right)=$$

$$-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot\ldots(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right]^2\frac{nk^{2n-1}}{2n-1}=-\frac{\pi}{4}\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2^{n-1}n!}\right]^2\frac{nk^{2n-1}}{2n-1}.$$

At $k=0$: $d\mathrm{K}/dk=d\mathrm{E}/dk=0$; at $k=1$: $d\mathrm{K}/dk\uparrow\infty$; $d\mathrm{E}/dk\downarrow(-\infty)$. Applying the previous two exact relations and using the four definitions from chapter 2 one gets the expansions:

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_0=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225.75}{1024}k^6+\ldots\right);$$

$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_0=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{174.25}{1024}k^6+\ldots\right),$$

for the 1st set of approximate functions ($\mathrm{K}_0$, $\mathrm{E}_0$), and resp.

$$\left[\frac{d\mathrm{K}(k)}{dk}\right]_1=\frac{\pi}{4}k\left(1+\frac{9}{8}k^2+\frac{75}{64}k^4+\frac{1225}{1024}k^6+\frac{19845}{16384}k^8+\frac{160083}{131072}k^{10}+\frac{1288287}{1048576}k^{12}+\frac{41409226125}{33554432}k^{14}+\ldots\right);$$

$$\left[\frac{d\mathrm{E}(k)}{dk}\right]_1=-\frac{\pi}{4}k\left(1+\frac{3}{8}k^2+\frac{15}{64}k^4+\frac{175}{1024}k^6+\frac{2205}{16384}k^8+\frac{14553}{131072}k^{10}+\frac{99099}{1048576}k^{12}+\frac{276061425}{33554432}k^{14}+\ldots\right),$$

for the 2nd set of approximate functions ($\mathrm{K}_1$, $\mathrm{E}_1$). The difference with respect to the expansions of the 1st derivatives of the exact functions (K, E) begins at the terms in $k^7$ for the 1st set, and at the terms in $k^{15}$ for the 2nd one, so much lesser than that for the expansions of the respective sets ($\mathrm{K}_{0,1}$, $\mathrm{E}_{0,1}$). One can also easily find the analytic expressions and series representations for the 2nd derivatives of all K, $\mathrm{K}_{0,1}$, E, $\mathrm{E}_{0,1}$, with similar results, but a lesser precision than for K, E, K′, E′. Besides the above definitions of the derivatives $\mathrm{K}'(=d\mathrm{K}/dk)$, $\mathrm{E}'(=d\mathrm{E}/dk)$, there is a useful functional relation (Legendre's): $\mathrm{K}(k)\cdot\mathrm{E}(k')+\mathrm{E}(k)\cdot\mathrm{K}(k')-\mathrm{K}(k)\cdot\mathrm{K}(k')=\pi/2$.

## V. GRAPHIC COMPARISON

The variation curves of Legendre complete elliptic integrals, as well as that of the two sets of closed analytic functions are graphically represented in the comparative figures 1 and 2, all vs. $\theta$, in sexagesimal degrees, and given by $\theta=\sin^{-1}k$. In both figures the exact functions K($k$), E($k$) were represented by solid (continuous) black lines, the 1st set of approximation [$\mathrm{K}_0(k)$, $\mathrm{E}_0(k)$] by dashed black lines, and the 2nd set of approximation [$\mathrm{K}_1(k)$, $\mathrm{E}_1(k)$] by solid red lines. At $k=1$ the graphs of all $\mathrm{K}_{0,1}(k)$ fall to $(-\infty)$; the graphs of all $\mathrm{E}_{0,1}(k)$ pass through $(1, 3\pi/8)$.

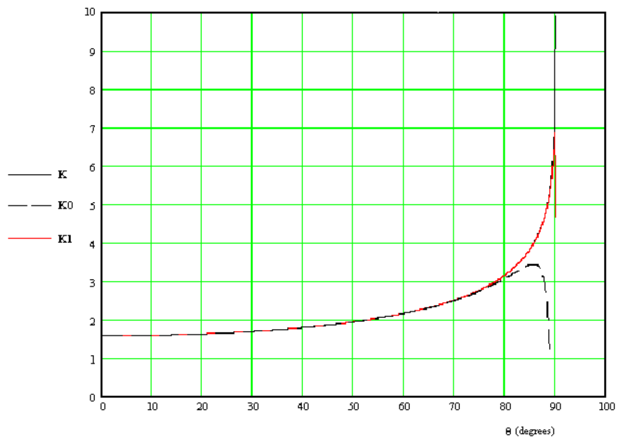


Fig. 1. Comparison of K($k$) with the closed analytic functions $\mathrm{K}_0(k)$, $\mathrm{K}_1(k)$; also see the 2nd part of remark 1 in the appendix

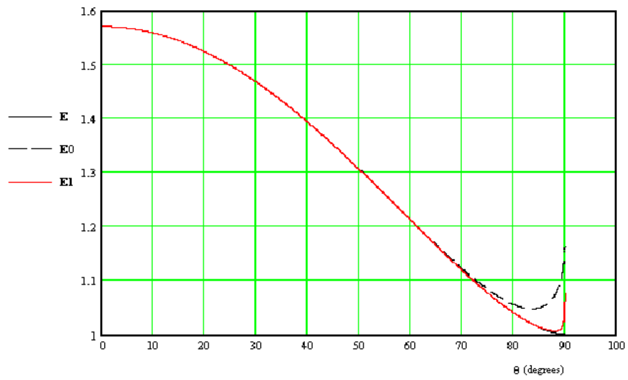


Fig. 2. Comparison of E($k$) with the closed analytic functions $\mathrm{E}_0(k)$, $\mathrm{E}_1(k)$; also see the 2nd part of remark 1 in the appendix

## VI. CONCLUSIONS

As for simplicity, the formulas in $k/k'$ do not need mathematical tables (are purely algebraic). As for accuracy, in mathematical/technical applications, it must use the 1st set until $\theta=70°.5$ ($k=0.94264$) only, and (for a better accuracy or a greater upper limit of the validity domain) the 2nd set, until $\theta=88°.2$ ($k=0.99951$).

## VII. NOTES; OTHER METHODS; FUTURE RESEARCH

Without the comparative tables 1 and 2, the errors table becoming so table 1, this work was published previously in a proceedings volume (scientific bulletin), in Romanian [8]. For the first English version of this work see [9], [10]. Approximations for the complete elliptic integrals based on the trapezoidal-type numerical integration formulas discussed in [11], are developed in [12], [13] (a mixed numerical-analytic method). Newer formulas (using Γ function – not an elementary, but a special one, like K & E, even if these formulas are the most accurate) are in [14], [15]; as stated in their abstracts, the works [9], [14] do not have the same goal. An *original fast converging recurrent-iterative scheme* to get a 3rd (and higher) set of closed analytic formulas (seemingly intricate) with desired accuracy is given in article's appendix. This article represents a fully extended version of the paper [9]. Notable *special functions* suitable for applying such an approximate method of calculation are: Si(x); Ci(x); Ei(x); li(x).

