Performance Evaluation of HWMA Control Chart based on AR(p) with Trend Model to Detect Shift Process Mean

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Abstract: - The main goal of this study is to establish explicit solutions for the average run length (ARL) of the Homogenously Weighted Moving Average control chart when subjected to autoregressive with trend process. The accuracy of the explicit formula for the ARL is evaluated in comparison to the numerical integral equation method. To evaluate the two approaches, the accuracy percentage was employed. A determination is carried out of the HWMA control chart's effectiveness using the median run length (MRL), the standard deviation of run length (SDRL), and the average run length (ARL). A comprehensive comparison is performed between the HWMA control chart, the Extended Exponentially Weighted Moving Average (EEWMA), and the cumulative sum (CUSUM) control charts with mean process shifts to illustrate the design and implementation of the HWMA control chart. As criteria for various values of design parameters, the performance of these control charts can also be evaluated using the relative mean index (RMI), the average extra quadratic loss (AEQL), and the performance comparison index (PCI). To evaluate the effectiveness of our explicit formula approach, we employ this formula on copper price data.

Key-Words: - Average run length, median run length, numerical integral equation, explicit formula, autoregressive with trend process, Extended EWMA control chart, CUSUM control chart.

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1 Introduction

Control charts are indeed a fundamental tool in Statistical Process Control (SPC) and are considered essential for monitoring and managing processes in various industries. They play a crucial role in quality control and process improvement. Control charts are of great importance in SPC including the detection of variability, process enhancement, cost reduction of inspections, and achieve their quality and performance objectives. Specifically, studies have examined the benefits and limitations of statistical process control (SPC) about quality improvement, with implications for the industrial sector, financial institutions, and healthcare, [1], [2]. The Shewhart control chart by, [3], is one of the most commonly used types of control charts in SPC. It is a graphical tool designed to monitor and analyze process data over time to determine whether a process is in a state of statistical control. The study, [4], proposed a Cumulative Sum (CUSUM) control chart which is particularly useful when monitoring processes where small changes. CUSUM charts can be more sensitive to small shifts compared to Shewhart control charts, making them valuable tools for proactive quality management. The Exponentially Weighted Moving Average (EWMA) control chart was later introduced in the study, [5]. It is very effective at detecting small changes in the process means. There is lots of evidence supporting the advantages of using the EWMA control chart, [6], [7]. The Extended Exponentially Weighted Moving Average (EEWMA) control chart was developed by [8], as an extension of the statistics provided by the EWMA control chart. The purpose of this design is to detect changes in both the mean and standard deviation of the process. The study, [9], recently introduced the Homogeneous Weighted Moving Average (HWMA) control chart as a control chart with weighting of historical and current data. An investigation is conducted into the effect of non-normal data on the performance of the HWMA control chart. It appears that the parameters of the HWMA control chart can be adjusted to be more effective when data is non-normal. Furthermore, the study, [9], demonstrated that the HWMA control chart shown superior performance compared to the CUSUM and EWMA control charts. As a result, the authors intended to compare the effectiveness of the control charts in identifying process changes by providing an explicit formula for the average run length of the HWMA control chart.

Data that is often found in the present situations is frequently interconnected, including economic data. These processes are often derived from econometric models such as autoregressive (AR) and moving average (MA) models. The proper control charts must be applied to these data. Moreover, in general, in the case of residual, there is often a form of white noise. However, in some data, other forms may occur, such as exponential white noise, [10], [11], [12]. In this research, we are interested in studying data that has the AR model with trends and residuals from exponential distribution. The average run length (ARL), which is described in the control chart as a performance evaluation metric, consists of two distinct components. To begin with, ARL_0 refers to the average of observations before going outside of the control limit. The ARL_0 value should be large in general. In contrast, ARL_1 denotes the average of observation accumulated from the starting point of the change procedure until it exceeds the control threshold. Consequently, the ARL_1 value should be as small as possible.

A variety of techniques, including Monte Carlo simulation, the Markov Chain approach (MCA), numerical integral equation (NIE), and explicit formulations can potentially be applied to evaluate ARL. For instance, [13], constructed a Cumulative Sum (CUSUM) control chart that presents a numerical integral equation to evaluate ARL for Long-Memory data based on FIA process with exogenous variables. The study, [14], employed the Markov chain approach to resolve the ARL of EWMA and CUSUM control chart based on Zero-inflated negative binomial model. The study, [15], determined the average run length (ARL) for the EWMA and the CUSUM control charts using the Markov chain approach and the numerical integral equation approach. The findings demonstrated that both methods produce equivalent approximations for the ARL while the integral in the integral equation is approximated using the product midpoint technique. ARL was proposed in the study, [16], for the CUSUM control chart employing a trend model along with $SAR(P)_L$. The performance of the explicit formula could outperform that of the numerical integration. Subsequently, [17], employing the CUSUM control chart, demonstrated the explicit formulas and numerical integral equation of ARL for the SARX (P,r)L model. The study, [18], stated the exact run length computation on the EWMA control chart for moving average process with exogenous variable. Furthermore, an enhanced CUSUM control chart was implemented to monitor process changes along with seasonal AR processes with exogenous variables, [19]. In addition to developing the EWMA control chart to monitor the process mean, [20], established the explicit formula of ARL for the seasonal moving average process with an external variable. Using the SARFIMA $(p,d,q)(P,D,Q)_L$ model, [21], suggested the explicit formula of ARL for an upper-sided CUSUM control chart. Recently, [22], demonstrated the explicit ARL of a Double EWMA control chart for autocorrelated data and compared its precision to that of the NIE method. The goal of this study is to construct an explicit formula for the ARL on the HWMA control chart for AR(p) with a trend model. In addition, an analysis of the efficacy of the CUSUM and EEWMA control charts is also provided. Furthermore, the copper price dataset has been enhanced to determine the effectiveness of the HWMA control chart.

2 Materials and Methods

2.1 The Homogenously Weighted Moving Average Control Chart

The Homogenously Weighted Moving Average control chart (HWMA) statistic under the assumption $\{H_t; t = 1, 2, 3, ...\}$, as a sequence of i.i.d continuous random variables with common probability density function, is considered. The HWMA statistic (H_t) is referred to as an upper HWMA statistic, based on AR(p) with trend process. The statistic (H_t) of the HWMA control chart can be expressed by the recursive formula as in Eq. (1)

$$H_t = \lambda Y_t + (1-\lambda)\bar{Y}_{t-1}, \text{ for } t = 1,2,3,... \quad (1)$$

where Y_t is a sequence of the AR(p) with a trend process with exponential white noise, and the starting value $\bar{Y}_0 = \psi$ is an initial value.

