Optimal Exchange Rates: A Stochastic Control Approach

AHMED S. ABUTALEB
Currently with Cairo University, School of Engineering, Giza, Egypt, and
previously with MIT Lincoln Lab, Lexington, Massachusetts,
MA 02421, USA

MICHAEL G. PAPAIOANNOU
Visiting Scholar, International Monetary Fund,
700 19th Street, NW, Washington,
DC 20431, USA

Abstract: We present novel formulas for the determination of the optimal values of the exchange rate and the external (foreign currency) debt. A stochastic differential equation is developed that relates the net worth of an economy to both its exchange rate and level of external debt. Using the martingale optimality principle, the optimal values for the exchange rate and the external debt are derived. Applied to Egypt, we find that its actual nominal exchange rate has stayed above its optimal value since the early 1990s, except for two years, and that the actual external debt-to-net worth ratio is lower than the optimal one for most of the time. Since 2013 the actual external debt-to-net worth ratio is close to the risky levels of the optimal values.

Keywords: Exchange Rate, Martingales, Martingale Optimality Principle, Optimal Stochastic Control, Optimal Debt

Received: July 23, 2021. Revised: November 13, 2022. Accepted: December 5, 2022. Published: December 31, 2022.

1 Introduction

We study the interaction between the real exchange rate and external debt in an environment where both the return on capital and the real rate of interest are stochastic variables. Our model of such dynamic interaction reveals that an "overvalued" exchange rate, i.e., the cost of an identical basket of goods is more expensive domestically than abroad at the prevailing nominal exchange rate, leads to a steady rise in the external debt. In turn, the accumulation of debt due to ensuing trade account deficits and the interest rate payments on the debt exert downward pressure on the exchange rate, which may lead to a currency (balance of payments) crisis. In particular, a significant depreciation of the currency increases the debt burden and increases the probability of a debt crisis, [1].

The vast literature on exchange rates devotes considerable attention to the determinants of the equilibrium real exchange rate, e.g., ([2], [3], [4], [5], [6], and [7]), focusing on three main methodologies: (1) the fundamental equilibrium exchange rate (FEER) or macroeconomic balance (MB) approach, (2) the behavioral equilibrium exchange rate (BEER) approach, and (3) the stock-flow approach to the equilibrium real exchange rate. Further, long-term equilibrium real exchange rates are assumed to depend on exogenous and policy variables that need to be sustainable, or on “long-run fundamentals.” Regarding exogenous variables, their sustainability is not under the control of domestic policymakers (the empirical estimation of their sustainable levels and the assessment of the effect of those levels on the long-run equilibrium real exchange rates present problems). With regard to policy variables, policies are sustainable if they are optimal. In this case, long-run equilibrium real exchange rates are also “desirable” ones, as they are based on optimal policies. However, if sustainable (optimal) policies are unlikely to be implemented, i.e., if actual policies are likely to be inappropriate and therefore the dynamics of the real exchange rate would be determined by these policies, the “desirable” long-run equilibrium real exchange rates would provide a misleading indication of where the real exchange rate is heading. This will make the long-rum equilibrium real exchange rate the relevant concept for formulating exchange rate policy. In particular, if the policies in place are not sustainable (non-optimal), e.g., trade or capital account restrictions, or other distortionary policies, then the long-term equilibrium real exchange rate can be argued that it is not a true equilibrium rate. Under these conditions, the distinction between the true long-run equilibrium real exchange rate and "desirable" long-run equilibrium real exchange rate becomes indistinguishable. Accordingly, the challenge is to identify the relevant set of fundamental determinants of the long-run equilibrium real exchange rate, determine the appropriate sustainable policies (the optimal policy set), and estimate the "desirable" long-run equilibrium real exchange rate.However, although the link between the real (and nominal) exchange rate, productivity, and external debt is crucial for the economic and financial stability of a country, we are not aware of any studies that have dealt with this issue.[1] Since both exchange rates and external debt are dynamic concepts, representation of their interlinks warrants a suitable dynamic approach. Typically, the analysis of the dynamic interaction between variables is carried out in optimal value terms. This paper aims to develop such a framework of dynamic behavior and to apply it to Egypt. This framework, appropriately calibrated for specific country conditions, can serve as a policy tool for government authorities to assess the state of their external debt and draw conclusions on the following exchange rate policies.