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## APPENDIX – A FAST CONVERGING RECURRENTITERATIVE SCHEME TO GET A 3RD (AND HIGHER) SET OF ANALYTIC FORMULAS WITH DESIRED ACCURACY

The formulas for transforming the modulus (Landen, [16]-[18]) are:

$$\mathrm{K}(k)=\frac{2}{1+\sqrt{1-k^2}}\mathrm{K}\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)=\frac{2}{1+k'}\mathrm{K}\left(\frac{1-k'}{1+k'}\right),$$

$$\text{or}: \mathrm{K}(\theta)=\mathrm{K}[\tan^2(\theta/2)]/\cos^2(\theta/2), \text{ and, respectively}:$$

$$\mathrm{E}(k)=(1+\sqrt{1-k^2})\mathrm{E}\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)-\sqrt{1-k^2}\,\mathrm{K}(k)=$$

$$(1+k')\mathrm{E}[(1-k')/(1+k')]-k'\mathrm{K}(k), \text{ with } k'=(1-k^2)^{1/2},$$

$$\text{or}: \mathrm{E}(\theta)=2\cos^2(\theta/2)\cdot\mathrm{E}[\tan^2(\theta/2)]-\cos\theta\cdot\mathrm{K}(\theta),$$

(passing from $k$ to $k_1=(1-k')/(1+k')\le k$ and from $\theta$ to $\theta_1=\sin^{-1}[\tan^2(\theta/2)]\le\theta$; $k_1=k$ $(\theta_1=\theta)$, for: $k=0;\ 1$ $(\theta=0;\ \pi/2)$), which can be transcribed in recurrent form, as follows:

$$\mathrm{K}_2(k)=\frac{2}{1+\sqrt{1-k^2}}\mathrm{K}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)=\frac{2}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right),$$

$$\text{or}: \mathrm{K}_2(\theta)=\mathrm{K}_1[\tan^2(\theta/2)]/\cos^2(\theta/2), \text{ and, respectively}:$$

$$\mathrm{E}_2(k)=(1+\sqrt{1-k^2})\mathrm{E}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)-\sqrt{1-k^2}\,\mathrm{K}_2(k)=$$

$$=(1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right)-\frac{2k'}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right), \quad \text{or}: \quad \mathrm{E}_2(\theta)=$$

$$2\cos^2(\theta/2)\mathrm{E}_1[\tan^2(\theta/2)]-[\cos\theta/\cos^2(\theta/2)]\mathrm{K}_1[\tan^2(\theta/2)],$$

expressing the 3rd set ($\mathrm{K}_2$, $\mathrm{E}_2$) in function of the 2nd one ($\mathrm{K}_1$, $\mathrm{E}_1$), so starting a recurrent-iterative scheme; it allows writing

$$\text{for the } (\mathrm{n}+1)^{\text{th}} \text{ set}: \quad \mathrm{K}_\mathrm{n}(k)=\frac{2}{1+k'}\mathrm{K}_{\mathrm{n}-1}\left(\frac{1-k'}{1+k'}\right), \text{and}$$

$$\mathrm{E}_\mathrm{n}(k)=(1+k')\mathrm{E}_{\mathrm{n}-1}\left(\frac{1-k'}{1+k'}\right)-\frac{2k'}{1+k'}\mathrm{K}_{\mathrm{n}-1}\left(\frac{1-k'}{1+k'}\right), \text{resp.}$$

Starting from the newly found closed analytic formulas, which connect the 3rd set ($K_2$, $E_2$) with the 2nd one ($K_1$, $E_1$), by applying the new recurrent-iterative scheme previously presented, the comparative tables 1 and 2 from chapter 2 were remade, suppressing the column "$k = \sin \theta$", and inserting the new columns "$K_2(k)$" and "$E_2(k)$" with 4*D* approximate values, resp., so getting the new tables 4 and 5, given below, resp., keeping for comparison the columns "$\theta$(°)", "K(*k*)", "$K_0(k)$" and "$K_1(k)$" (in table 4), and "$\theta$(°)", "E(*k*)", "$E_0(k)$" and "$E_1(k)$" (in table 5), resp.

Table 4. Values of the functions K (part one)
(this table completes and replaces table 1)