The control limits of the HWMA control chart consist of

Upper control limit:

$$UCL_t = \begin{cases} \mu + B_1\sqrt{\frac{\sigma^2}{n}\lambda^2}, t = 1 \\ \mu + B_1\sqrt{\frac{\sigma^2}{n}[\lambda^2 + \frac{(1-\lambda)^2}{(t-1)}]}, t > 1 \end{cases}$$

Center Line: $CL = \mu$

Lower control limit:

$$LCL_t = \begin{cases} \mu - B_1\sqrt{\frac{\sigma^2}{n}\lambda^2}, t = 1 \\ \mu - B_1\sqrt{\frac{\sigma^2}{n}[\lambda^2 + \frac{(1-\lambda)^2}{(t-1)}]}, t > 1 \end{cases}$$

where B_1 is the width of the control limits.

The HWMA stopping time (τ_h)is defined as

$$\tau_h = \{t > 0; H_t \geq h\}, \text{ for } h > \psi.$$

where τ_h is the stopping time and h is UCL.

2.2 The EEWMA Control Chart

The EEWMA control chart was presented by, [8]. The EEWMA control chart supplements the fundamental EWMA control chart concept through the addition of supplementary functions or adjustments, to enhance its efficacy under particular conditions. The EEWMA statistic is given by:

$$E_t = \lambda_1 Y_t - \lambda_2 Y_{t-2} + (1-\lambda_1-\lambda_2)E_{t-1}, t = 1,2,... \quad (2)$$

where λ_1 and λ_2 are exponential smoothing parameters with $(0 < \lambda_1 \leq 1)$ and $(0 \leq \lambda_2 \leq \lambda_1)$ and the initial value is a constant, $E_0 = u$. The upper control limit (UCL) and lower control limit (LCL) of the EEWMA control chart are given by:

$$UCL = \mu_0 + B_2\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

$$LCL = \mu_0 - B_2\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

where μ_0 is the target mean, σ is the process standard deviation, and B_2 is width of the control limits. The stopping time of the EEWMA control chart (τ_b) is given by:

$$\tau_b = \{t > 0; E_t \geq b\},$$

where τ_b is the stopping time and b is UCL.

2.3 The Cumulative Sum Control Chart

The Cumulative Sum (CUSUM) control chart, which, [2], developed, is a quality control tool utilized to identify small differences in the process mean. The statistics (C_t) of the CUSUM control chart can be mathematically represented as follows, utilizing the algorithm described in Eq. (3)

$$C_t = C_{t-1} + Y_t - \upsilon, t = 1,2,3,.. \quad (3)$$

where υ is non-zero constant, $C_0 = \varsigma$ is the initial value of CUSUM; $\varsigma \in [0,l]$ and the stopping time of the CUSUM control chart is defined as $\tau_s = \{t > 0; C_t > l\}$ and l is UCL.

3 The ARL of HWMA Control Chart

3.1 The Exact Solution of ARL the HWMA Control Chart for AR(p) with trend process

An AR (p) with a trend process can be derived as

$$Y_t = \delta + \gamma T_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t \quad (4)$$

where Y_t is a sequence of the AR(p) with trend process with exponential white noise, γ is trend parameter, ϕ_i is autoregressive parameter, the starting value $\bar{Y}_0 = \psi$ is an initial value; $\psi \in [0,h]$ where h is a control limit of HWMA control chart.

From the recursion of HWMA statistics in Eq. (1),

$$H_t = \lambda Y_t + (1-\lambda)\bar{Y}_{t-1}$$

and $Y_t = \delta + \gamma T_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t$

Therefore, the HEWMA control chart for the AR(p) with trend process can be written as,

$$H_t = \lambda(\delta + \gamma T_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t) + (1-\lambda)\bar{Y}_{t-1}$$

For t=1,

$$H_1 = \lambda(\delta + \gamma T_1 + \phi_1 Y_0 + \phi_2 Y_{-1} + \ldots + \phi_p Y_{1-p} + \varepsilon_1) + (1-\lambda)\bar{Y}_0$$

$$H_1 = \lambda(\delta + \gamma T_1 + \phi_1 Y_0 + \phi_2 Y_{-1} + \ldots + \phi_p Y_{1-p}) + \lambda\varepsilon_1 + (1-\lambda)\bar{Y}_0$$

Let $N = \lambda(\delta + \gamma T_1 + \phi_1 Y_0 + \phi_2 Y_{-1} + \ldots + \phi_p Y_{1-p})$

Consider the in-control process, given LCL=0, UCL= h and initial value $\bar{Y}_0 = \psi$ that is

$$0 < H_t < h$$

$$0 < \lambda N + \lambda\varepsilon_1 + (1-\lambda)\bar{Y}_0 < h$$

The change-point time at $t=1$ is studied, therefore $S(\psi)$ can be expressed by Fredholm integral equation of the second kind as follows,

$$S(\psi) = 1 + \int_0^{\frac{h-(1-\lambda)\psi - N}{\lambda}} S(N + \lambda y + (1-\lambda)\psi) f(y)dy \qquad (5)$$

Let $w = N + \lambda y + (1-\lambda)\psi$, then $dy = \frac{1}{\lambda}dw.$

After changing the variable in (5) it can be rewritten as

$$S(\psi) = 1 + \frac{1}{\lambda}\int_0^h S(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}[\frac{w-(1-\lambda)\psi - N}{\lambda}]}dw$$

Since we determine ε_1 is $Exp(\alpha)$ then $f(y) = \frac{1}{\alpha}e^{-\frac{y}{\alpha}}.$

Thus,

$$S(\psi) = 1 + \frac{e^{\frac{(1-\lambda)u+N}{\alpha\lambda}}}{\alpha\lambda}\int_0^h S(w)\frac{1}{\alpha}e^{-\frac{w}{\alpha\lambda}}dw$$

We setting that $Q(\psi) = \frac{e^{\frac{(1-\lambda)\psi+N}{\alpha\lambda}}}{\alpha\lambda}$ and

$$R = \int_0^h S(w)\frac{1}{\alpha}e^{-\frac{w}{\alpha\lambda}}dw$$

So that $S(\psi) = 1 + Q(\psi)R.$ (6)

Since $R = \int_0^h S(w)\frac{1}{\alpha}e^{-\frac{w}{\alpha\lambda}}dw,$ then

$$R = \int_0^h \left(1 + Q(w)R\right)e^{\frac{-w}{\alpha\lambda}}dw = \int_0^h e^{\frac{-w}{\alpha\lambda}}dw + \frac{Re^{\frac{N}{\alpha\lambda}}}{\alpha\lambda}\int_0^h e^{\frac{-\lambda w}{\alpha\lambda}}dw$$

$$R = \frac{-\alpha\lambda[e^{\frac{-h}{\alpha\lambda}} - 1]}{[1 + \frac{e^{\frac{N}{\alpha\lambda}}}{\lambda}(e^{\frac{-h}{\alpha}} - 1)]}$$

