Comparison of the Minimum Initial Capital of Investment Discrete Time Surplus Process in Case Separated Claims

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Abstract: - In this paper, we suggest the bare minimum initial capital a firm providing insurance must hold to avoid going bankrupt. A case-separated investment discrete-time surplus process in motor insurance claims serves as the study's model. The 50th, 60th, 70th, and 80th percentiles are used as the dividing line between a claim's standard claim and large claim situations. We also discover a link between an insurance company's initial capital and the likelihood of ruin. The least squares regression method is utilized to calculate the minimum initial capital, and the simulation approach would be used to determine the ruin probability. The results indicate that the least initial capital is better provided by the 70th percentile than the 50th, 60th, and 80th percentiles, respectively.

Key-Words: - ruin probability, discrete-time surplus process, minimum initial capital, insurance company reserve.

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1 Introduction

According to [1], the first actuary to define a modern excess process, a surplus process is one that: Initial capital + premium - outflow equals the surplus process. The surplus process is given under three assumptions.

1. Claims occur at times T_i that satisfy the conditions $0 \le T_1 \le T_2 \le \dots$. We refer to these as claim times.

2. The claim size or claim severity Y_i is caused by the i-th claim arriving at a time T_i. An independent and identically distributed (i.i.d.) sequence of non-negative random variables is the sequence of claim sizes $\{Y_i, i \in \mathbb{N}\}$.

3. The claim sizes process $\{Y_i, i \in \mathbb{N}\}$ and the claim arrival process $\{T_i, i \in \mathbb{N}\}$ are mutually independent. The discrete-time surplus process

$$U_n = u + cn - \sum_{k=1}^{n} Y_k, \ \ U_0 = u; \ n \in \mathbb{N}. \tag{1}$$

is studied by [2], in the scenario of claim sizes occurring daily ($T_n = n, n \in \mathbb{N}$), where u is the initial capital, cn is the premium, and $\{Y_i, i = 1, 2, \dots n\}$ is a set of claim sizes (outflow) that are i.i.d. random variables. It is assumed that the set of claim sizes $\{Y_i, i = 1, 2, \dots n\}$ is made up of geometric and exponential random variables. Let

$$\Phi(u) = P\left(\{U(n) < 0 \ \exists n > 0, n \in \mathbb{N}\} \mid U(0) = u\right)$$

be the ruin probability(insolvency), and, [2], suggests a recursive version of this equation. The model of [2], is generalized in, [3], by considering $\{Y_i, i = 1, 2, \dots n\}$ as an arbitrary i.i.d. sequence (the sequence need not be exponential nor does it require geometric random variables). While defining an explicit form of the ruin probability is challenging, it is possible for it to take the recursive form.

$$\Phi_n(u) = \Phi_1(u) + \int_{-\infty}^{u+c} \Phi_n(u + c - y) dF_{Y_1}(y), n \in \mathbb{N}.$$

When $T_n = n, n \in \mathbb{N}$ and $Z_k = T_k - T_{k-1}, k \in \mathbb{N}$ is the inter-arrival time, the surplus process (1) can be expressed using the notation

$$U_n = U_{n-1} + cZ_n - Y_n, U_0 = u. \tag{2}$$

The discrete-time surplus process, according to the authors in [4], [5], incorporates the constant interest rate r and expresses Lundberg's bounds of the ruin probabilities. The model has the form of a

$$U_n = U_{n-1}(1 + r) - Y_n, \ \ U_0 = u, n \in \mathbb{N}.$$

In a discrete-time surplus process with proportional reinsurance and investment, [6], [7] provide lower and upper bounds for the ruin probability by applying the techniques of, [8], [9]. As shown by [10], using the martingale method, the exponential upper bounds of the model's ruin probability can be presented as:

$$U_n = U_{n-1}(1+I_n) + \alpha Y_n - \min\{X_n, M\},$$
$$U_0 = u,\ n \in \mathbb{N}$$

where I_n is the interest rate sequence, $\alpha, M \in \mathbb{R}$, and X_n is the sequence of claim sizes.