| $\theta$(°) | K(*k*) | $K_0(k)$ | $K_1(k)$ | $K_2(k)$ |
|---|---|---|---|---|
| 0 | 1.5708 | 1.5708 | 1.5708 | 1.5708 |
| 1 | 1.5709 | 1.5709 | 1.5709 | 1.5709 |
| 2 | 1.5713 | 1.5713 | 1.5713 | 1.5713 |
| 3 | 1.5719 | 1.5719 | 1.5719 | 1.5719 |
| 4 | 1.5727 | 1.5727 | 1.5727 | 1.5727 |
| 5 | 1.5738 | 1.5738 | 1.5738 | 1.5738 |
| 6 | 1.5751 | 1.5751 | 1.5751 | 1.5751 |
| 7 | 1.5767 | 1.5767 | 1.5767 | 1.5767 |
| 8 | 1.5785 | 1.5785 | 1.5785 | 1.5785 |
| 9 | 1.5805 | 1.5805 | 1.5805 | 1.5805 |
| 10 | 1.5828 | 1.5828 | 1.5828 | 1.5828 |
| 11 | 1.5854 | 1.5854 | 1.5854 | 1.5854 |
| 12 | 1.5882 | 1.5882 | 1.5882 | 1.5882 |
| 13 | 1.5913 | 1.5913 | 1.5913 | 1.5913 |
| 14 | 1.5946 | 1.5946 | 1.5946 | 1.5946 |
| 15 | 1.5981 | 1.5981 | 1.5981 | 1.5981 |
| 16 | 1.6020 | 1.6020 | 1.6020 | 1.6020 |
| 17 | 1.6061 | 1.6061 | 1.6061 | 1.6061 |
| 18 | 1.6105 | 1.6105 | 1.6105 | 1.6105 |
| 19 | 1.6151 | 1.6151 | 1.6151 | 1.6151 |
| 20 | 1.6200 | 1.6200 | 1.6200 | 1.6200 |
| 21 | 1.6252 | 1.6252 | 1.6252 | 1.6252 |
| 22 | 1.6307 | 1.6307 | 1.6307 | 1.6307 |
| 23 | 1.6365 | 1.6365 | 1.6365 | 1.6365 |
| 24 | 1.6426 | 1.6426 | 1.6426 | 1.6426 |
| 25 | 1.6490 | 1.6490 | 1.6490 | 1.6490 |
| 26 | 1.6557 | 1.6557 | 1.6557 | 1.6557 |
| 27 | 1.6627 | 1.6627 | 1.6627 | 1.6627 |
| 28 | 1.6701 | 1.6701 | 1.6701 | 1.6701 |
| 29 | 1.6777 | 1.6777 | 1.6777 | 1.6777 |
| 30 | 1.6858 | 1.6857 | 1.6858 | 1.6858 |
| 31 | 1.6941 | 1.6941 | 1.6941 | 1.6941 |
| 32 | 1.7028 | 1.7028 | 1.7028 | 1.7028 |
| 33 | 1.7119 | 1.7119 | 1.7119 | 1.7119 |
| 34 | 1.7214 | 1.7214 | 1.7214 | 1.7214 |
| 35 | 1.7312 | 1.7312 | 1.7312 | 1.7312 |
| 36 | 1.7415 | 1.7415 | 1.7415 | 1.7415 |
| 37 | 1.7522 | 1.7522 | 1.7522 | 1.7522 |
| 38 | 1.7633 | 1.7632 | 1.7633 | 1.7633 |
| 39 | 1.7748 | 1.7748 | 1.7748 | 1.7748 |
| 40 | 1.7868 | 1.7867 | 1.7868 | 1.7868 |
| 41 | 1.7992 | 1.7992 | 1.7992 | 1.7992 |
| 42 | 1.8122 | 1.8121 | 1.8122 | 1.8122 |
| 43 | 1.8256 | 1.8256 | 1.8256 | 1.8256 |
| 44 | 1.8396 | 1.8395 | 1.8396 | 1.8396 |
| 45 | 1.8541 | 1.8540 | 1.8541 | 1.8541 |
| 46 | 1.8691 | 1.8691 | 1.8691 | 1.8691 |
| 47 | 1.8848 | 1.8847 | 1.8848 | 1.8848 |
| 48 | 1.9011 | 1.9009 | 1.9011 | 1.9011 |
| 49 | 1.9180 | 1.9178 | 1.9180 | 1.9180 |
| 50 | 1.9356 | 1.9354 | 1.9356 | 1.9356 |
| 51 | 1.9539 | 1.9536 | 1.9539 | 1.9539 |
| 52 | 1.9729 | 1.9726 | 1.9729 | 1.9729 |
| 53 | 1.9927 | 1.9923 | 1.9927 | 1.9927 |
| 54 | 2.0133 | 2.0128 | 2.0133 | 2.0133 |
| 55 | 2.0347 | 2.0341 | 2.0347 | 2.0347 |
| 56 | 2.0571 | 2.0564 | 2.0571 | 2.0571 |
| 57 | 2.0804 | 2.0795 | 2.0804 | 2.0804 |
| 58 | 2.1047 | 2.1037 | 2.1047 | 2.1047 |
| 59 | 2.1300 | 2.1288 | 2.1300 | 2.1300 |
| 60 | 2.1565 | 2.1551 | 2.1565 | 2.1565 |
| 61 | 2.1842 | 2.1825 | 2.1842 | 2.1842 |
| 62 | 2.2132 | 2.2111 | 2.2132 | 2.2132 |
| 63 | 2.2435 | 2.2410 | 2.2435 | 2.2435 |
| 64 | 2.2754 | 2.2723 | 2.2754 | 2.2754 |
| 65 | 2.3088 | 2.3051 | 2.3088 | 2.3088 |
| 66 | 2.3439 | 2.3394 | 2.3439 | 2.3439 |
| 67 | 2.3809 | 2.3754 | 2.3809 | 2.3809 |
| 68 | 2.4198 | 2.4132 | 2.4198 | 2.4198 |
| 69 | 2.4610 | 2.4530 | 2.4610 | 2.4610 |
| 70 | 2.5046 | 2.4948 | 2.5045 | 2.5046 |
| 70.5 | 2.5273 | 2.5165 | 2.5273 | 2.5273 |
| 71 | 2.5507 | 2.5389 | 2.5507 | 2.5507 |
| 71.5 | 2.5749 | | 2.5749 | 2.5749 |
| 72 | 2.5998 | | 2.5998 | 2.5998 |
| 72.5 | 2.6256 | | 2.6255 | 2.6256 |
| 73 | 2.6521 | | 2.6521 | 2.6521 |
| 73.5 | 2.6796 | | 2.6796 | 2.6796 |
| 74 | 2.7081 | | 2.7081 | 2.7081 |
| 74.5 | 2.7375 | | 2.7375 | 2.7375 |
| 75 | 2.7681 | | 2.7680 | 2.7681 |
| 75.5 | 2.7998 | | 2.7997 | 2.7998 |
| 76 | 2.8327 | | 2.8326 | 2.8327 |
| 76.5 | 2.8669 | | 2.8669 | 2.8669 |
| 77 | 2.9026 | | 2.9025 | 2.9026 |
| 77.5 | 2.9397 | | 2.9397 | 2.9397 |
| 78 | 2.9786 | | 2.9785 | 2.9786 |
| 78.5 | 3.0192 | | 3.0191 | 3.0192 |
| 79 | 3.0617 | | 3.0616 | 3.0617 |
| 79.5 | 3.1064 | | 3.1063 | 3.1064 |
| 80 | 3.1534 | | 3.1533 | 3.1534 |
| 80.2 | 3.1729 | | 3.1727 | 3.1729 |
| 80.4 | 3.1928 | | 3.1927 | 3.1928 |
| 80.6 | 3.2132 | | 3.2130 | 3.2132 |
| 80.8 | 3.2340 | | 3.2338 | 3.2340 |
| 81 | 3.2553 | | 3.2551 | 3.2553 |

Table 4. Values of the functions K (part two)

| | | | | |
|---|---|---|---|---|
| 81.2 | 3.2771 | | 3.2769 | 3.2771 |
| 81.4 | 3.2995 | | 3.2992 | 3.2995 |
| 81.6 | 3.3223 | | 3.3221 | 3.3223 |
| 81.8 | 3.3458 | | 3.3455 | 3.3458 |
| 82 | 3.3699 | | 3.3696 | 3.3699 |
| 82.2 | 3.3946 | | 3.3942 | 3.3946 |
| 82.4 | 3.4199 | | 3.4196 | 3.4199 |
| 82.6 | 3.4460 | | 3.4456 | 3.4460 |
| 82.8 | 3.4728 | | 3.4724 | 3.4728 |
| 83 | 3.5004 | | 3.4999 | 3.5004 |
| 83.2 | 3.5288 | | 3.5283 | 3.5288 |
| 83.4 | 3.5581 | | 3.5575 | 3.5581 |
| 83.6 | 3.5884 | | 3.5877 | 3.5884 |
| 83.8 | 3.6196 | | 3.6188 | 3.6196 |
| 84 | 3.6519 | | 3.6510 | 3.6519 |
| 84.2 | 3.6852 | | 3.6843 | 3.6852 |
| 84.4 | 3.7198 | | 3.7187 | 3.7198 |
| 84.6 | 3.7557 | | 3.7545 | 3.7557 |
| 84.8 | 3.7930 | | 3.7916 | 3.7930 |
| 85 | 3.8317 | | 3.8302 | 3.8317 |
| 85.2 | 3.8721 | | 3.8704 | 3.8721 |
| 85.4 | 3.9142 | | 3.9122 | 3.9142 |
| 85.6 | 3.9583 | | 3.9560 | 3.9583 |
| 85.8 | 4.0044 | | 4.0018 | 4.0044 |
| 86 | 4.0528 | | 4.0498 | 4.0528 |
| 86.2 | 4.1037 | | 4.1003 | 4.1037 |
| 86.4 | 4.1574 | | 4.1535 | 4.1574 |
| 86.6 | 4.2142 | | 4.2097 | 4.2142 |
| 86.8 | 4.2744 | | 4.2692 | 4.2744 |
| 87 | 4.3387 | | 4.3325 | 4.3387 |
| 87.2 | 4.4073 | | 4.4001 | 4.4073 |
| 87.4 | 4.4811 | | 4.4726 | 4.4811 |
| 87.6 | 4.5609 | | 4.5507 | 4.5609 |
| 87.8 | 4.6477 | | 4.6354 | 4.6477 |
| 88 | 4.7427 | | 4.7277 | 4.7427 |
| 88.2 | 4.8478 | | 4.8293 | 4.8478 |
| 88.4 | 4.9654 | | | 4.9654 |
| 88.6 | 5.0988 | | | 5.0987 |
| 88.8 | 5.2527 | | | 5.2527 |
| 89 | 5.4349 | | | 5.4349 |
| 89.1 | 5.5402 | | | 5.5402 |
| 89.2 | 5.6579 | | | 5.6579 |
| 89.3 | 5.7914 | | | 5.7913 |
| 89.4 | 5.9455 | | | 5.9454 |
| 89.5 | 6.1278 | | | 6.1276 |
| 89.6 | 6.3509 | | | 6.3506 |
| 89.7 | 6.6385 | | | 6.6380 |
| 89.8 | 7.0440 | | | 7.0428 |
| 89.9 | 7.7371 | | | 7.7336 |
| 90 | ∞ | – ∞ | – ∞ | – ∞ |