In this novel study, we examine the determination of the optimal real exchange rate that stems from the maximization of a utility function as opposed to the equilibrium real exchange rate. We find that the optimal real exchange rate is proportional to the return on domestic investments, and inversely related to the U.S. interest rate. This result was obtained using the Martingale Optimality Principle ([9], [10], [11], [12], and [13]). The paper is organized as follows: Section 2 presents the problem formulation of the dynamic interaction between the exchange rate and external debt, while section 3 highlights the optimization problem and gives the optimal values of the exchange rate and foreign debt. Section 4 discusses the results and provides some concluding remarks on the applicability and shortcomings of the proposed framework and points out some areas for future research. An Appendix presents all necessary derivations.

2 Problem Formulation

Our prototype model is a simplification of a complex economy that focuses on shocks (economic and financial disturbances) that have led to crises. The model is analytically tractable, with all derived equations having an economic interpretation. However, introduction of more realistic assumptions generates a less transparent solution and economic interpretation. The prototype model is proposed as a "benchmark" model. We assume two sources of uncertainty: one source concerns the value of GDP and the return on capital, and the second concerns the interest rate on loans and bonds. It is important to recognize that there is a correlation between these two sources of uncertainty.Adopting a stochastic calculus formulation, the model is expressed in real terms. The net worth or wealth, X(t), in nominal terms is defined as [1]:

$$X(t) = K(t) - L(t) \tag{1}$$

Taking the derivative of both sides we get:

$$dX(t) = dK(t) - dL(t)$$

where K(t) is the capital owned by the residents of the country, and

L(t) is the country's external debt, denominated in the $US

The change in capital, dK(t), is the rate of investment I(t):

$$dK(t) = I(t)dt \tag{2}$$

Also, real consumption C(t), real investments I(t), and real GDP Y(t) are related through the equation:

$$p(t)C(t)dt = \begin{bmatrix} p(t)Y(t)dt - e(t)p^*(t)i(t)L(t)dt \\ - p(t)I(t)dt + e(t)dL(t) \end{bmatrix} \tag{3}$$

[1] [8] estimate optimal levels of foreign debt for Egypt during 1985-2008, without taking explicitly into account the dynamic interactions between foreign debt and the exchange rate.

where p(t) is the domestic price index, $p^*(t)$ is the US price index, e(t) is the nominal exchange rate (domestic price of foreign exchange or units of domestic currency per unit of $US). Note that the formulation of this equation excludes government expenditures and assumes that the external sector is represented by the change in the external debt due to the trade account, dL(t), and the interest rate payments on the debt, i(t)L(t).

In particular, as we assume that the external debt L(t) is denominated in $US, the term i(t)L(t) stands for the interest payments in $US, at the rate of interest i(t), on US dollar denominated loans and bonds.

Further, we assume that the accumulation of debt refers to annual intervals.

By dividing eqn. (3) by p(t), we obtain the real consumption C(t) (measured in domestic-goods units) as:

$$C(t)dt = \left[\begin{array}{l} Y(t)dt - \dfrac{e(t)p^*(t)}{p(t)} i(t)L(t)dt \\ -I(t)dt + \dfrac{p^*(t)e(t)}{p(t)} dL(t) \end{array}\right] \quad (4)$$

Note that in this study we do not assume "Purchasing Power Parity" (PPP) or the "Law of One Price", i.e., e(t)/p(t) = 1. As the PPP hypothesis states that the prices of a good (or a basket of goods) at home and abroad should be the same, when measured in a common currency, this hypothesis in essence assumes that the nominal exchange rate, e(t), adjusts to equalize domestic and foreign prices. If we denote the foreign price index as p*(t), then PPP states that e(t)p*(t) = p(t). When p*(t) is normalized at p*(t) = 1, then PPP is reduced to e(t)/p(t) = 1.
Further, by rearranging terms in equation (4), we get:

$$dL(t) = i(t)L(t)dt + \frac{p(t)}{p^*(t)e(t)}\big(C(t) + I(t) - Y(t)\big)dt \quad (5)$$

Now, we develop the SDE for the net worth, X(t):

$$dX(t) = dK(t) - \frac{p^*(t)e(t)}{p(t)} dL(t)$$

$$= I(t)dt - \left(\frac{p^*(t)e(t)}{p(t)}\right)\left[\begin{array}{l} i(t)L(t) + \\ \left(\dfrac{p(t)}{p^*(t)e(t)}\right)\big[C(t)+I(t)-Y(t)\big] \end{array}\right]dt$$

$$= I(t)\left[1 - \left(\frac{e(t)}{p(t)}\right)\left(\frac{p(t)}{e(t)}\right)\right]dt - \left[\begin{array}{l} \left(\dfrac{e(t)}{p(t)}\right) i(t)L(t) + \\ \left(\dfrac{e(t)}{p(t)}\right)\left(\dfrac{p(t)}{e(t)}\right)\big[C(t)-Y(t)\big] \end{array}\right]dt$$

$$= -\left[\left(\frac{p^*(t)e(t)}{p(t)}\right) i(t)L(t) + \big[C(t)-Y(t)\big]\right]dt \quad (6)$$