When claims occur exactly once per day ($Z_n = 1$), [11], examine the investment discrete-time surplus process (2) and divide claim sizes into standard and large claims under the condition that standard and large claims do not occur at the same time. The criteria used by them is the 80th percentile. The description of this model is as follows:

$$U_n = \begin{cases} U_{n-1}(1+r) + c - V_n,\ n \neq T_k^L, \\ \quad \text{for all } k = 1,2,3,\cdots, \\ U_{n-1}(1+r) + c - W_n,\ n = T_k^L, \\ \quad \text{for some } k = 1,2,3,\cdots, \end{cases} \qquad (3)$$
$$U_0 = u,\ n \in \mathbb{N},$$

where r is the daily interest rate, c is the premium rate, T_k^L is the arrival time of large claims, and the set of standard claims $\{V_n, n \in \mathbb{N}\}$ and the set of large claims $\{W_n, n \in \mathbb{N}\}$ are i.i.d.

It should be highlighted that the 80th percentile is the only criterion used in the proposed model (3) to distinguish between standard and large claim sizes.

The research of, [4], [5], [6], [7], [10], leads us to the conclusion that the ruin probability is challenging to express explicitly. As a result, they propose using the ruin probability as an implicit form or to express the upper and lower bounds of the ruin probability. It is not possible to utilize the conclusions of, [4], [5], [6], [7], [10], to calculate the minimum initial capital that an insurance firm must hold in reserve to avoid bankruptcy. Consequently, we must obtain the ruin probability as a number. The 80th percentile has been the sole criterion employed by the writers of [11], to distinguish between standard and large claims.

In this study, the ruin probability is calculated as a number and used to determine the minimum initial capital using the simulation approach. Not only is the 80th percentile used as a criterion for dividing claims into standard and large claims, but also the 50th, 60th, and 70th percentiles. We are interested in how the minimum initial capital is impacted by the percentile difference. In addition, we analyze the minimum initial capital that an insurance company must reserve to avoid bankruptcy for each percentile in the model and determine the relationship between initial capital and the ruin probability (3). The study's boundary is determined by our assumption that the insurance company's probability of bankruptcy is less than or equal to 0.01 and that it can invest in risk-free assets at a constant interest rate of $r = (1+r_0)^{1/365} - 1$, where r_0 is the compound interest rate and $r_0 = 0.02$ annually.

2 Methodology

2.1 Estimation of Parameters

Daily motor claims with a 365-claim sample size have been the source of the data for this study (see Figure 1). P_r should be the percentile ($r = 50, 60, 70, 80$) and $i \in \mathbb{N}$. If the claim size is less than or equal to P_r, it is referred to as a standard claim and is represented by the symbol V_i. When a claim size exceeds P_r, we refer to it as a large claim, and this is indicated by the symbol W_i. Let $|S|$ and $|L|$ represent a collection of standard and large claims, respectively, in terms of the number of elements. We determine $P_{50} = 63833$ Baht, $|S| = 183$, $|L| = 182$, $P_{60} = 82768.2$ Baht, $|S| = 219$, $|L| = 146$, $P_{70} = 104150.2$ Bath, $|S| = 257$, $|L| = 109$, and $P_{80} = 1043618.2$ Bath, $|S| = 292$, $|L| = 73$ from Figure 1. Moreover, Table 1 presents, for each percentile, the distributions of the standard and large claims at a 95% confidence level.

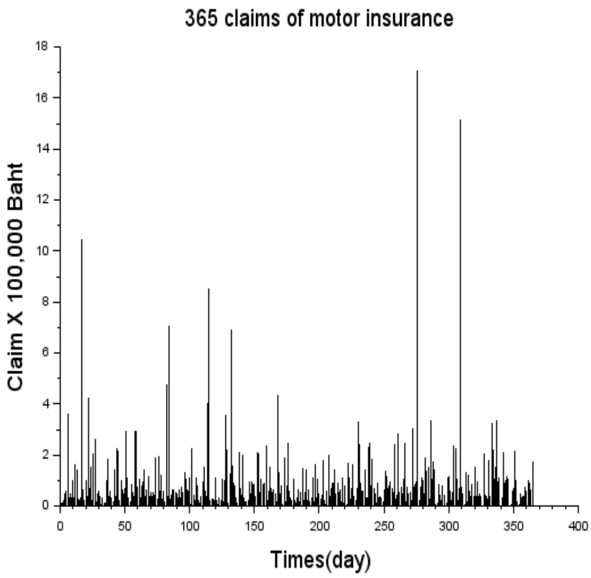

Fig. 1: The 365 Claims of motor insurance

Furthermore, for each percentile, the distributions of the standard and large claims at a 95% confidence level are expressed in Table 1.