The values string in the last column is given by:

$$\mathrm{K}_2(k)=\frac{2}{1+\sqrt{1-k^2}}\mathrm{K}_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)=\frac{2}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right),$$

$$\text{with: }\ \mathrm{K}_1(k_1)=\frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right)=$$

$$=\frac{\pi\sqrt{2}}{\sqrt{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}{\sqrt[4]{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\right),$$

and finally the algebraic formula: $\mathrm{K}_2(k) = 2\mathrm{K}_1(k_1)/(1 + k')$.

Table 5. Values of the functions E (part one)
(this table completes and replaces table 2)

| $\theta$ (°) | E($k$) | $E_0(k)$ | $E_1(k)$ | $E_2(k)$ |
|---|---|---|---|---|
| 0 | 1.5708 | 1.5708 | 1.5708 | 1.5708 |
| 1 | 1.5707 | 1.5707 | 1.5707 | 1.5707 |
| 2 | 1.5703 | 1.5703 | 1.5703 | 1.5703 |
| 3 | 1.5697 | 1.5697 | 1.5697 | 1.5697 |
| 4 | 1.5689 | 1.5689 | 1.5689 | 1.5689 |
| 5 | 1.5678 | 1.5678 | 1.5678 | 1.5678 |
| 6 | 1.5665 | 1.5665 | 1.5665 | 1.5665 |
| 7 | 1.5649 | 1.5649 | 1.5649 | 1.5649 |
| 8 | 1.5632 | 1.5632 | 1.5632 | 1.5632 |
| 9 | 1.5611 | 1.5611 | 1.5611 | 1.5611 |
| 10 | 1.5589 | 1.5589 | 1.5589 | 1.5589 |
| 11 | 1.5564 | 1.5564 | 1.5564 | 1.5564 |
| 12 | 1.5537 | 1.5537 | 1.5537 | 1.5537 |
| 13 | 1.5507 | 1.5507 | 1.5507 | 1.5507 |
| 14 | 1.5476 | 1.5476 | 1.5476 | 1.5476 |
| 15 | 1.5442 | 1.5442 | 1.5442 | 1.5442 |
| 16 | 1.5405 | 1.5405 | 1.5405 | 1.5405 |
| 17 | 1.5367 | 1.5367 | 1.5367 | 1.5367 |
| 18 | 1.5326 | 1.5326 | 1.5326 | 1.5326 |
| 19 | 1.5283 | 1.5283 | 1.5283 | 1.5283 |
| 20 | 1.5238 | 1.5238 | 1.5238 | 1.5238 |
| 21 | 1.5191 | 1.5191 | 1.5191 | 1.5191 |
| 22 | 1.5141 | 1.5141 | 1.5141 | 1.5141 |
| 23 | 1.5090 | 1.5090 | 1.5090 | 1.5090 |
| 24 | 1.5037 | 1.5037 | 1.5037 | 1.5037 |
| 25 | 1.4981 | 1.4981 | 1.4981 | 1.4981 |
| 26 | 1.4924 | 1.4924 | 1.4924 | 1.4924 |
| 27 | 1.4864 | 1.4864 | 1.4864 | 1.4864 |
| 28 | 1.4803 | 1.4803 | 1.4803 | 1.4803 |
| 29 | 1.4740 | 1.4740 | 1.4740 | 1.4740 |
| 30 | 1.4675 | 1.4675 | 1.4675 | 1.4675 |
| 31 | 1.4608 | 1.4608 | 1.4608 | 1.4608 |
| 32 | 1.4539 | 1.4539 | 1.4539 | 1.4539 |
| 33 | 1.4469 | 1.4469 | 1.4469 | 1.4469 |
| 34 | 1.4397 | 1.4397 | 1.4397 | 1.4397 |
| 35 | 1.4323 | 1.4323 | 1.4323 | 1.4323 |
| 36 | 1.4248 | 1.4248 | 1.4248 | 1.4248 |
| 37 | 1.4171 | 1.4171 | 1.4171 | 1.4171 |
| 38 | 1.4092 | 1.4093 | 1.4092 | 1.4092 |
| 39 | 1.4013 | 1.4013 | 1.4013 | 1.4013 |
| 40 | 1.3931 | 1.3932 | 1.3931 | 1.3931 |

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Table 5. Values of the functions E (part two)