By substituting

$$Y(t)dt = K(t)b(t)dt = \big(X(t) + L(t)\big)b(t)dt$$

in eqn. (6),
we get:

$$dX(t) = -\left[\begin{array}{l} \left(\dfrac{p^*(t)e(t)}{p(t)}\right) i(t)L(t) \\ + \big[C(t) - \big(X(t)+L(t)\big)b(t)\big] \end{array}\right]dt$$

Dividing by X(t), we get:

$$\frac{dX(t)}{X(t)} = -\left[\begin{array}{l} \left(\dfrac{p^*(t)e(t)}{p(t)}\right) i(t)\dfrac{L(t)}{X(t)} \\ + \left[\dfrac{C(t)}{X(t)} - \left(1 + \dfrac{L(t)}{X(t)}\right)b(t)\right] \end{array}\right]dt \quad (7)$$

If we define $l(t) = \dfrac{L(t)}{X(t)}$, $c(t) = \dfrac{C(t)}{X(t)}$, (8)

then,

$$\frac{dX(t)}{X(t)} = -\left[\begin{array}{l} \left(\dfrac{p^*(t)e(t)}{p(t)}\right) i(t)l(t) \\ + \big[c(t) - \big(1 + l(t)\big)b(t)\big] \end{array}\right]dt$$

$$= -c(t)dt - \left(\frac{p^*(t)e(t)}{p(t)}\right) i(t)l(t)dt + \big(1 + l(t)\big)b(t)dt \quad (9)$$

Assuming that the rate of interest i(t) on US dollar-denominated loans and bonds can be represented by the following process:

$$i(t)dt = idt + \sigma_i dW_i(t) \quad (10)$$

The other source of uncertainty is the return on investments, b(t), which can be represented by the following process:

$$b(t)dt = bdt + \sigma_b dW_b(t) \quad (11)$$

By substituting equations (10) and (11) in equation (9), we get:

$$\frac{dX(t)}{X(t)} = -c(t)dt - \left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\left(idt + \sigma_i dW_i(t)\right) + \left(1 + l(t)\right)\left(bdt + \sigma_b dW_b(t)\right)$$

Collecting terms, we get:

$$\frac{dX(t)}{X(t)} = \left[\left(b - c(t)\right) - i\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t) + bl(t)\right]dt - \left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \quad (12)$$

Then, the unknowns are: (1) c(t)=C(t)/X(t), (2) *l*(t)=L(t)/X(t), , and (3) e(t).

3 Problem Solution (The Optimization Problem)

The objective of the policy maker is to find the normalized consumption, the normalized foreign debt, and the real exchange rate that maximizes the expected value of the utility of consumption and the utility of the networth at time T, the end of the optimization period. Specifically, we need to find:

$$\max_{c(s),x(T)} E\left\{\int_0^T e^{-\rho s} U(c(s))ds + U_x\left(x(T)\right)\right\} \quad (13)$$

subject to the dynamic constraints:

$$\frac{dX(t)}{X(t)} = \left[\left(b - c(t)\right) - i\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t) + bl(t)\right]dt - \left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \quad (12)$$

and

$$X(0) = x$$

where ρ is the discount rate, $U(c(s))$ is the utility function of households, and $U_x\left(x(T)\right)$ is the utility function of the final value of the economy's net worth. Thus, the two utility functions, $U\left(c(s)\right)$ and $U_x\left(x(T)\right)$, reflect, respectively, the desire to increase the public welfare or consumption, and the desire to increase the net worth of the society at some future time T.

We use the martingale optimality principle to get c(t), *l*(t), and $p^*(t)$ e(t)/p(t).

After some manipulations (see Appendix A), we obtain the optimal values as:

$$\left(p^*(t)\frac{e(t)}{p(t)}\right) \approx \frac{i}{\sigma_i^2}\frac{\sigma_b^2}{b} = \frac{i/\sigma_i^2}{b/\sigma_b^2} \quad (A.\ 19)$$

Recall that p(t) is the price index of the home country, e(t) is the nominal exchange rate between the home currency and the US Dollar, i is the mean value of the US interest rate, with variance σ_i^2, and b is the mean value of the return on investment for the home country, with variance σ_b^2.