Substituting R in (6), we have

$$S(\psi) = 1 - \frac{[e^{\frac{-h}{\alpha\lambda}} - 1]e^{\frac{(1-\lambda)\psi+N}{\alpha\lambda}}}{1 + \frac{e^{\frac{N}{\alpha\lambda}}}{\lambda}(e^{\frac{-h}{\alpha}} - 1)}.$$

$$S(\psi) = 1 - \frac{[e^{\frac{-h}{\alpha\lambda}} - 1]e^{\frac{(1-\lambda)\psi + \lambda(\delta+\gamma T_1+\phi_1 Y_0+\phi_2 Y_{-1}+\ldots+\phi_p Y_{1-p})}{\alpha\lambda}}}{1 + \frac{e^{\frac{\lambda(\delta+\gamma T_1+\phi_1 Y_0+\phi_2 Y_{-1}+\ldots+\phi_p Y_{1-p})}{\alpha\lambda}}}{\lambda}(e^{\frac{-h}{\alpha}} - 1)}. \qquad (7)$$

Banach's Fixed-point Theorem provides theoretical support for the validity of the ARL equation, ensuring that there is a unique solution to the integral equation for explicit formulas. Let J be an operation on the class of all continuous functions defined by:

$$S(\psi) = 1 + \frac{1}{\lambda}\int_0^h S(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}[\frac{w-(1-\lambda)\psi-N}{\lambda}]}dw$$

(8)

where $N = \lambda(\delta + \gamma T_1 + \phi_1 Y_0 + \phi_2 Y_{-1} + \ldots + \phi_p Y_{1-p}).$

According to Banach's Fixed-point Theorem, if an operator J is a contraction, and then the fixed-point equation $J(S(\psi)) = S(\psi)$ has a unique solution. To show that Eq. (7) exists and has a unique solution, theorem can be used as follows below.

Theorem 1: Banach's Fixed-point Theorem

Let (X,d) defined on a complete metric space and $J: X \to X$ satisfies the conditions of a contraction mapping with contraction constant $0 \le r < 1$ such that $\|J(S_1) - J(S_2)\| \le r\|S_1 - S_2\|,\ \forall S_1, S_2 \in X.$ Then there exists a unique $S(\cdot) \in X$ such that $J(S(\psi)) = S(\psi)$, i.e., a unique fixed-point in X.

Proof: Let J defined in Eq. (7) as a contraction mapping for $S_1, S_2 \in F[0,h],$ such that

$$\|J(S_1) - J(S_2)\| \le r\|S_1 - S_2\|,\ \forall S_1, S_2 \in F[0,h]$$

with $0 \le r < 1$ under the norm $\|S\|_\infty = \sup_{\psi\in[0,h]}|S(\psi)|$, so

$$\|J(S_1) - J(S_2)\|_\infty = \sup_{\psi\in[0,h]}\left|\frac{e^{\frac{(1-\lambda)u+N}{\alpha\lambda}}}{\alpha\lambda}\int_0^h (S_1(w) - S_2(w))\frac{1}{\alpha}e^{-\frac{w}{\alpha\lambda}}dw\right|$$

$$\le \sup_{\psi\in[0,h]}\left|\|S_1 - S_2\|\frac{1}{\alpha\lambda}\cdot e^{\frac{(1-\lambda)u+N}{\alpha\lambda}}(-\alpha\lambda)(e^{-\frac{h}{\alpha\lambda}} - 1)\right|$$

$$= \|S_1 - S_2\|_\infty \sup_{\psi\in[0,h]}\left|e^{\frac{(1-\lambda)u+N}{\alpha\lambda}}\right|\left|1 - e^{-\frac{h}{\alpha\lambda}}\right| \le r\|S_1 - S_2\|_\infty$$

where $r = \sup_{\psi\in[0,h]}\left|e^{\frac{(1-\lambda)u+N}{\alpha\lambda}}\right|\left|1-e^{-\frac{h}{\alpha\lambda}}\right|;\ 0\le r<1$.

Thus, $\|J(S_1)-J(S_2)\|_\infty \le r\|S_1-S_2\|_\infty$ where a positive constant $r\in[0,1)$ and J represents the contraction, such that a mapping of contractions can have at most one fixed point. By applying the Banach contraction principle, a unique solution of the $S(\psi)$ is thus verified.

3.2 The NIE for the ARL of AR(p) with Trend Process on HWMA Control Chart

The NIE method is utilized extensively in the examination of the ARL. It can be obtained using a variety of quadrature rules, involving midpoint, trapezoidal, Simpson's rule, and Gauss-Legendre, all of which yield very similar ARL, [7]. The current investigation employs the Gauss-Legendre rule to determine the ARL. In this study, we use the Gauss-Legendre rule to evaluate the ARL on the HWMA control chart for the AR with the trend process as follows.

$$S(\psi)=1+\frac{1}{\lambda}\int_0^h S(w)f\left(\frac{w-(1-\lambda)-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)dw$$

The approximation for an integral is evaluated by the quadrature rule as follows;

$$\int_0^h f(x)dx \approx \sum_{k=1}^n w_k f(a_k)$$

where a_k is a point and w_k is a weight that is determined by the different rules.

Using the quadrature formula, we obtain

$$\tilde{S}(a_h)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{w-(1-\lambda)u-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right),\ h=1,2,...,n$$

The system of n linear equations is as follows;

$$\tilde{S}(a_h)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{w-(1-\lambda)\psi-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right),\ h=1,2,...,n$$

$$\tilde{S}(a_1)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{a_k-(1-\lambda)\psi-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)$$

$$\tilde{S}(a_2)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{a_k-(1-\lambda)\psi-\lambda(\delta+\gamma T_2+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)$$

$$\vdots$$

$$\tilde{S}(a_n)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{a_k-(1-\lambda)\psi-\lambda(\delta+\gamma T_n+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)$$

This system can be shown as

$$\mathbf{S}_{n\times1}=\left(\mathbf{I}_n-\mathbf{R}_{n\times n}\right)^{-1}\mathbf{1}_{n\times1},$$

where

$$\mathbf{S}_{n\times1}=\begin{bmatrix}\tilde{S}(a_1)\\ \tilde{S}(a_2)\\ \vdots\\ \tilde{S}(a_n)\end{bmatrix},\mathbf{I}_n=diag(1,1,...,1)\text{ and }\mathbf{1}_{n\times1}=\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}.$$

Let $\mathbf{R}_{n\times n}$ is a matrix and define the n to n^{th} as an element of the matrix $\mathbf{R}$ as follows;

$$[\mathbf{R}_{hk}]\approx\frac{1}{\lambda}w_kf\left(\frac{w-(1-\lambda)u-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)$$

If $(\mathbf{I}-\mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation is the term of the matrix,

$$\mathbf{S}_{n\times1}=(\mathbf{I}_{n\times1}-\mathbf{R}_{n\times n})^{-1}\mathbf{1}_{n\times1}$$