Table 1. Distribution of standard and large claims with a 95% level of confidence

P_r	Distributions of standard claims	Distributions of Large claims
P_{50}	Weibull(α,β) $\alpha=2.1075$ $\beta=40963.8212$	Log-normal(μ,σ,γ) $\mu=10.9174, \sigma=1.229$ $\gamma=61323$
P_{60}	Weibull(α,β) $\alpha=1.9382$ $\beta=47864.8482$	Log-normal(μ,σ,γ) $\mu=10.8925, \sigma=1.3033$ $\gamma=81078$
P_{70}	Weibull(α,β) $\alpha=1.7513$ $\beta=55986.4935$	Weibull(α,β,γ) $\alpha=0.7743$ $\beta=102610.9392$ $\gamma=104280$
P_{80}	Weibull(α,β) $\alpha=1.5818$ $\beta=65134.7091$	Log-normal(μ,σ,γ) $\mu=11.1282, \sigma=1.3489$ $\gamma=141080$

2.2 Simulation

First, we should be notified of the ruin probability $\Phi(u)$,

$$\Phi(u)=P\left(\{U_n<0\ \exists n>0,\ n\in\mathbb{N}\}\mid U_0=u\right).$$

In other words, the ruin probability is the possibility that U_n will eventually fall below zero. This section U_n is determined by (3), which is

$$U_n=\begin{cases}U_{n-1}(1+r)+c-V_n,\ n\neq T_k^L, \\ \quad \text{for all}\ k=1,2,3,\cdots, \\ U_{n-1}(1+r)+c-W_n,\ n=T_k^L, \\ \quad \text{for some}\ k=1,2,3,\cdots,\end{cases}$$
$$U_0=u,\ n\in\mathbb{N},$$

where $r=(1+r_0)^{1/365}-1$ r_0 is the compound interest rate with $r_0=2\%$ per annum, c is determined using the expected value premium principle, that is,

$$c=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1), \tag{4}$$

$\{V_n,n\in\mathbb{N}\}$ and $\{W_n,n\in\mathbb{N}\}$ are i.i.d., along with θ, the safety loading, Z_k^L, the inter-arrival time of a large claim, T_k^L, and the arrival time of large claims, so $\{T_k^L,k\in N\}$ is assumed as i.i.d., and $Z_1^L\sim\text{Poisson}(\lambda^L)$.

The premium rate of (4) is then determined using the parameters listed in Table 1 and the knowledge that, $X\sim\text{Weibull}(\alpha,\beta),\ E(X)=\beta\Gamma(1+\frac{1}{\alpha}),$ $X\sim\text{Weibull}(\alpha,\beta,\gamma),\ E(X)=\beta\Gamma(1+\frac{1}{\alpha})+\gamma,$ and if $X\sim\text{Log-normal}(\mu,\sigma,\gamma),\ E(X)=\exp(\mu+\frac{1}{2}\sigma^2)+\gamma.$

Table 2. Premium rate c for each percentile

P_r	Premium rate c
P_{50}	$c=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1)=(1+\theta)(125355.0265),$ $EZ_1^L=\frac{365}{182}=2.0055$
P_{60}	$c=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1)=(1+\theta)(125167.1633),$ $EZ_1^L=\frac{365}{146}=2.500$
P_{70}	$c=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1)=(1+\theta)(116571.3903),$ $EZ_1^L=\frac{365}{109}=3.3486$
P_{80}	$c=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1)=(1+\theta)(120487.796),$ $EZ_1^L=\frac{365}{73}=5$

An example of calculating the premium rate c is shown in Table 2 after that. When considering the 50th percentile as the criterion,