(Note that the left-hand-side of the real exchange rate equation (A.19) is e(t)p*(t)/p(t), with p*(t) being the US consumer price index set at 1.) Further, equation (A.19) states that as the return on investments in the home country increases above the US interest rate, indicating strength of an economy, the exchange rate will appreciate, i.e., e(t)/p(t) will decrease. If the variance σ_b^2 increases drastically, i.e., if the uncertainty regarding the return on domestic investment increases, the exchange rate will depreciate, i.e., e(t)/p(t) will increase.

The optimal value of the debt (bonds and loans) ratio $l(t) = L(t)/X(t)$ is derived as:

$$l(t) = \frac{(1/\gamma_x)\left(b - \xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)}{\sigma_b^2} - 1,\ 0 < \xi < 1,\quad \gamma_x > 0 \qquad \text{(A. 16)}$$

where ξ is a tuning parameter, controlled by the policy maker, and γ_x is the degree of risk aversion, being small for a risk taker. Equation (A. 16) states that the optimal value of debt (foreign-currency denominated bonds and loans) is related to the productivity of the economy, as represented by the mean return on investment, b, with the economy allowed to borrow more as its performance improves. The optimal exchange rate is related to the optimal external debt as follows:

$$\left(\frac{p^*(t)e(t)}{p(t)}\right) = (1/\gamma_x)\frac{(1-\xi)i}{\sigma_i^2}\frac{1}{(1-l(t))},\qquad \gamma_x > 0,\quad 0 < \xi < 1 \qquad \text{(A. 17)}$$

This equation can be used to calculate the optimal external debt, l(t)=L(t)/X(t):

$$l(t) = \frac{(1/\gamma_x)\left(b - \xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)}{\sigma_b^2} - 1,$$

$$(1/\gamma_x)\left(b - \xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right) >> \sigma_b^2, \qquad 0 \xi < 1,\quad 0 < \gamma_x < 1$$

with ξ and γ_x used as tuning parameters.

An exact but complicated expression, for the optimal exchange rate, is also obtained as:

$$\left(\frac{p^*(t)e(t)}{p(t)}\right) = \frac{\left(\frac{b}{\xi i} + \gamma_x\sigma_b^2\right)}{2} \pm \frac{1}{2}\sqrt{\left(\frac{b}{\xi i} + \gamma_x\sigma_b^2\right)^2 - 4\frac{\sigma_b^2}{\sigma_i^2}\gamma_x\frac{(1-\xi)}{\xi}},\ 0 < \xi < 1,\quad \gamma_x > 0 \qquad \text{(A. 21)}$$

Using the approximate expressions, the results, however, are close to the ones obtained from the exact expression. The optimal value of the external debt, l(t), is different from the value obtained from the Hamilton-Jacobi-Bellman (HJB) equation-based optimal control approach [8]. Note that [8] use the PPP assumption, i.e., the actual exchange rate was not taken into consideration.

4 Results and Conclusions

In this section, we apply the above derived optimal values to the Egyptian economy. Thus, equation (A. 19) could be written as:

$$e(t) \approx \frac{i/}{\sigma_i^2}\frac{\sigma_b^2}{b} = \frac{i/\sigma_i^2}{b/\sigma_b^2}\frac{p(t)}{p^*(t)} \qquad (14)$$

Taking the logarithm (ln) of both sides, we get:

$$\ln e(t) \approx [\ln p(t) - \ln p^*(t)] + \left\lfloor \ln\frac{i}{\sigma_i^2} - \ln\frac{b}{\sigma_b^2}\right\rfloor \qquad (15)$$

Equation (15) is similar to other results reported in the literature using empirical models [2] and [3].

Below, using data for Egypt, we present our model's findings about the return on domestic investment, average interest paid on external debt, actual and optimal debt ratios, and actual, optimal, and PPP exchange rates. First, the return on investment b((t), defined as $b(t) = \Delta Y(t)/K(t)$, where K(t) is gross fixed capital formation (GFCF) and Y(t) is the GDP, both in current $US, is shown in Figure 1. We also present the estimate of b(t) using a Moving Average model of order 2.