Finally, we substitute a_h by ψ in $\tilde{S}(a_h)$, the approximation of numerical integral for the function $S(\psi)$ is,

$$\tilde{S}(\psi)=1+\frac{1}{\lambda}\sum_{k=1}^n w_kS(a_k)f\left(\frac{a_k-(1-\lambda)\psi-\lambda(\delta+\gamma T_1+\phi_1Y_0+\phi_2Y_{-1}+...+\phi_pY_{1-p})}{\lambda}\right)$$

In this study, we compare the outcomes obtained for ARL_0 and ARL_1 through the use of explicit formulas and the NIE method for AR(p) into a trend process carried out on a HWMA control chart. The accuracy of the ARL is compared with the accuracy percentage which can be obtained from

$$\%Accuracy=100-\left|\frac{S(\psi)-\tilde{S}(\psi)}{S(\psi)}\right|\times100\%.$$

Furthermore, performance metrics such as the Median Run Length (MRL) and Standard Deviation Run Length (SDRL) are employed to measure the efficacy of control charts, [23]. The calculation for SDRL and MRL for the in-control process is as follows.

$$ARL_0=\frac{1}{\alpha},SDRL_0=\sqrt{\frac{1-\alpha}{\alpha^2}},MRL_0=\frac{\log(0.5)}{\log(1-\alpha)},\quad(15)$$

where "α" denotes an error of type I. The present investigation established ARL_0 at 370, and it can be computed using Equation (15) as $SDRL_0$ and MRL_0 at approximations 370 and 256, correspondingly. Conversely, $SDRL_1$ and MRL_1 are computed by replacing α with γ, where γ signifies type II error. A minimum value of the ARL_1, $SDRL_1$, and MRL_1 indicates enhanced capability in promptly detecting changes in the process mean.

To compare the performance of the HWMA, EEWMA, and CUSUM control charts for AR(p) with the trend model along with the ARL, SDRL, and MRL values, the RMI value is computed as described below, [24]:

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i} - Min[ARL_{shift,i}]}{Min[ARL_{shift,i}]}\right)$$

where $ARL_{shift,i}$ denotes the ARL of the control chart corresponding to the shift size of row i, while $Min[ARL_{shift,i}]$ indicates the ARL at the same level that is the smallest among all control charts.

In addition, the performance measurements can be used to assess a control chart's success throughout a variety of changes $(\delta_{\min} \le \delta \le \delta_{\max})$ .Moreover, the average extra quadratic loss (AEQL) may refer to the average extra loss incurred due to an out-of-control condition. This comparison might involve different control chart types to find the most effective approach for a particular process. AEQL can be calculated as follows, [25],

$$AEQL = \frac{1}{\Delta}\sum_{\delta_i=\delta_{\min}}^{\delta_{\max}}\left(\delta_i^2 \times ARL(\delta_i)\right) \quad (17)$$

where δ denotes the specific change in the process, and Δ denotes the aggregate of a number of divisions from $\delta_{\min}$ to $\delta_{\max}$ represents the sum of number of divisions from $\delta_{\min}$ to $\delta_{\max}$. In this study, $\Delta = 10$ is determined from $\delta_{\min} = 0.001$ to $\delta_{\max} = 1.00$. The most effective control chart is the one with the lowest AEQL values. In addition, performance evaluation metrics including the Performance Comparison Index (PCI) can be employed to evaluate the performance of control charts. The PCI value is calculated as the ratio of the control chart's AEQL to that of the control chart with the lowest AEQL, which represents the most efficient control chart. The mathematical expression describing the PCI is

$$PCI = \frac{AEQL}{AEQL_{lowest}} \quad (18)$$

4 Numerical Results

In Table 1 (Appendix), the comparison of the ARL$_1$ values using the explicit formula and NIE method on the HEWMA control chart for AR(1), AR(2) and AR(3) with trend processes with $\phi_1 = 0.1, \phi_2 = 0.2, \phi_3 = 0.3,$ and $\gamma = 1.5$ $\lambda = 0.05, 0.10,$ $\text{ARL}_0 = 370$ is implemented so that the computation time (CPU time) and percentage accuracy are utilized to compare the two methods. The findings indicate that the ARL of both are highly similar with a percentage accuracy of one hundred, which is utilized to verify this explicit precise formula. Additionally, the explicit formula requires less than 0.01 seconds of CPU time, which is significantly less than the NIE method. The outcomes for control limit (*h*) on HWMA control charts for AR(p) with trend processes are presented in Table 2 (Appendix). As an illustration, when $\delta = 0.05, \gamma = 1.5,$ $\phi_1 = 0.1,$ and $\phi_2 = 0.2$ the control limit for AR(2) with the trend is 0.000212. According to Table 3 (Appendix), the comparison of the ARL on HWMA control charts for AR(2) with the trend model against EEWMA and CUSUM control charts given $\delta = 0.05,$ $\phi_1 = 0.1, \phi_2 = 0.2,$ $\alpha_0 = 1,$ $\lambda = 0.05,$ $0.1, 0.2$ and $\text{ARL}_0 = 370$ is presented. The ARL of the HWMA control chart are almost lower than the EEWMA and CUSUM control charts for all λ. Therefore, the HWMA control chart has a higher performance than the EEWMA and CUSUM control charts. Moreover, the performance of the HWMA control chart is better when the λ increases. Additionally, the RMI, AEQL, and PCI values gained from each control chart are utilized to assess the effectiveness of the indicated charts. The HWMA control chart was determined to have the most effective results, with the lowest RMI, AEQL, and PCI all equal to 1. Table 4 (Appendix) illustrates the ARL of the HWMA control chart for AR(3) with a trend model calculated using an explicit formula, in comparison to the EEWMA and CUSUM control charts given $\delta = 0.05,$ $\phi_1 = 0.1, \phi_2 = 0.2, \phi_3 = 0.3$ and $\alpha_0 = 1$. The findings agree with the conclusions presented in Table 3 (Appendix). As a result, the HWMA control chart demonstrates superior performance in comparison to the EEWMA and CUSUM control charts.

4.1 Application

Using the quarterly copper price from January to August 2023, the efficacy of the explicit formulas for the ARL on the HWMA control chart is evaluated in comparison to the EEWMA and CUSUM control charts. The following coefficient parameters are derived for AR(1) with a trend model, based on the model estimation performed using maximum likelihood estimation: $\delta = 4.146,$ $\gamma = -0.012,$ $\phi_1 = 0.534,$ and the in-control parameter equal to 0.8054 as shown in Table 5 (Appendix). By applying the parameter of this forecasting model, the following can be represented:

$$\hat{Y}_t = 4.146 - 0.012T_t + 0.534Y_{t-1}$$

Using the explicit formula method, the ARL values for AR(1) with the trend model on the HWMA, EWMA, and CUSUM control charts are compared for efficiency in terms of ARL, SDRL, and MRL. The results are summarized in Table 6 (Appendix); it is evident that the results are consistent with those in Tables 3 (Appendix) and Table 4 (Appendix). As shown in Figure 1 (Appendix), Table 6 (Appendix) indicates that the HWMA control chart has the lowest RMI, AEQL, and PCI of all λ levels. In summary, the explicit formula approach proves to be an effective alternative for practical applications involving the detection of changes in the process mean using the HWMA control chart.