| | | | | |
|---|---|---|---|---|
| 41 | 1.3849 | 1.3849 | 1.3849 | 1.3849 |
| 42 | 1.3765 | 1.3765 | 1.3765 | 1.3765 |
| 43 | 1.3680 | 1.3680 | 1.3680 | 1.3680 |
| 44 | 1.3594 | 1.3594 | 1.3594 | 1.3594 |
| 45 | 1.3506 | 1.3507 | 1.3506 | 1.3506 |
| 46 | 1.3418 | 1.3419 | 1.3418 | 1.3418 |
| 47 | 1.3329 | 1.3330 | 1.3329 | 1.3329 |
| 48 | 1.3238 | 1.3239 | 1.3238 | 1.3238 |
| 49 | 1.3147 | 1.3148 | 1.3147 | 1.3147 |
| 50 | 1.3055 | 1.3057 | 1.3055 | 1.3055 |
| 51 | 1.2963 | 1.2964 | 1.2963 | 1.2963 |
| 52 | 1.2870 | 1.2872 | 1.2870 | 1.2870 |
| 53 | 1.2776 | 1.2778 | 1.2776 | 1.2776 |
| 54 | 1.2681 | 1.2684 | 1.2681 | 1.2681 |
| 55 | 1.2587 | 1.2590 | 1.2587 | 1.2587 |
| 56 | 1.2492 | 1.2496 | 1.2492 | 1.2492 |
| 57 | 1.2397 | 1.2401 | 1.2397 | 1.2397 |
| 58 | 1.2301 | 1.2307 | 1.2301 | 1.2301 |
| 59 | 1.2206 | 1.2212 | 1.2206 | 1.2206 |
| 60 | 1.2111 | 1.2118 | 1.2111 | 1.2111 |
| 61 | 1.2015 | 1.2024 | 1.2015 | 1.2015 |
| 62 | 1.1920 | 1.1930 | 1.1920 | 1.1920 |
| 63 | 1.1826 | 1.1838 | 1.1826 | 1.1826 |
| 64 | 1.1732 | 1.1745 | 1.1732 | 1.1732 |
| 65 | 1.1638 | 1.1654 | 1.1638 | 1.1638 |
| 66 | 1.1545 | 1.1564 | 1.1545 | 1.1545 |
| 67 | 1.1453 | 1.1475 | 1.1453 | 1.1453 |
| 68 | 1.1362 | 1.1387 | 1.1362 | 1.1362 |
| 69 | 1.1272 | 1.1301 | 1.1273 | 1.1272 |
| 70 | 1.1184 | 1.1217 | 1.1184 | 1.1184 |
| 70.5 | 1.1140 | 1.1176 | 1.1140 | 1.1140 |
| 71 | 1.1096 | 1.1135 | 1.1096 | 1.1096 |
| 71.5 | 1.1053 | | 1.1053 | 1.1053 |
| 72 | 1.1011 | | 1.1011 | 1.1011 |
| 72.5 | 1.0968 | | 1.0968 | 1.0968 |
| 73 | 1.0927 | | 1.0927 | 1.0927 |
| 73.5 | 1.0885 | | 1.0885 | 1.0885 |
| 74 | 1.0844 | | 1.0844 | 1.0844 |
| 74.5 | 1.0804 | | 1.0804 | 1.0804 |
| 75 | 1.0764 | | 1.0764 | 1.0764 |
| 75.5 | 1.0725 | | 1.0725 | 1.0725 |
| 76 | 1.0686 | | 1.0686 | 1.0686 |
| 76.5 | 1.0648 | | 1.0648 | 1.0648 |
| 77 | 1.0611 | | 1.0611 | 1.0611 |
| 77.5 | 1.0574 | | 1.0574 | 1.0574 |
| 78 | 1.0538 | | 1.0538 | 1.0538 |
| 78.5 | 1.0502 | | 1.0503 | 1.0502 |
| 79 | 1.0468 | | 1.0468 | 1.0468 |
| 79.5 | 1.0434 | | 1.0435 | 1.0434 |
| 80 | 1.0401 | | 1.0402 | 1.0401 |
| 80.2 | 1.0388 | | 1.0389 | 1.0388 |
| 80.4 | 1.0375 | | 1.0376 | 1.0375 |
| 80.6 | 1.0363 | | 1.0364 | 1.0363 |
| 80.8 | 1.0350 | | 1.0351 | 1.0350 |

| | | | | |
|---|---|---|---|---|
| 81 | 1.0338 | | 1.0339 | 1.0338 |
| 81.2 | 1.0326 | | 1.0327 | 1.0326 |
| 81.4 | 1.0314 | | 1.0315 | 1.0314 |
| 81.6 | 1.0302 | | 1.0303 | 1.0302 |
| 81.8 | 1.0290 | | 1.0292 | 1.0290 |
| 82 | 1.0278 | | 1.0280 | 1.0278 |
| 82.2 | 1.0267 | | 1.0269 | 1.0267 |
| 82.4 | 1.0256 | | 1.0258 | 1.0256 |
| 82.6 | 1.0245 | | 1.0247 | 1.0245 |
| 82.8 | 1.0234 | | 1.0236 | 1.0234 |
| 83 | 1.0223 | | 1.0226 | 1.0223 |
| 83.2 | 1.0213 | | 1.0215 | 1.0213 |
| 83.4 | 1.0202 | | 1.0205 | 1.0202 |
| 83.6 | 1.0192 | false min. | 1.0196 | 1.0192 |
| 83.8 | 1.0182 | | 1.0186 | 1.0182 |
| 84 | 1.0172 | | 1.0176 | 1.0172 |
| 84.2 | 1.0163 | | 1.0167 | 1.0163 |
| 84.4 | 1.0153 | | 1.0158 | 1.0153 |
| 84.6 | 1.0144 | | 1.0150 | 1.0144 |
| 84.8 | 1.0135 | | 1.0141 | 1.0135 |
| 85 | 1.0127 | | 1.0133 | 1.0127 |
| 85.2 | 1.0118 | | 1.0125 | 1.0118 |
| 85.4 | 1.0110 | | 1.0118 | 1.0110 |
| 85.6 | 1.0102 | | 1.0110 | 1.0102 |
| 85.8 | 1.0094 | | 1.0103 | 1.0094 |
| 86 | 1.0086 | | 1.0097 | 1.0086 |
| 86.2 | 1.0079 | | 1.0091 | 1.0079 |
| 86.4 | 1.0072 | | 1.0085 | 1.0072 |
| 86.6 | 1.0065 | | 1.0080 | 1.0065 |
| 86.8 | 1.0059 | | 1.0075 | 1.0059 |
| 87 | 1.0053 | | 1.0071 | 1.0053 |
| 87.2 | 1.0047 | | 1.0067 | 1.0047 |
| 87.4 | 1.0041 | | 1.0064 | 1.0041 |
| 87.6 | 1.0036 | | 1.0062 | 1.0036 |
| 87.8 | 1.0031 | | 1.0060 | 1.0031 |
| 88 | 1.0026 | | 1.0060 | 1.0026 |
| 88.2 | 1.0021 | | 1.0061 | 1.0021 |
| 88.4 | 1.0017 | | | 1.0017 |
| 88.6 | 1.0014 | | | 1.0014 |
| 88.8 | 1.0010 | | | 1.0011 |
| 89 | 1.0008 | | | 1.0008 |
| 89.1 | 1.0006 | | | 1.0006 |
| 89.2 | 1.0005 | | | 1.0005 |
| 89.3 | 1.0004 | | | 1.0004 |
| 89.4 | 1.0003 | | | 1.0003 |
| 89.5 | 1.0002 | | | 1.0003 |
| 89.6 | 1.0001 | | | 1.0002 |
| 89.7 | 1.0001 | | | 1.0002 |
| 89.8 | 1.0000 | | | 1.0003 |
| 89.9 | 1.0000 | | | 1.0007 |
| 90 | 1.0000 | 1.1781 | 1.1781 | 1.1781 |