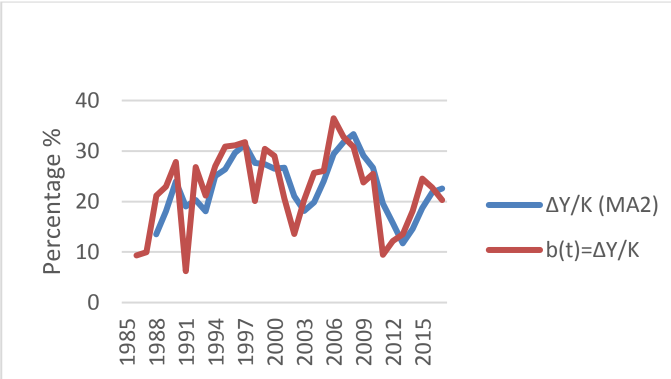

Fig. 1: Return on Investments b(t), True and Estimate

Further, Figure 2 shows that the interest paid on Egypt's foreign-currency denominated bonds and loans had an elliptic shape.

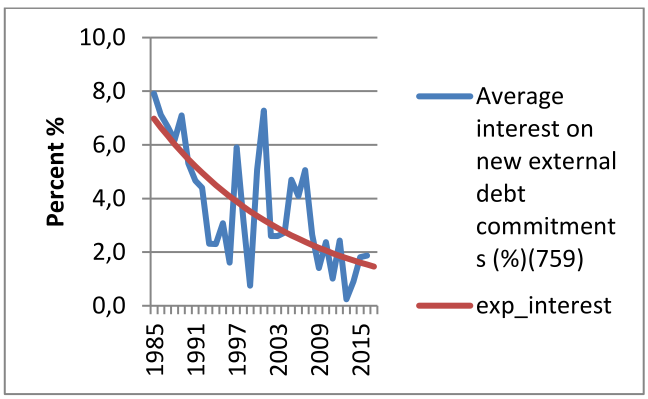

Fig. 2: Average interest on new external debt commitments (%)

In addition, Figure 3 shows the actual and approximate optimal external debt ratios, $l(t)=L(t)/X(t)$, for Egypt. In particular, both conservative (high value for γ) and risky (low value for γ) estimates are displayed for the approximate optimal external debt. It is observed that the actual external debt-to-net worth ratio is lower than the risky approximate optimal external debt-to-net worth ratio for almost the entire period analyzed, 1985-2017 (with the exception of 1991 and 2011). However, the actual external debt-to-net worth ratio is higher than the conservative approximate optimal external debt-to-net worth ratio for the period 1985-1995; is about equal between 1996-2010; and is higher again during 2011-2017. These results indicate that Egypt's actual external debt-to-net worth ratio was higher than the conservative approximate optimal external debt-to-net worth ratio for 18 out of the 33 years in the sample, implying that the country had contracted more debt than it should during these years.

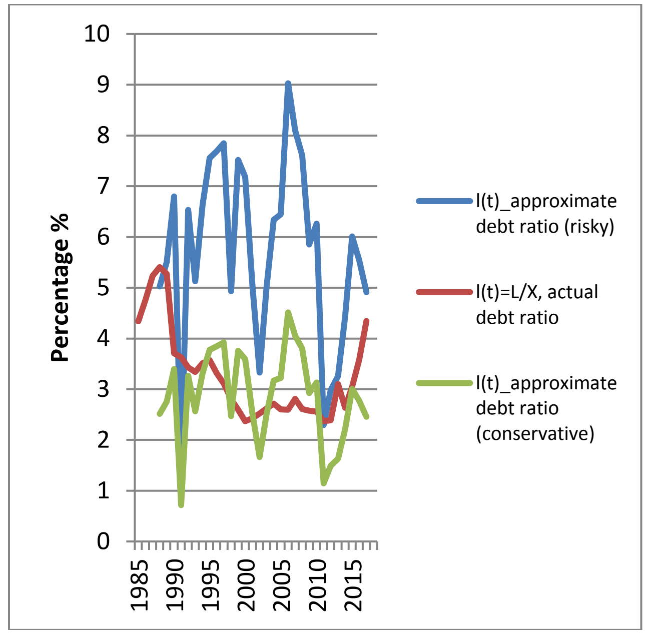

Fig. 3: Actual, Optimal (conservative), and Optimal (risky) debt ratio l(t)=L(t)/X(t)

Finally, Figure 4 depicts the estimated approximate optimal exchange rate for Egypt, the nominal (official) exchange rate and the purchasing power parity (PPP) exchange rate (using WDI data for annual inflation P(t), the PPP and the nominal (official) exchange rates).