5 Conclusion

The ARL explicit formula for the AR(p) with trend model on the HWMA control chart was derived in this study. In terms of reduced computation time, the explicit formula is a highly practical method for determining the precise value of the ARL. When comparing the ARL values using the absolute percentage relative error (APRE) criterion between the explicit formula and the numerical integral equation (NIE) method, no significant differences in the results were observed.

Moreover, the explicit formula is computed in a significantly shorter period than the NIE method, as demonstrated by the results. When considering the performance of HWMA, EEWMA, and CUSUM control charts for detecting process changes, the findings indicate that the HWMA control chart shows superior performance comparing to other types of control charts. This is evidenced by the lowest values of RMI and AEQL, as well as a PCI value of 1. The present study revealed that the outcomes of investigating simulation and its implementation on real-world data are consistent. In future research, it is also possible to develop formulas for ARL values on HWMA control chart for new control charts or other interesting models.

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APPENDIX

Table 1. The ARL values of the explicit formula against the NIE method for AR(p) with trend on the HWMA control chart with $\delta = 0.05$, $\gamma = 1.5$ $\phi_1 = 0.1, \phi_2 = 0.2, \phi_3 = 0.3,$ and $\gamma = 1.5$ under different conditions.

σ	Model	Trend AR(1)		Trend AR(2)		Trend AR(3)	
	λ	0.05	0.10	0.05	0.10	0.05	0.10
	h	0.000259	0.01577	0.000212	0.012865	0.000157	0.00949
0.000	$S(\psi)$	370.3704	370.0396	370.4523	370.4759	370.5264	370.8901
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	370.3704	370.0396	370.4523	370.4759	370.5264	370.8901
	CPU time	(1.640)	(1.546)	(1.641)	(1.609)	(1.672)	(1.593)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.001	$S(\psi)$	365.8881	363.2419	365.8940	363.3279	365.8546	363.2189
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	365.8881	363.2419	365.8940	363.3279	365.8546	363.2189
	CPU time	(1.594)	(1.672)	(1.641)	(1.656)	(1.625)	(1.625)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.003	$S(\psi)$	357.1113	350.2197	356.9709	349.6678	356.7137	348.6120
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	357.1113	350.2197	356.9709	349.6678	356.7137	348.6120
	CPU time	(1.671)	(1.609)	(1.594)	(1.609)	(1.625)	(1.672)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.005	$S(\psi)$	348.5781	337.9123	348.2992	336.7971	347.8357	334.9130
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	348.5781	337.9123	348.2992	336.7970	347.8357	334.9130
	CPU time	(1.640)	(1.625)	(1.672)	(1.609)	(1.657)	(1.625)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.01	$S(\psi)$	328.2618	309.9265	327.6679	307.6743	326.7367	304.1431
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	328.2618	309.9265	327.6679	307.6743	326.7367	304.1431
	CPU time	(1.719)	(1.641)	(1.641)	(1.656)	(1.641)	(1.641)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.03	$S(\psi)$	259.6541	227.1554	258.1671	222.7142	255.9207	216.1737
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	259.6541	227.1554	258.1672	222.7142	255.9207	216.1737
	CPU time	(1.672)	(1.640)	(1.656)	(1.640)	(1.625)	(1.594)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.05	$S(\psi)$	207.2052	173.6670	205.2445	168.7604	202.3140	161.7086
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	207.2052	173.6670	205.2445	168.7604	202.3140	161.7086
	CPU time	(1.625)	(1.640)	(1.578)	(1.610)	(1.656)	(1.640)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.1	$S(\psi)$	122.1558	99.85197	119.9482	95.60361	116.6998	89.68831
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	122.1558	99.85197	119.9482	95.60361	116.6998	89.68831
	CPU time	(1.609)	(1.640)	(1.641)	(1.657)	(1.625)	(1.656)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.3	$S(\psi)$	22.41078	23.68498	21.43418	22.07040	20.05137	19.91859
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	22.41078	23.68498	21.43418	22.07040	20.05137	19.91859
	CPU time	(1.610)	(1.625)	(1.657)	(1.625)	(1.641)	(1.671)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
0.5	$S(\psi)$	6.892140	9.780320	6.508800	9.017013	5.980040	8.018595
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	6.892141	9.780320	6.508797	9.017010	5.980038	8.018590
	CPU time	(1.625)	(1.641)	(1.625)	(1.657)	(1.656)	(1.656)
	%Acc	100.00	100.00	100.00	100.00	100.00	100.00
1.0	$S(\psi)$	1.686280	2.999960	1.620619	2.778160	1.533740	2.495710
	CPU time	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
	$\hat{S}(\psi)$	1.686280	2.999955	1.620620	2.778160	1.533740	2.495710

CPU time	(1.641)	(1.656)	(1.609)	(1.625)	(1.625)	(1.672)
%Acc	100.00	100.00	100.00	100.00	100.00	100.00

Note: The numerical results in parentheses are computational times in seconds

Table 2. Control limits of HWMA control chart for AR(p) with trend processes

Models	Coefficients					$ARL_0 = 370$			
	δ	γ	ϕ_1	ϕ_2	ϕ_3	λ=0.05	λ=0.1	λ=0.2	λ= 0.3
AR(1)	0.05	1.5	0.1			0.0002590	0.0157700	0.0384400	0.0588490
AR(2)	0.05	1.5	0.1	0.2		0.0002120	0.0128650	0.0313540	0.0479160
AR(3)	0.05	1.5	0.1	0.2	0.3	0.0001570	0.0094900	0.0231250	0.0352710
AR(1)	0.05	1.5	-0.1			0.0003165	0.0193600	0.0471700	0.0723690
AR(2)	0.05	1.5	-0.1	-0.2		0.0003870	0.0237800	0.0579430	0.0891390
AR(3)	0.05	1.5	-0.1	-0.2	-0.3	0.0005230	0.0324500	0.0790000	0.1223370
	Coefficients					$ARL_0 = 500$			
	δ	γ	ϕ_1	ϕ_2	ϕ_3	λ=0.05	λ=0.1	λ=0.2	λ= 0.3
AR(1)	0.05	1.5	0.1			0.0003470	0.0165850	0.0386270	0.0589760
AR(2)	0.05	1.5	0.1	0.2		0.0002840	0.0135350	0.0315070	0.0480210
AR(3)	0.05	1.5	0.1	0.2	0.3	0.0002103	0.0099850	0.0232400	0.0353490
AR(1)	0.05	1.5	-0.1			0.0004247	0.0203470	0.0473950	0.0725240
AR(2)	0.05	1.5	-0.1	-0.2		0.0005190	0.0249800	0.0582160	0.0893280
AR(3)	0.05	1.5	-0.1	-0.2	-0.3	0.0007010	0.0340500	0.0794530	0.1225920

Table 3. The *ARL* of HWMA control chart for AR(**2**) with the trend using explicit formula against EEWMA and CUSUM control charts given $\delta = 0.05$, $\phi_1 = 0.1, \phi_2 = 0.2$ and $\alpha_0 = 1$.