$V_1\sim\text{weibull}(\alpha=2.1075,\beta=40963.8212)$ and
$W_1\sim\text{Log-normal}(\mu=10.9174,\sigma=1.229,\gamma=61323)$

$$\begin{aligned}EV_1&=\beta\Gamma(1+\frac{1}{\alpha})\\&=(40963.8212)\Gamma(1+1/2.1075)\\&=36280.5842\end{aligned}$$

$$\begin{aligned}EW_1&=\exp(\mu+(1/2)\sigma^2)+\gamma\\&=\exp(10.9174+(1/2)(1.229)^2)+61323\\&=178638.3046\end{aligned}$$

Z_1^L is the inter-arrival time of a large claim and $Z_1^L\sim Poisson(\lambda)$ and $EZ_1^L=\lambda=365/182=2.0055$. As a result,

$$\begin{aligned}c&=(1+\theta)(\frac{EW_1}{EZ_1^L}+EV_1)\\&=(1+\theta)(\frac{178638.3046}{2.0055}+36280.5842)\\&=(1+\theta)(125355.0265).\end{aligned}$$

These are the simulation procedures with the 50th percentile as the criterion.

step 1: set the number of simulations $N = 20000$, $u = 0, 20000, \ldots, 1320000$, safety loading $\theta = 0, 0.1, \ldots, 1$ and $r = (1+0.02)^{1/365} - 1$.

step 2: generate variate v_i and w_i (sample from Weibull and log-normal distribution) and generate variate Z_i^L (sample from Poisson distribution).

step 3: calculate the inter-arrival time of Z_i^L summation. If $\text{sum}(Z_i^L) < 365$, move on to step 4. We begin the next simulation if $\text{sum}(Z_i^L) \geq 365$.

step 4: compute $u_i = u_{i-i}(1+r) + c - v_i$ or
compute $u_i = u_{i-i}(1+r) + c - w_i$

step 5: if $u_i \geq 0$, set $i = i+1$ and move on to step 2 from there. If $u_i < 0$, then the current time is in ruin (insolvency), and the subsequent simulation will begin.

step 6: repeat this process up to a total of 20000 times. The ruin probability is equal to the sum of all instances where $u_i < 0$ divided by 20000.

In Figure 2, the flowchart is presented.

When used as a criterion, the 60th, 70th, and 80th percentile simulation steps are equivalent to the 50th percentile simulation steps.

The exponential relationships between the ruin probabilities and the initial capitals are obtained from Figure 2 and shown in Figure 3; the highest curve is plotted based on $\theta = 0.1$, followed by the next curve at $\theta = 0.2$, and so on, ending with the lowest curve at $\theta = 1$. When we divide claims using the 60th, 70th, and 80th percentiles, the relationships between the ruin probabilities and the initial capitals are still exponential relationships. The results are represented in Figure 4, Figure 5, and Figure 6, respectively.