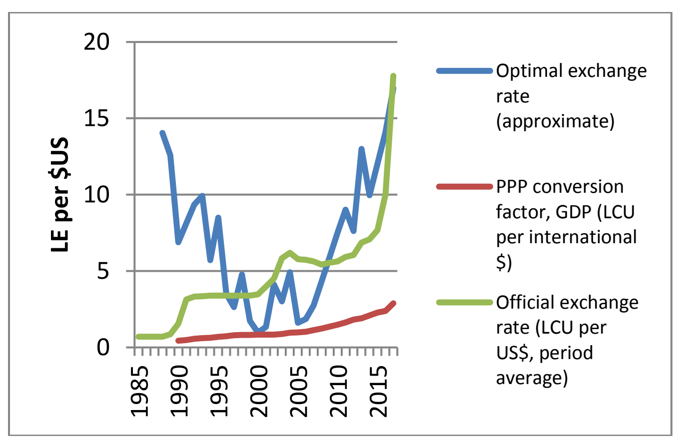

Fig. 4: Optimal, Official, and PPP exchange rate

As shown in Fig. 4, Egypt's nominal (official) exchange rate (LE/$US) was much less than its optimal values during 1985-1994. Then, it stayed consistently above its optimal values during 1998-2008, while, starting in 2009, the official exchange rate is less than its optimal values, but converging in 2015-2016 and became almost equal in 2017.

Ultimately, for these results to be used for policy analysis, further refining in our derived formulas needs to be contacted and continuous update of the empirical findings to be performed. In particular, future research could focus on the relationship of optimal exchange rates, based on the optimization of various social utility functions, and optimal sovereign debt paths, as well as on the applicability and potential shortcomings of the proposed novel framework.

Appendix A (The Martingale Approach)

In this appendix we use the martingale optimality principle [11] and [12] to find the optimal value of the normalized external debt and consequently the optimal level of the exchange rate.

First, we employ the SDE of the net worth, X(t), as given by eqn. (14):

$$\frac{dX(t)}{X(t)} = \left[\left(b - c(t)\right) + \left(b - i\left(\frac{p^*(t)e(t)}{p(t)} \right) l(t) \right) \right] dt - \left(\frac{p^*(t)e(t)}{p(t)} \right) l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \qquad \text{(A. 1)}$$

The unknowns are: (1) c(t)=C(t)/X(t), (2) *l*(t)=L(t)/X(t), and (3) e(t). .

For the purposes of this analysis, we define the utility function from normalized consumption as:

$$U\left(c(t)\right) = \frac{c^{1-\gamma}}{1-\gamma}, \ \gamma > 0, \gamma \neq 1 \qquad \text{(A. 2)}$$

Also, we define the objective function from net worth as:

$$V\left(X(0)\right) = \max_{c(s), x(T)} E\left\{ \int_0^T e^{-\rho s} U(c(s))ds + U_x\left(x(T)\right) \right\} \qquad \text{(A. 3)}$$

Thus, we have two utility functions, $U\left(c(t)\right)$ and $U_x\left(x(T)\right)$. The first reflects the desire to increase the public welfare or consumption, while the second one reflects the desire to increase the net worth of the society at some future time T.

In the martingale approach, we need to find the process H(t) such that

$H(t)X(t) + \int_0^t H(s)X(s)\left[c(s) - b\right]ds$ is a martingale.

Assume that H(t) has the following SDE:

$$\frac{dH(t)}{H(t)} = \alpha_H(t)dt + \beta_{Hb}(t)dW_b(t) + \beta_{Hi}(t)dW_i(t) \qquad \text{(A. 4)}$$

where $W_b(t)$ and $W_i(t)$ are Wiener processes. The drift and diffusion coefficients are unknowns to be estimated.

Using Ito's Lemma, we get:

$$d(HX) = HdX + XdH + dHdX$$

$$= H(t)X(t)\left\{ \begin{array}{l} \left[\left(b - c(t)\right) + \left(b - i\left(\frac{e(t)}{p(t)} \right) l(t) \right) \right] dt \\ - \left(\frac{e(t)}{p(t)} \right) l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \end{array} \right\}$$
$$+ X(t)H(t)\left[\alpha_H dt + \beta_{Hb} dW_b(t) + \beta_{Hi} dW_i(t)\right]$$
$$+ H(t)\left[\alpha_H dt + \beta_{Hb} dW_b(t) + \beta_{Hi} dW_i(t)\right]X(t)$$
$$\left\{ \begin{array}{l} \left[\left(b - c(t)\right) + \left(b - i\left(\frac{e(t)}{p(t)} \right) l(t) \right) \right] dt \\ - \left(\frac{e(t)}{p(t)} \right) l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \end{array} \right\}$$