λ		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$			$\lambda_1 = 0.2$		
σ	Control Chart	HWMA	EEWMA λ_2 =0.01	CUSUM υ =5	HWMA	EEWMA λ_2 =0.05	CUSUM υ =5	HWMA	EEWMA λ_2 =0.1	CUSUM υ =5
	UCL	0.000212	0.00101	2.195	0.012865	0.09365	2.195	0.031354	0.23029	2.195
0.001	ARL_1	365.8940	366.7705	367.8900	363.3279	366.8937	367.8900	320.8731	351.5811	367.8900
	$SDRL_1$	365.3936	366.2701	367.3897	362.8276	366.3934	367.3897	320.3727	351.0808	367.3897
	MRL_1	253.2716	253.8792	254.6552	251.4930	253.9646	254.6552	222.0655	243.3507	254.6552
0.003	ARL_1	356.9709	358.9695	363.5070	349.6678	360.0783	363.5070	252.4758	318.2737	363.5070
	$SDRL_1$	356.4706	358.4691	363.0067	349.1674	359.5779	363.0067	251.9753	317.7733	363.0067
	MRL_1	247.0867	248.4720	251.6171	242.0245	249.2405	251.6171	174.6561	220.2637	251.6171
0.005	ARL_1	348.2992	351.3636	359.1930	336.7971	353.4375	359.1930	207.8141	290.5678	359.1930
	$SDRL_1$	347.7988	350.8632	358.6927	336.2967	352.9371	358.6927	207.3135	290.0673	358.6927
	MRL_1	241.0759	243.1999	248.6269	233.1032	244.6374	248.6269	143.6989	201.0594	248.6269
0.01	ARL_1	327.6679	333.1663	348.703	307.6743	337.5622	348.7030	143.4672	238.1563	348.7030
	$SDRL_1$	327.1675	332.6659	348.2026	307.1739	337.0618	348.2026	142.9664	237.6558	348.2026
	MRL_1	226.7753	230.5865	241.3558	212.9168	233.6335	241.3558	99.09693	164.7306	241.3558
0.03	ARL_1	258.1671	270.6797	310.6020	222.7142	283.0606	310.6020	62.39193	136.106	310.6020
	$SDRL_1$	257.6667	270.1793	310.1016	222.2136	282.5601	310.1016	61.88991	135.6051	310.1016
	MRL_1	178.6010	187.2741	214.9461	154.0269	195.8559	214.9461	42.89928	93.99450	214.9461
0.05	ARL_1	205.2445	221.6125	277.8760	168.7604	240.0105	277.8760	38.84405	93.62050	277.8760
	$SDRL_1$	204.7439	221.1119	277.3755	168.2596	239.5099	277.3755	38.34079	93.11916	277.3755
	MRL_1	141.9178	153.2632	192.2622	116.6289	166.0158	192.2622	26.57656	64.54559	192.2622
0.10	ARL_1	119.9482	138.6623	214.0930	95.60361	165.3944	214.0930	18.97324	50.52562	214.0930
	$SDRL_1$	119.4471	138.1614	213.5924	95.10230	164.8936	213.5924	18.46647	50.02312	213.5924
	MRL_1	82.79469	95.76638	148.0511	65.92019	114.2957	148.0511	12.80155	34.67396	148.0511
0.30	ARL_1	21.43418	30.50467	92.11630	22.07040	54.25104	92.11630	5.343272	15.27733	92.11630
	$SDRL_1$	20.92821	30.00051	91.61494	21.56461	53.74872	91.61494	4.817393	14.76887	91.61494
	MRL_1	14.50771	20.79573	63.50295	14.94879	37.25631	63.50295	3.345140	10.23895	63.50295
0.50	ARL_1	6.508800	10.34249	49.63070	9.017010	25.2575	49.63070	3.003670	8.277210	49.63070
	$SDRL_1$	5.987961	9.829780	49.12816	8.502321	24.75245	49.12816	2.453235	7.761121	49.12816
	MRL_1	4.155352	6.816420	34.05363	5.896753	17.15826	34.05363	1.712090	5.383316	34.05363
1.0	ARL_1	1.620620	2.364731	18.24850	2.778162	7.831112	18.24850	1.642920	3.741368	18.24850
	$SDRL_1$	1.002890	1.796447	17.74146	2.222617	7.314042	17.74146	1.027748	3.202572	17.74146
	MRL_1	0.722145	1.260939	12.29907	1.553413	5.073651	12.29907	0.738797	2.228810	12.29907
RMI		0	0.1789	2.1644	0	0.6687	1.5779	0	1.0561	6.5031
AEQL		0.7167	0.9926	4.2109	0.8640	2.1583	4.2109	0.3240	0.8082	4.2109
PCI		1	1.3849	5.8755	1	2.4981	4.8739	1	2.4945	12.9972

Table 4. The *ARL* of HWMA control chart for AR(3) with trend using explicit formula against EEWMA and CUSUM control charts given $\delta = 0.05$, $\phi_1 = 0.1, \phi_2 = 0.2, \phi_3 = 0.3$ and $\alpha_0 = 1$.