The values string in the last column is given by:

$$E_2(k) = (1+\sqrt{1-k^2})E_1\left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right) - \sqrt{1-k^2}K_2(k) =$$

$$= (1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right) - k'\mathrm{K}_2(k) =$$
$$= (1+k')\mathrm{E}_1\left(\frac{1-k'}{1+k'}\right) - \frac{2k'}{1+k'}\mathrm{K}_1\left(\frac{1-k'}{1+k'}\right), \text{with:}$$
$$\frac{1-k'}{1+k'} = k_1 \text{(descending Landen transformation), getting}$$
$$\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2}\sqrt{1+k_1'}\sqrt[4]{k_1'}\right] - k_1'\cdot \mathrm{K}_1(k_1),$$
$$\text{and: } \mathrm{K}_1(k_1) = \frac{\pi\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right),$$

previously given, thus getting:

$$\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2}\sqrt{(1+k_1')\sqrt{k_1'}}\right] -$$
$$-\frac{\pi k_1'\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(1-\frac{\sqrt[4]{2}}{4}\frac{1+\sqrt{k_1'}}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right) =$$
$$= \frac{\pi}{4}\left[\frac{3}{2}\left(1+\sqrt{k_1'}\right)^2 - \sqrt{2(1+k_1')\sqrt{k_1'}} -\right.$$
$$\left. -\frac{k_1'\sqrt{2}}{\sqrt{(1+k_1')\sqrt{k_1'}}}\left(4-\frac{\sqrt[4]{2}\left(1+\sqrt{k_1'}\right)}{\sqrt[4]{(1+k_1')\sqrt{k_1'}}}\right)\right].$$

Expressing $k_1'(k')$: $k_1' = (1-k_1^2)^{1/2} = 2(k')^{1/2}/(1+k')$, (ascending Landen transformation), and replacing it:

$$\mathrm{E}_1(k_1) = \frac{\pi}{4}\left[\frac{3}{2}\left(1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}\right)^2 - \sqrt{2\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}\right.$$
$$\left. -\frac{\sqrt{2}\cdot\frac{2\sqrt{k'}}{1+k'}}{\sqrt{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\left(4-\frac{\sqrt[4]{2}\left(1+\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}\right)}{\sqrt[4]{\left(1+\frac{2\sqrt{k'}}{1+k'}\right)\frac{\sqrt{2}\sqrt[4]{k'}}{\sqrt{1+k'}}}}\right)\right],$$

and finally: $\mathrm{E}_2(k) = (1+k')\mathrm{E}_1(k_1) - k'\mathrm{K}_2(k)$, where $\mathrm{K}_2(k)$ was given just before table 5, so getting another purely algebraic formula (the most accurate, just seemingly intricate), the 3rd set of closed analytic formulas being given by the recurrences: $\mathrm{K}_2(k) = 2\mathrm{K}_1(k_1)/(1+k')$; $\mathrm{E}_2(k) = (1+k')\mathrm{E}_1(k_1) - k'\mathrm{K}_2(k)$. Noting: $k_1' = x$ and $[(1+x)\cdot x^{1/2}]^{1/2} = y$, one can write:

$$\mathrm{K}_2(k) = \pi(2/k')^{1/2}\cdot(x/y)\left[1-(2^{1/4}/4)(1+x^{1/2})/y^{1/2}\right];$$
$$\mathrm{E}_2(k) = \pi(k')^{1/2}/(2x)\cdot\left\{(3/2)(1+x^{1/2})^2 - 2^{1/2}y -\right.$$
$$\left. -2^{1/2}(x/y)[4-2^{1/4}(1+x^{1/2})/y^{1/2}]\right\} - k'\mathrm{K}_2(k), \text{resp.,}$$

much simpler than previous ones (for calculation only). The validity of all approximate sets is limited to $k \in [0, k_{extr})$; $k_{extr} \le 1$, "extr" ≡ extremum (max. for K, and min. for E; $k_{max} \ne k_{min}$) (see figs. 1 & 2 – the dashed black lines, and the solid red ones, resp.). The higher the "n" index (of the ($\mathrm{K}_{n-1}$, $\mathrm{E}_{n-1}$) approximation set) value is, the better this approximation is and the closer to the right domain's end ($k = 1$) the extremum is located. We will cancel the recurrent-iterative scheme (stopping it to a specific "n" index value) when the maximum relative error (over the whole valid domain of variation $k \in [0, k_{extr})$) becomes lesser than the desired (required) accuracy. The first important application of the results obtained in chapter 4 consists in determining the locations of the extrema values $k_{extr}$ ($k_{max}$ for $\mathrm{K}_{n-1}(k)$ and $k_{min}$ for $\mathrm{E}_{n-1}(k)$), corresponding to the annulment of their first derivatives with respect to $k$, using the relations:

$$\mathrm{K}_{n-1}'(k) = d\mathrm{K}_{n-1}(k)/dk = 0; \mathrm{E}_{n-1}'(k) = d\mathrm{E}_{n-1}(k)/dk = 0,$$

and adding the recurrent definitions for $\mathrm{K}_{n-1}(k)$ and $\mathrm{E}_{n-1}(k)$. The 1st ODE above gives the value $k_{max}$ and the 2nd one gives the value $k_{min}$. Each of these ODEs has really two solutions. Besides the searched for one, both ODEs admit the solution $k = 0$, corresponding to a minimum for $\mathrm{K}_{n-1}(k)$ and to a maximum for $\mathrm{E}_{n-1}(k)$, both with the value $\pi/2$ (for both approximate and exact functions: $\mathrm{K}_{n-1}(0) = \mathrm{E}_{n-1}(0) = \mathrm{K}(0) = \mathrm{E}(0) = \pi/2$, with:

$$\mathrm{K}_{n-1}'(0) = \mathrm{E}_{n-1}'(0) = \mathrm{K}'(0) = \mathrm{E}'(0) = 0), \text{ but with:}$$
$$\mathrm{K}_{n-1}''(0) > 0 \text{ and } \mathrm{K}''(0) > 0 - \text{a minimum, while:}$$
$$\mathrm{E}_{n-1}''(0) < 0 \text{ and } \mathrm{E}''(0) < 0 - \text{a maximum}).$$

Thus one knows now the values $k_{max}$ and $k_{min}$ (the right ends of the validity domains of the approximate functions). In order to evaluate the accuracy of the 3rd set ($\mathrm{K}_2$, $\mathrm{E}_2$), similarly as for the previous two sets, ($\mathrm{K}_0$, $\mathrm{E}_0$) and ($\mathrm{K}_1$, $\mathrm{E}_1$), we will define the following relative error functions: $\varepsilon_{K2}(k) = \mathrm{K}_2(k)/\mathrm{K}(k) - 1$, and: $\varepsilon_{E2}(k) = \mathrm{E}_2(k)/\mathrm{E}(k) - 1$, for the approximate formulas of 1st & 2nd kind integrals. Their values are given in table 6, expressed in thousandths (‰). These errors were calculated for the 3rd set ($\mathrm{K}_2$, $\mathrm{E}_2$) only, with an increment of 0°.2 in the field $\theta \in [84°, 89°]$ of the domain, and of 0°.1 beyond 89°. To get table 6, in table 3 were suppressed the columns $\varepsilon_{K0}$(‰), $\varepsilon_{E0}$(‰) (the most inaccurate) and were inserted the columns $\varepsilon_{K2}$(‰), $\varepsilon_{E2}$(‰), keeping for comparison the columns "$\theta$(°)", "$k = \sin\theta$", "$\varepsilon_{K1}$(‰)" and "$\varepsilon_{E1}$(‰)" (from table 3), only.