$$= H(t)X(t)\left\{ \begin{array}{l} \left[\left(b - c(t)\right) + \left(b - i\left(\frac{p^*(t)e(t)}{p(t)} \right) l(t) \right) \right] dt \\ - \left(\frac{p^*(t)e(t)}{p(t)} \right) l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \end{array} \right\}$$
$$+ X(t)H(t)\left[\alpha_H dt + \beta_{Hb} dW_b(t) + \beta_{Hi} dW_i(t)\right]$$
$$+ H(t)X(t)\left[\left(1 + l(t)\right)\sigma_b \beta_{Hb} dt - \left(\frac{p^*(t)e(t)}{p(t)} \right) l(t)\sigma_i \beta_{Hi} dt \right]$$

Collecting terms, we get:

$$\frac{d\left(H(t)X(t)\right)}{H(t)X(t)} = \left\{ \left[\begin{array}{l} \left(b - c(t)\right) + \left(b - i\left(\frac{e(t)}{p(t)} \right) l(t) \right) \\ + \alpha_H + \left(1 + l(t)\right)\sigma_b \beta_{Hb} - \left(\frac{e(t)}{p(t)} \right) l(t)\sigma_i \beta_{Hi} \end{array} \right] dt \right\}$$
$$+ \left[\beta_{Hi} - \left(\frac{e(t)}{p(t)} \right) l(t)\sigma_i \right] dW_i(t) + \left[\beta_{Hb} + \left(1 + l(t)\right)\sigma_b\right] dW_b(t)$$

Moving $(b-c(t))$dt to the left-hand side, we get:

$$d(HX)+HX(c(t)-b)dt = HX\left\{\left[\begin{array}{l}\left(b-i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)l(t) \\ +\alpha_H+(1+l(t))\sigma_b\beta_{Hb}-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\beta_{Hi}\end{array}\right]dt\right\}$$

$$+H(t)X(t)\left[\beta_{Hi}-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\right]dW_i(t)+H(t)X(t)\left[\beta_{Hb}+(1+l(t))\sigma_b\right]dW_b(t)$$

or

$$d(HX)+HX(c(t)-b-\sigma_b\beta_{Hb}-\alpha_H)dt =$$

$$HX\left\{\left[\left(b-i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)l(t)+l(t)\sigma_b\beta_{Hb}-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\beta_{Hi}\right]dt\right\}$$

$$+H(t)X(t)\left[\beta_{Hi}-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\right]dW_i(t)$$

$$+H(t)X(t)\left[\beta_{Hb}+(1+l(t))\sigma_b\right]dW_b(t)$$

For

$$H(t)X(t)+\int H(s)X(s)(c(s)-b-\sigma_b\beta_{Hb}-\alpha_H)ds$$

to be a martingale, we need the drift term to be 0.

This requires that:

$$\left(b-i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)l(t)+l(t)\sigma_b\beta_{Hb}$$

$$-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\beta_{Hi}=0$$

A possible solution is $\alpha_H=0$,
or $\alpha_H+\sigma_b\beta_{Hb}=0$

We can now divide $i\left(\frac{p^*(t)e(t)}{p(t)}\right)$ into two parts:

$$i\left(\frac{p^*(t)e(t)}{p(t)}\right)=(1-\xi)i\left(\frac{p^*(t)e(t)}{p(t)}\right)+\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)$$

Thus,

$$\left(b-\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)l(t)-(1-\xi)i\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)$$

$$+l(t)\sigma_b\beta_{Hb}-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i\beta_{Hi}=0$$

Collecting terms, we get:

$$\left(b-\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)+\sigma_b\beta_{Hb}\right)l(t)$$

$$-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)(\sigma_i\beta_{Hi}+(1-\xi)i)=0$$

Then, a possible solution is:

$$\left(b-\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)+\sigma_b\beta_{Hb}\right)=0,$$ which

yields $$\beta_{Hb}=\frac{b-\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)}{\sigma_b} \quad \text{(A.5a)}$$

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)(\sigma_i\beta_{Hi}+(1-\xi)i)=0$$

which yields $$\beta_{Hi}=\frac{(1-\xi)i}{\sigma_i} \quad \text{(A.5b)}$$

and $$\alpha_H=-\sigma_b\beta_{Hb}=-\sigma_b\frac{b-\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)}{\sigma_b}$$

$$=-b+\xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)$$

(A.5c)