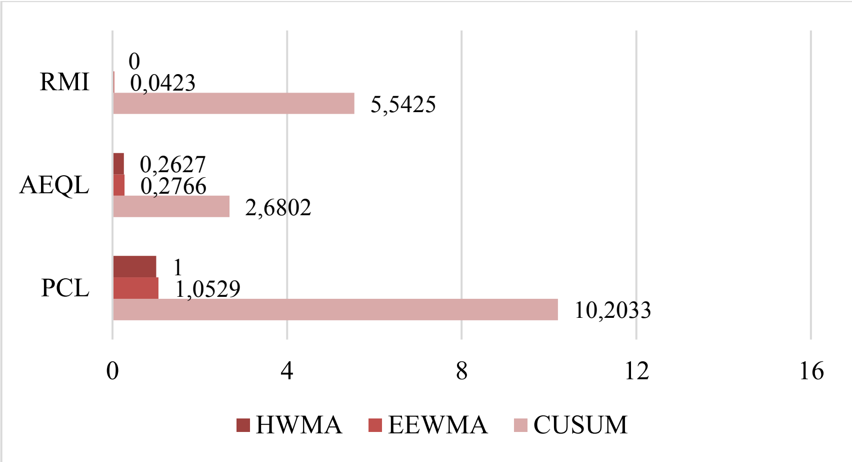
λ		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$			$\lambda_1 = 0.2$		
σ	Control Chart	HWMA	EEWMA λ_2 =0.01	CUSUM υ =5	HWMA	EEWMA λ_2 =0.05	CUSUM υ =5	HWMA	EEWMA λ_2 =0.1	CUSUM υ =5
	UCL	0.000157	0.000745	2.516	0.00949	0.06763	2.516	0.023125	0.16767	2.516
0.001	ARL_1	365.8546	366.0049	367.937	363.2189	366.579	367.937	316.5741	347.5467	367.937
	$SDRL_1$	365.3542	365.5045	367.4367	362.7186	366.0786	367.4367	316.0737	347.0463	367.4367
	MRL_1	253.2443	253.3485	254.6878	251.4174	253.7464	254.6878	219.0857	240.5543	254.6878
0.003	ARL_1	356.7137	357.9997	363.513	348.612	358.8589	363.513	245.1766	308.5874	363.513
	$SDRL_1$	356.2133	357.4994	363.0127	348.1116	358.3586	363.0127	244.6761	308.087	363.0127
	MRL_1	246.9083	247.7998	251.6213	241.2927	248.3953	251.6213	169.5967	213.5497	251.6213
0.005	ARL_1	347.8357	350.1996	359.159	334.913	351.3678	359.159	199.7389	277.2939	359.159
	$SDRL_1$	347.3354	349.6992	358.6587	334.4126	350.8674	358.6587	199.2383	276.7934	358.6587
	MRL_1	240.7546	242.3931	248.6033	231.7972	243.2028	248.6033	138.1016	191.8587	248.6033
0.01	ARL_1	326.7367	331.5578	348.572	304.1431	333.5836	348.572	135.8538	220.6911	348.572
	$SDRL_1$	326.2363	331.0574	348.0716	303.6427	333.0832	348.0716	135.3529	220.1906	348.0716
	MRL_1	226.1299	229.4716	241.265	210.4692	230.8758	241.265	93.81966	152.6246	241.265
0.03	ARL_1	255.9207	267.7808	310.147	216.1738	273.856	310.147	57.91622	119.3118	310.147
	$SDRL_1$	255.4202	267.2803	309.6466	215.6732	273.3555	309.6466	57.41405	118.8107	309.6466
	MRL_1	177.0439	185.2647	214.6308	149.4934	189.4757	214.6308	39.79689	82.35357	214.6308
0.05	ARL_1	202.314	217.9964	277.176	161.7086	228.1336	277.176	35.804	80.21244	277.176
	$SDRL_1$	201.8134	217.4958	276.6755	161.2079	227.633	276.6755	35.30046	79.71087	276.6755
	MRL_1	139.8865	150.7567	191.777	111.741	157.7833	191.777	24.46923	55.25173	191.777
0.10	ARL_1	116.6998	134.6085	213.025	89.68831	151.9863	213.025	17.34291	42.26754	213.025
	$SDRL_1$	116.1987	134.1076	212.5244	89.18691	151.4855	212.5244	16.83549	41.76454	212.5244
	MRL_1	80.54308	92.95653	147.3108	61.81998	105.0019	147.3108	11.67119	28.94967	147.3108
0.30	ARL_1	20.05137	28.43238	90.9396	19.91859	46.55677	90.9396	4.84225	12.52101	90.9396
	$SDRL_1$	19.54498	27.92791	90.43822	19.41215	46.05406	90.43822	4.313367	12.01061	90.43822
	MRL_1	13.54902	19.35918	62.68732	13.45697	31.92287	62.68732	2.996468	8.327524	62.68732
0.50	ARL_1	5.98004	9.418813	48.7499	8.01859	21.05717	48.7499	2.737078	6.786677	48.7499
	$SDRL_1$	5.457182	8.904787	48.24731	7.501946	20.55109	48.24731	2.180486	6.266762	48.24731
	MRL_1	3.78791	6.175568	33.44308	5.203798	14.24633	33.44308	1.524451	4.348389	33.44308
1.0	ARL_1	1.53374	2.169231	17.8392	2.49571	6.416029	17.8392	1.538	3.13361	17.8392
	$SDRL_1$	0.904775	1.592587	17.33199	1.932061	5.894863	17.33199	0.90964	2.585711	17.33199
	MRL_1	0.656666	1.12155	12.01529	1.353877	4.090897	12.01529	0.659902	1.803332	12.01529
RMI		**0**	0.1710	2.2850	**0**	0.6091	1.7591	**0**	0.9210	7.0579
AEQL		**0.6781**	0.9260	4.1361	**0.7831**	1.8253	4.1361	**0.2994**	0.6720	4.1361
PCI		**1**	1.3656	6.0992	**1**	2.3308	5.2817	**1**	2.2443	13.8135

Table 5. The coefficients for the AR(p) with trend model using the real-world dataset.

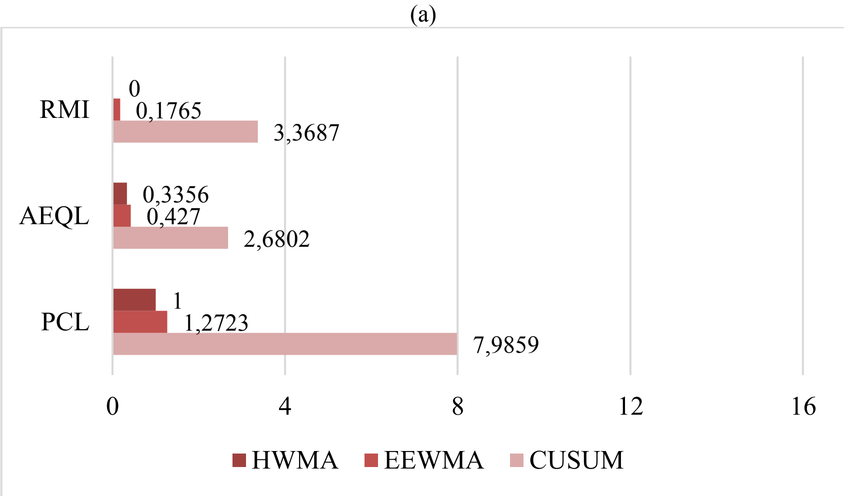
model	**AR(1) model**			
parameters	δ	γ	ϕ_1	p-value
AR(1)	4.146	-0.012	0.534	0.000
RMSE		0.088		
MAPE		1.844		
Residual of Application	**Residual AR(1) model**			
Exponential parameter		0.08054		
One-sample Kolmogorov-Smirnov test		1.114		
p-value		0.167		

Table 6. The *ARL* of HWMA control chart for AR(1) with trend using explicit formula against EEWMA and CUSUM control charts given $\delta = 4.146$, $\phi_1 = 0.534$, $\gamma = -0.012$ $\gamma = 1.5$ and $\alpha_0 = 0.8054$