Table 6. Relative errors ε distribution
(this table completes and replaces table 3)

| $\theta$(°) | $k = \sin\theta$ | $\varepsilon_{K1}$(‰) | $\varepsilon_{K2}$(‰) | $\varepsilon_{E1}$(‰) | $\varepsilon_{E2}$(‰) |
|---|---|---|---|---|---|
| 84.8 | 0.99588 | – 0.369 | 0 | + 0.607 | 0 |
| 85 | 0.99619 | – 0.396 | 0 | + 0.592 | 0 |
| 85.2 | 0.99649 | – 0.451 | 0 | + 0.705 | 0 |
| 85.4 | 0.99678 | – 0.500 | 0 | + 0.748 | 0 |
| 85.6 | 0.99705 | – 0.582 | 0 | + 0.823 | 0 |
| 85.8 | 0.99731 | – 0.652 | 0 | + 0.932 | 0 |
| 86 | 0.99756 | – 0.737 | 0 | + 1.076 | 0 |
| 86.2 | 0.99780 | – 0.832 | 0 | + 1.160 | 0 |
| 86.4 | 0.99803 | – 0.945 | 0 | + 1.284 | 0 |
| 86.6 | 0.99824 | – 1.077 | 0 | + 1.453 | 0 |
| 86.8 | 0.99844 | – 1.214 | 0 | + 1.571 | 0 |
| 87 | 0.99863 | – 1.421 | 0 | + 1.743 | 0 |
| 87.2 | 0.99881 | – 1.626 | 0 | + 1.976 | 0 |
| 87.4 | 0.99897 | – 1.894 | 0 | + 2.275 | 0 |
| 87.6 | 0.99912 | – 2.234 | 0 | + 2.553 | 0 |
| 87.8 | 0.99926 | – 2.655 | 0 | + 2.922 | 0 |
| 88 | 0.99939 | – 3.156 | 0 | + 3.397 | 0 |
| 88.2 | 0.99951 | – 3.808 | 0 | + 4.004 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| 88.4 | 0.99961 | - | 0 | - | 0 |
| 88.6 | 0.99970 | - | 0 | - | 0 |
| 88.8 | 0.99978 | - | 0 | - | 0 |
| 89 | 0.99985 | - | 0 | - | 0 |
| 89.1 | 0.99988 | - | 0 | - | 0 |
| 89.2 | 0.99990 | - | 0 | - | 0 |
| 89.3 | 0.99993 | - | 0 | - | 0 |
| 89.4 | 0.99995 | - | 0 | - | 0 |
| 89.5 | 0.99996 | - | | - | |
| 89.6 | 0.99998 | - | | - | |
| 89.7 | 0.99999 | - | | - | |
| 89.8 | 0.99999 | - | | - | |
| 89.9 | 1.00000 | - | | - | |
| 90 | 1.00000 | – 2000 | – 2000 | 178.097 | 178.097 |

The errors strings are stopped if their modulus is ≥ 4 ‰. From the tables 3 and 6 one can see that, for any n$^{th}$ set of approximation and at any $k$ value, $\varepsilon_K < 0$ ($K_n < K$) and $\varepsilon_E > 0$ ($E_n > E$), i.e. K is approximated by lack, while E – by excess. Similarly to the 3$^{rd}$ set [$K_2(k)$, $E_2(k)$], expressed in algebraic functions, one can build the 3$^{rd}$ set [$K_2(\theta)$, $E_2(\theta)$], expressed in trigonometric functions, replacing $k'$ in [$K_2(k)$, $E_2(k)$] set by $\cos\theta$ and applying usual trigonometric identities. The comparative series representations and the graphic comparison are superfluous, due to the great accuracy of the approximate values given by the 3$^{rd}$ set (practically identical to the exact ones, which could be already noticed from the analysis of the 2$^{nd}$ set, this showing the fast converging character of this recurrent-iterative scheme). Except for the right domain's end ($k = 1$), the 3$^{rd}$ set of approximation ($K_2$, $E_2$), even more accurate than the 2$^{nd}$ one ($K_1$, $E_1$), may be considered and successfully used instead of the exact values of $K(k)$ and $E(k)$ from mathematical tables. A false minimum takes place for all $E_n(k)$: for $E_2(k)$, at $\theta = 89°.7$ ($k = 0.99999$); for $E_1(k)$, at $\theta = 88°$ ($k = 0.99939$), and for $E_0(k)$, at $\theta = 83°.62$ ($k = 0.99381$). The graphs of all $E_n(k)$ pass through the point (1, $3\pi/8 = 1.178097$); for $k$ tending to unity, the graphs of all $K_n(k)$ go toward ($-\infty$); the higher n$^{th}$ sets ($n \geq 4$) give a much better accuracy). Unlike the mathematical tables (and in addition to them), all approximation sets (the 1$^{st}$, 2$^{nd}$, 3$^{rd}$ and the higher n$^{th}$ ($n \geq 4$) ones) allow performing the *analytic study of variation* of the functions in which $K(k)$ and / or $E(k)$ appear /s, *using the derivatives of the 1$^{st}$ and 2$^{nd}$ order* (with respect to $k$).