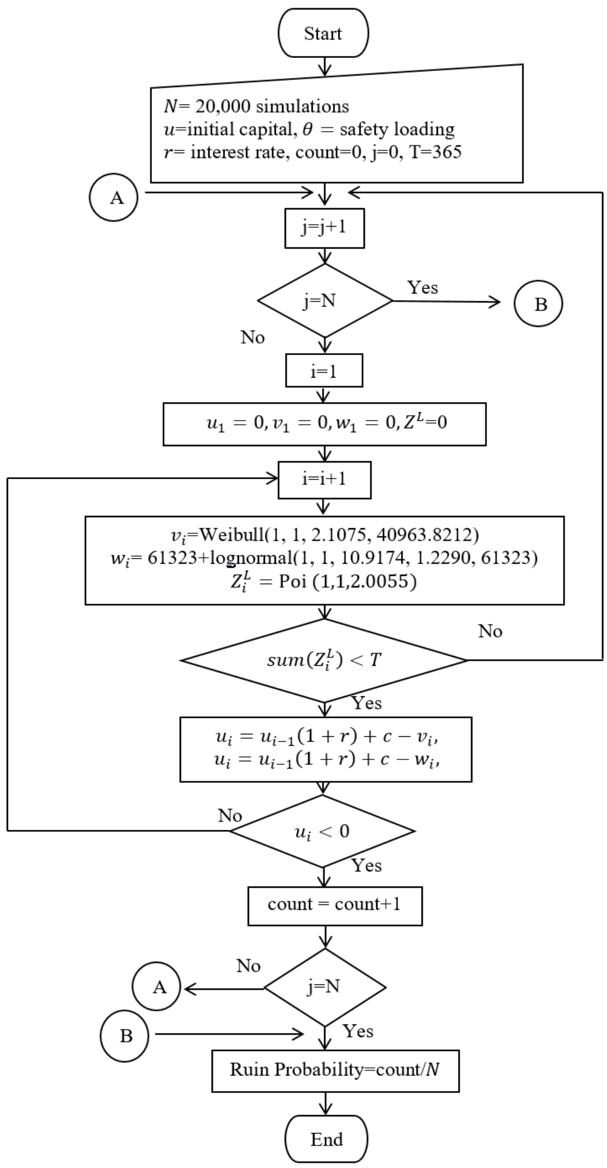

Fig. 2: The simulation flowchart uses the 50th percentile as the criterion.

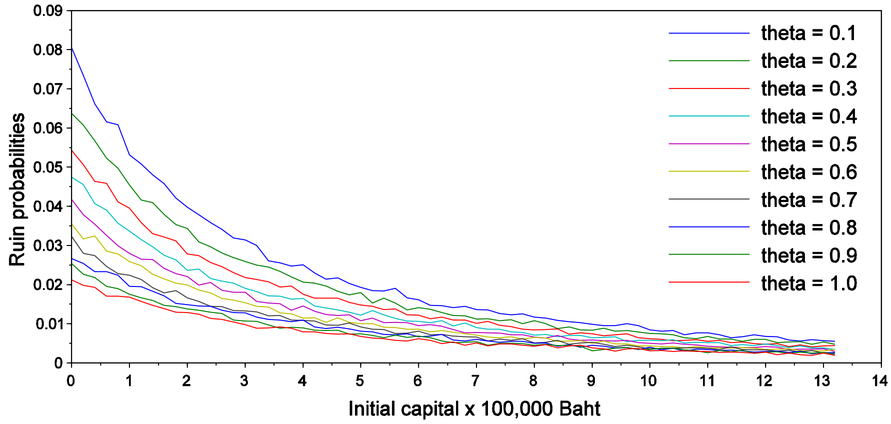

Fig. 3: Relations between initial capitals and ruin probabilities using the P_{50} as a criterion

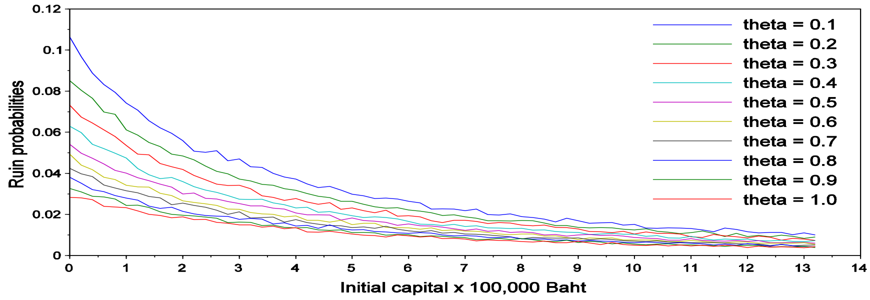

Fig. 4: Relations between initial capitals and ruin probabilities using the P_{60} as a criterion

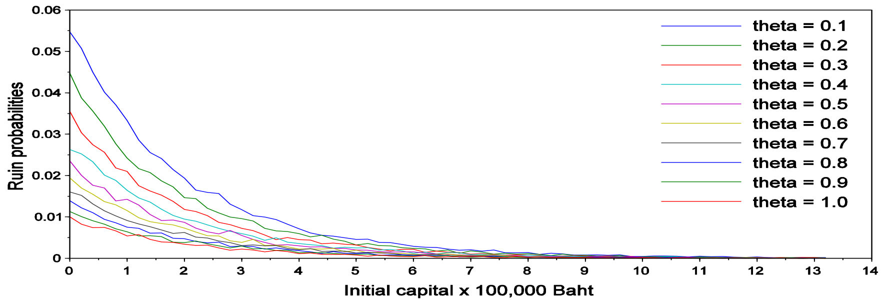

Fig. 5: Relations between initial capitals and ruin probabilities using the P_{70} as a criterion