λ		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$			$\lambda_1 = 0.2$		
σ	Control Chart	HWMA	EEWMA λ_2 =0.01	CUSUM υ =5	HWMA	EEWMA λ_2 =0.05	CUSUM υ =5	HWMA	EEWMA λ_2 =0.1	CUSUM υ =5
	UCL	0.00000034	0.000000742	3.07	0.0001422	0.0004426	3.07	0.00047446	0.0016826	3.07
0.001	ARL_1	361.9666	362.2568	367.1720	360.7148	362.5468	367.1720	290.9099	317.5049	367.1720
	$SDRL_1$	361.4662	361.7564	366.6717	360.2145	362.0464	366.6717	290.4094	317.0045	366.6717
	MRL_1	250.5494	250.7505	254.1575	249.6817	250.9515	254.1575	201.2966	219.7309	254.1575
0.003	ARL_1	345.7416	346.6883	361.2320	342.1580	347.7789	361.2320	202.6896	246.1954	361.2320
	$SDRL_1$	345.2412	346.1879	360.7317	341.6577	347.2786	360.7317	202.1890	245.6949	360.7317
	MRL_1	239.3031	239.9592	250.0402	236.8191	240.7153	250.0402	140.1469	170.3028	250.0402
0.005	ARL_1	330.3189	331.8607	355.4170	324.9118	333.7974	355.4170	154.7645	200.2863	355.4170
	$SDRL_1$	329.8185	331.3604	354.9166	324.4114	333.2970	354.9166	154.2637	199.7856	354.9166
	MRL_1	228.6128	229.6816	246.0096	224.8650	231.0240	246.0096	106.9276	138.4810	246.0096
0.01	ARL_1	294.9953	297.7870	341.4030	286.7642	301.9455	341.4030	95.96641	135.0727	341.4030
	$SDRL_1$	294.4949	297.2865	340.9026	286.2637	301.4450	340.9026	95.46510	134.5718	340.9026
	MRL_1	204.1284	206.0634	236.2958	198.4230	208.9459	236.2958	66.17167	93.27827	236.2958
0.03	ARL_1	190.2312	195.6015	292.0650	183.0071	207.9757	292.0650	35.35186	54.75820	292.0650
	$SDRL_1$	189.7306	195.1009	291.5646	182.5064	207.4751	291.5646	34.84828	54.25589	291.5646
	MRL_1	131.5114	135.2338	202.0973	126.5040	143.8109	202.0973	24.15581	37.60785	202.0973
0.05	ARL_1	125.2718	131.0742	251.6920	123.8776	148.5857	251.6920	20.21537	32.11782	251.6920
	$SDRL_1$	124.7708	130.5733	251.1915	123.3766	148.0848	251.1915	19.70903	31.61387	251.1915
	MRL_1	86.48474	90.50670	174.1128	85.51838	102.6448	174.1128	13.66272	21.91398	174.1128
0.10	ARL_1	48.01291	52.25042	178.6320	54.87959	71.85317	178.6320	8.572470	13.80732	178.6320
	$SDRL_1$	47.51028	51.74800	178.1313	54.37729	71.35142	178.1313	8.056970	13.29792	178.1313
	MRL_1	32.93222	35.86954	123.4714	37.69200	49.45744	123.4714	5.588247	9.219589	123.4714
0.30	ARL_1	3.267553	3.802933	62.32660	6.635860	9.856721	62.32660	2.046366	2.964267	62.32660
	$SDRL_1$	2.722012	3.264868	61.82458	6.115454	9.343352	61.82458	1.463300	2.413009	61.82458
	MRL_1	1.897266	2.271822	42.85400	4.243623	6.479407	42.85400	1.033403	1.684397	42.85400
0.50	ARL_1	1.269406	1.363317	30.54930	2.271355	3.214531	30.54930	1.321042	1.653507	30.54930
	$SDRL_1$	0.584795	0.703787	30.04514	1.699323	2.668085	30.04514	0.651238	1.039509	30.04514
	MRL_1	0.447167	0.524158	20.82667	1.194477	1.860095	20.82667	0.489994	0.746684	20.82667
1.0	ARL_1	1.009641	1.014855	10.83050	1.127680	1.260085	10.83050	1.053698	1.125773	10.83050
	$SDRL_1$	0.098660	0.122784	10.31839	0.379450	0.572476	10.31839	0.237868	0.376287	10.31839
	MRL_1	0.149021	0.164092	7.154962	0.318192	0.439277	7.154962	0.232859	0.316254	7.154962
RMI		**0**	0.0423	5.5425	**0**	0.1765	3.3687	**0**	0.3525	10.4309
AEQL		**0.2627**	0.2766	2.6802	**0.3356**	0.4270	2.6802	**0.1752**	0.2095	2.6802
PCI		**1**	1.0529	10.2033	**1**	1.2723	7.9859	**1**	1.1957	15.2999



(a)



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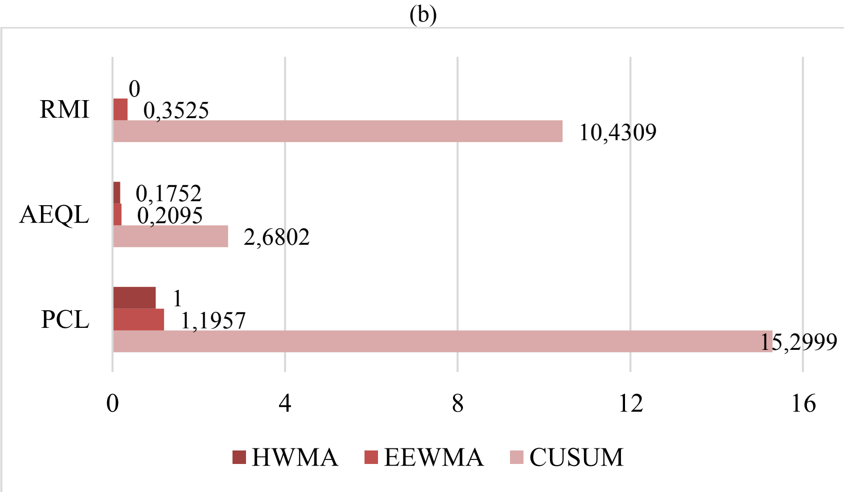

(c)

Fig. 1: Comparison of the RMI, AEQL, and PCI values among HWMA, EEWMA, and CUSUM charts for AR(1) when (a) $\lambda = 0.05$, (b) $\lambda = 0.1$ and (c) $\lambda = 0.2$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Rapin Sunthornwat carried out the conceptualization and software.
- Yupaporn Areepong has organized the writing-original draft, conceptualization, and validation
- Saowanit Sukparungsee has implemented the methodology and simulation.

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Conflicts of Interest

The authors declare no conflict of interest.