*Remarks:* 1. As a first step in applying the new recurrent-iterative scheme, just the obtaining of the 2$^{nd}$ set ($K_1$, $E_1$) as a function of the 1$^{st}$ one ($K_0$, $E_0$) (in ch. 2) may be considered, i.e. *this scheme starts really at the 2$^{nd}$ set.* It is to be highlighted the used method is a *purely analytic* one (neither numerical methods nor sophisticated software, at most using MatLab's (software package for engineers) "Symbolic Math" toolbox, for analytically solving the more intricate algebraic equations encountered). Its *simplicity, accuracy* and *fast convergence*, as well as its *limitations* depend *exclusively* on the *correct choice of its starting point* (approximation set) ($K_0$, $E_0$). It must be quite precise, and especially, *as simple as possible.* The starting approximate formula-definition giving $E_0(k)$ was suggested to the author by an old approximate formula (Peano, [19], [20]) for the perimeter $L$ of an ellipse of semiaxes $a$ and $b$ ($\leq a$): $L \approx \pi[1.5(a + b) - (ab)^{1/2}]$ – a good (& *simple*) approximation with the best accuracy for $b = a$ (circle): $L = 2\pi a$, and the worst one for $b = 0$ (plane plate): $L = 1.5\pi a$, instead of $L = 4a$ (or optimized Peano's law: $L_1 \approx \pi[1.32(a + b) - 0.64(ab)^{1/2}]$, *with the smallest overall error* [21](about 7 times smaller than that of the original law); for $b = a$: $L_1 = L = 2\pi a$, and for $b = 0$: $L_1 = 1.32\pi a$, much closer to the exact value $L = 4a$). For its behaviour at low $b/a$ ratios, this formula is not found on the list of the very accurate (but not simple) approximations [21] (Padé, Jacobsen, Ramanujan (2 expressions), Rackauckas), all expressed in terms of the particular ratio $h = [(a - b)/(a + b)]^2$. Thus a reliable approximate (by excess) formula-definition was obtained (see chapter 2) for the Legendre complete elliptic integral of the 2$^{nd}$ kind (in the 1$^{st}$ set of approximation): $E_0(k) = (\pi/4)[1.5(1 + k') - (k')^{0,5}]$, with $k' = (1 - k^2)^{0,5}$. It can be seen that the error committed if in the expansion in series of powers (of $k$) we stopped at the term of rank 5 (see chapter 4), is $(3/16384)k^8$ only, i.e. small enough. As for the pair approximate formula-definition giving $K_0(k)$, this was obtained using the previous one for $E_0(k)$ and applying the definition of the first derivative of $E(k)$ with respect to $k$: $dE(k)/dk = [E(k) - K(k)]/k$ (see chapter 4), thus getting: $K(k) = E(k) - k[dE(k)/dk]$; replacing $K(k)$ and $E(k)$ by their 1$^{st}$ approximations: $\mathcal{K}_0(k)$ and the previously given $E_0(k)$, one gets: $\mathcal{K}_0(k) = (\pi/8)[3/2(1 + 1/k') - (k')^{0,5}(1 + 1/(k')^2]$, of a lesser accuracy (esp. for $\theta > \pi/3$) than $E_0(k)$. To improve this, one uses a descending Landen transformation: $K(k) = (1 + k_1)K(k_1)$ with $k_1 = (1 - k')/(1 + k') \leq k$, and replacing in $K(k)$, one gets: $K_0(k) = \pi[1/(k')^{0,5} - (1/2^{1,5})(1 + k')^{0,5}/(k')^{0,75}] \geq \mathcal{K}_0(k)$ (see ch. 2), of an accuracy (in modulus) much closer to that of its pair $E_0(k)$. Being practically generated by the same mathematical source, $\mathcal{K}_0(k)$ and $E_0(k)$ vary (ordinates, slopes, asymptote, extrema, concavities, convexities, inflections) in perfectly correlated way. So, at the value $k_{extr}$ corresponding to a false minimum for $E_0(k)$, $\mathcal{K}_0(k)$ must equate $E_0(k)$, to satisfy the annulment of $dE_0(k)/dk$. To prepare this, $\mathcal{K}_0(k)$ must stop its vertiginous ascension to $\infty$, making a false inflection, followed by a false max. at $k_{Extr} < k_{extr}$ and a vertiginous ($k = 1$ – vertical asymptote) fall toward ($-\infty$); so $\mathcal{K}_0 = E_0$ at $k = 0$ and $k = k_{extr}$. But the new more accurate $K_0$ is not generated by the same mathematical source as $E_0$. To minimise the unwished events, limiting them to a very thin region in the neighbourhood of the right domain's end, one applies the descending Landen transformation, passing from $k$ to $k_1 \leq k$, where all goes well, maintaining all advantages of the asymptotic behaviour of the new approximate functions ($K_n$, $E_n$), i.e. applying a higher n$^{th}$ ($n \geq 2$) set of approximation (repeating this scheme until the desired accuracy for ($K_n$, $E_n$) is obtained; fortunately, this scheme is fast converging); though it keeps the limitation at $k = 1$, Peano's optimized law accelerates the scheme.

2. Besides the formulas for transforming the modulus using the descending Landen transformation, there are formulas using the ascending Landen transformation (not of interest here).

*Appendix' conclusions*

Some authors (e.g.: Bagis [14], [15]) *choose to start from more precise formulas for the perimeter of an ellipse* (similar to Ramanujan's "type π formulas" (1914) – see [22]): $L_I = \pi\{3(a + b) - [(a + 3b)(3a + b)]^{1/2}\} = \pi\{3(a + b) - [10ab + 3(a^2 + b^2)]^{1/2}\}$ – Ramanujan 1st approximation; $L_{II} = \pi(a + b)\{1 + 3h/[10 + (4 - 3h)^{1/2}]\}$; $h = [(a - b)/(a + b)]^2$ – the more famous Ramanujan 2nd approximation; the errors in these empirical relations, are of order $h^3$ and $h^5$ (both being very accurate, but not as simple as possible), *in order to obtain approximate formulas as accurate as possible for Legendre's complete elliptic integrals*.

We cite from [21]: "What makes Ramanujan's first formula interesting to this Author is the fact that, like the first form of Peano's approximation, it can be interpreted as a combination of the arithmetic mean with another one, denoted as $R(a, b, w)$ and defined by: $R(a, b, w) = [(a + wb)(b + wa)]^{1/2}/(1 + w)$. In Ramanujan's formula we have $w = 3$ and the two means are combined linearly with the relative weights + 3 and – 2, resp." Noteworthy are the fast converging power (of $h$) series [23], [24].

This appendix demonstrates that *even choosing as a starting point a "not so precise"* (with big problems at the right domain's end $k = 1$), *but especially simple formula* (like Peano's, or better, optimized Peano's one), *and applying the newly found original fast converging recurrent-iterative scheme* (also including Landen's descending transformation, to solve the unwished behaviour of $E_n(k)$ appeared in the neighbourhood of the value $k = 1$ of the modulus (the right domain's end), due to any of both Peano's approximate laws), this being a major method's limitation (see the 2nd part of remark 1), *similar results (from the viewpoint of their accuracy) for the values of Legendre's complete elliptic integrals* $K(k)$ *and* $E(k)$ (with very small values of the relative errors $\varepsilon_K$ and $\varepsilon_E$ – practically zero) *can be obtained*.

As regards the relations describing the recurrence (for the $(n + 1)^{th}$ set of approximation $[K_n(k), E_n(k)]$), they are:

$K_n(k) = [2/(1 + k')]K_{n-1}(k_1)$, and:

$E_n(k) = (1 + k')E_{n-1}(k_1) - [2k'/(1 + k')]K_{n-1}(k_1)$, resp., where: $k' = (1 - k^2)^{1/2}$ is the complementary modulus, and: $k_1 = (1 - k')/(1 + k') \le k$, this representing just the source of the descending Landen transformation; they express the values of the $(n + 1)^{th}$ set $[K_n(k), E_n(k)]$ in function of those of the $n^{th}$ one $[K_{n-1}(k_1), E_{n-1}(k_1)]$. The iterative scheme can continue until the desired (required) accuracy for the approximate set $(K_n, E_n)$ at a considered value of modulus $k = \sin\theta$ is obtained. As a rule, in the practical applications, the 3rd set of approximation $(K_2, E_2)$ is sufficiently accurate. It can be used until $\theta = 89°.7$ ($k = 0.99999$) – also see tables 4 – 6. Though *it* keeps the limitation at $k = 1$, *Peano's optimized law* is better to use; perhaps an example of calculation would have been useful, but we took as "overwhelming" *its* quality stated in [21].

Without these "appendix' conclusions", this work was published previously in a unitary form (main article + appendix), in English, as a scientific paper [25].

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