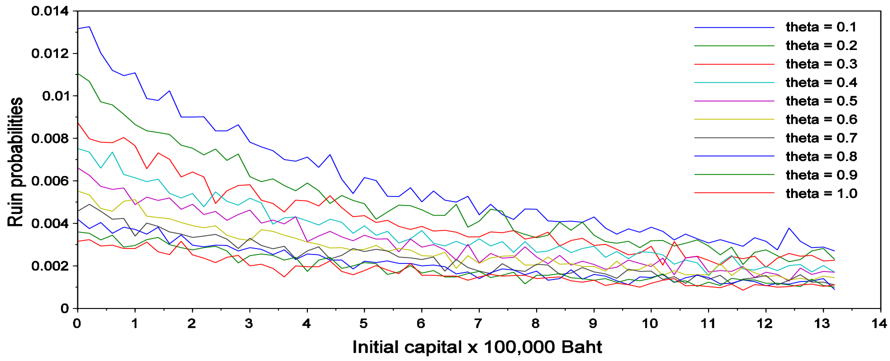

Fig. 6: Relations between initial capitals and ruin probabilities using the P_{80} as a criterion

2.3 Minimum Initial Capital

In Figures 3 to 6, the ruin probability, represented by y, and the initial capital u are expressed by the exponential function given by

$$y = ae^{-bu}. \tag{5}$$

Equation (5) can be written as the linear function $v = \beta_0 + \beta_1 u$, by taking the logarithm function,

$$\ln y = \ln a - bu,$$

where $v = \ln y$, $\beta_0 = \ln a$ and $\beta_1 = -b$.
The β_0 and β_1 can be calculated by the linear least squares regression:

$$\beta_0 = \frac{\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} \ln y_i\right) - \left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} u_i \ln y_i\right)}{n\sum_{i=1}^{n} u_i^2 - \left(\sum_{i=1}^{n} u_i\right)^2},$$

$$\beta_1 = \frac{n\sum_{i=1}^{n} u_i \ln y_i - \left(\sum_{i=1}^{n} u_i\right)\left(\sum_{i=1}^{n} \ln y_i\right)}{n\sum_{i=1}^{n} u_i^2 - \left(\sum_{i=1}^{n} u_i\right)^2},$$

where y_i is the ruin probability and u_i is the initial capital as in Figures 3 to 6. Thus we obtain $a = e^{\beta_0}$ and $b = -\beta_1$.

From (5), if we set the ruin probability y that is less than or equal to ε, i.e.,

$$ae^{-bu} \le \varepsilon,$$

then we obtain $u \ge -\frac{1}{b}\ln\frac{\varepsilon}{a}$. Therefore the minimum initial capital, denoted as MIC, is

$$\text{MIC} := u = -\frac{1}{b}\ln\frac{\varepsilon}{a}. \tag{6}$$

3 Results

Equation (6) demonstrates that for each safety loading θ, the MIC will be as given in Table 3, Table 4, Table 5, and Table 6 if an insurance company would like to reserve the MIC under the ruin probability less than or equal to $\varepsilon = 0.01$.

Table 3. The MIC using P_{50} as the criterion

θ	a	b	MIC(Baht)
0.1	0.6684	2.0716×10^{-6}	2028548
0.2	0.6387	2.0751×10^{-6}	2003259
0.3	0.6129	2.0642×10^{-6}	1993850
0.4	0.5909	2.0592×10^{-6}	1980880
0.5	0.5711	2.0420×10^{-6}	1980834
0.6	0.5517	2.0151×10^{-6}	1990225
0.7	0.5361	2.0125×10^{-6}	1978497
0.8	0.5207	1.9886×10^{-6}	1987597
0.9	0.5082	1.9994×10^{-6}	1964713
1	0.4942	1.9554×10^{-6}	1994660

Table 4. The MIC using P_{60} as the criterion

θ	a	b	MIC(Baht)
0.1	0.2630	1.6661×10^{-6}	1962494
0.2	0.2179	1.7056×10^{-6}	1806641
0.3	0.1821	1.7078×10^{-6}	1699245
0.4	0.1557	1.7109×10^{-6}	1604513
0.5	0.1343	1.7004×10^{-6}	1527443
0.6	0.1187	1.7081×10^{-6}	1448429
0.7	0.1039	1.6837×10^{-6}	1390153
0.8	0.0925	1.6780×10^{-6}	1325973
0.9	0.0830	1.6666×10^{-6}	1270002
1	0.0757	1.6656×10^{-6}	1214941

Table 5. The MIC using P_{70} as the criterion

θ	a	b	MIC(Baht)
0.1	0.3898	2.0471×10^{-6}	1789409
0.2	0.3079	2.2889×10^{-6}	1497381
0.3	0.2480	2.4433×10^{-6}	1313636
0.4	0.2031	2.5491×10^{-6}	1181252
0.5	0.1702	2.6401×10^{-6}	1073479
0.6	0.1444	2.6965×10^{-6}	990092
0.7	0.1239	2.7344×10^{-6}	920498
0.8	0.1081	2.7988×10^{-6}	850472
0.9	0.0933	2.7798×10^{-6}	803206
1	0.0822	2.8086×10^{-6}	749869

Table 6. The MIC using P_{80} as the criterion

θ	a	b	MIC(Baht)
0.1	0.1490	1.6661×10^{-6}	2423403
0.2	0.1115	1.6661×10^{-6}	2019312
0.3	0.0868	1.6661×10^{-6}	1749673
0.4	0.0698	1.6661×10^{-6}	1560972
0.5	0.0566	1.6661×10^{-6}	1398550
0.6	0.0480	1.6661×10^{-6}	1253184
0.7	0.0413	1.6661×10^{-6}	1138041
0.8	0.0358	1.6661×10^{-6}	1024725
0.9	0.0312	1.6661×10^{-6}	923151
1	0.0269	1.6661×10^{-6}	823280

The graph in Figure 7 illustrates the relationship between safety loading θ and MIC.

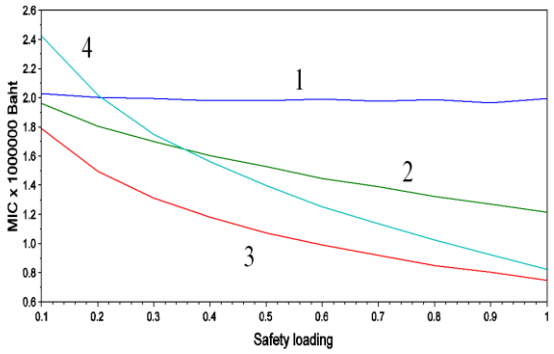

Fig. 7: Relation of safety loading θ and MIC for each percentile

1 Using the 50th percentile as the criterion
2 Using the 60th percentile as the criterion
3 Using the 70th percentile as the criterion
4 Using the 80th percentile as the criterion

According to Figure 7, the MIC is higher when using the 80th percentile as the criterion for safety loading $\theta = 0.1$ and $\theta = 0.2$ than when using the 50th, 60th, and 70th percentiles as the criterion. The MIC for safety loading $\theta = 0.3$ is less when the 80th percentile is used as the criterion than when the 50th percentile is used, but it is higher when the 60th and 70th percentiles are used as the criterion. The MIC is smaller than using the 50th and 60th percentiles as the criterion for safety loading $\theta = 0.4, 0.5, \ldots, 1,$ but it is still more than using the 70th percentile as the criterion under the ruin probability less than or equal to 0.01.

4 Conclusion

Figure 7 shows that for each safety loading θ, dividing the claims into standard and large claims according to the 70th percentile yields a better MIC than using the 50th, 60th, and 80th percentiles so long as the dividing threshold for the claims under the ruin probability is less than or equal to 0.01. The insurance company should therefore be reserved the MIC as shown in Table 5 in cases where the ruin probability is less than or equal to 0.01. This is beneficial for the financial management of the insurance company since it allows them to grow their investment in risk-free assets, which will enhance return. In our future research, we'll look at discrete-time surplus processes with minimal initial capital that involve investments in risky assets like the stock market and the gold market.

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Conflict of Interest
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