Substituting in the d(HX) equation above, we get:

$$d(HX)+HX(c(t)-b)dt =$$

$$H(t)X(t)\left[\beta_{Hi}-\left(\frac{e(t)}{p(t)}\right)l(t)\sigma_i\right]dW_i(t)$$

$$+H(t)X(t)\left[\beta_{Hb}+(1+l(t))\sigma_b\right]dW_b(t)$$

(A. 6)

and the SDE for H(t) becomes:

$$\frac{dH(t)}{H(t)} = \left[-b + \xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)\right]dt$$
$$+\left[\frac{b - \xi i\left(\frac{p^*(t)e(t)}{p(t)}\right)}{\sigma_b}\right]dW_b(t) + \left[\frac{(1-\xi)i}{\sigma_i}\right]dW_i(t)$$

The new Optimization Problem

The new system dynamics become:

$$\frac{d(HX)}{(HX)} + (c(t)-b)dt = \left[\beta_{Hi} - \left(\frac{e(t)}{p(t)}\right)l(t)\sigma_i\right]dW_i(t)$$
$$+ H(t)X(t)[\beta_{Hb} + (1+l(t))\sigma_b]dW_b(t)$$

Integrating both sides, between 0 and T, we get:

$$H(T)X(T) - H(0)X(0) - \int_0^T H(s)X(s)(c(s)-b)ds$$

$$= \int_0^T X(s)H(s)\left[\begin{array}{l}\left[\beta_{Hi} - \left(\frac{e(s)}{p(s)}\right)l(s)\sigma_i\right]dW_i(s) \\ + [\beta_{Hb} + (1+l(s))\sigma_b]dW_b(s)\end{array}\right]$$

Taking the expectation of both sides and using the fact that E{H(0)}=1, we get:

$$E\left\{H(T)X(T) - \int_0^T H(s)X(s)(c(s)-b)ds\right\} = E\{H(0)X(0)\}$$
$$= X(0)E\{H(0)\}$$
$$= X(0) \qquad \text{(A. 7)}$$

The optimization problem could now be stated as follows:

Find c(t) and X(T) that maximize $V(X(0))$:

$$V(X(0)) = \max_{c(s),x(T)} E\left\{\int_0^T e^{-\rho s}U(c(s))ds + U_x(X(T))\right\} \qquad \text{(A. 8)}$$

subject to the constraint:

$$E\left\{H(T)X(T) - \int_0^T H(s)X(s)c(s)ds\right\} = X(0)$$

with $$U(c(t)) = \frac{c^{1-\gamma}}{1-\gamma},\ \gamma > 0, \gamma \neq 1 \quad \text{(A. 2)}$$

Using the method of the Lagrange multiplier, we need to find:

$$\max_{c(s),X(T)} E\left\{\int_0^T e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma}ds + U_x(X(T))\right\} -$$
$$\Lambda\left[E\left\{H(T)X(T) - \int_0^T H(s)X(s)(c(s)-b)ds\right\} - X(0)\right]$$

,

which has the form:

$$\max_{c(s),X(T)} E\left\{\begin{array}{l}\int_0^T\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)X(s)(c(s)-b)\right]ds \\ +U_x(X(T)) \\ -\Lambda[H(T)X(T) - X(0)]\end{array}\right\}$$

where Λ is the Lagrange multiplier. Assuming that the conditions for the exchange of derivative and expectation are satisfied, taking the derivative for c(t) we get:

$$\frac{\partial}{\partial c(s)}E\left\{\int_0^T\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)X(s)(c(s)-b)\right]ds\right\} = 0$$

i.e.,

$$\frac{\partial}{\partial c(s)}\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)X(s)(c(s)-b)\right] = 0$$

which yields:

$$e^{-\rho s}c(s)^{-\gamma} - \Lambda H(s)X(s) = 0$$

i.e., $$c(s)^{-\gamma} = e^{\rho s}\Lambda H(s)X(s)$$

Taking the ln of both sides, we get:
$$-\gamma \ln c(s) = \rho s + \ln \Lambda H(s)X(s)$$

and $$\ln c(s) = -\frac{\rho s}{\gamma} + \left(\frac{-1}{\gamma}\right)\ln \Lambda H(s)X(s)$$

Thus,

$$c(t)=e^{-\rho t/\gamma}\left(\Lambda\right)^{(-1/\gamma)}\left(H(t)X(t)\right)^{(-1/\gamma)},\ 0<t\le T\ , \qquad \text{(A. 9)}$$

with Λ being a constant deterministic value.
We can now find an expression for the optimal X(T):

$$\frac{\partial}{\partial X(T)}E\left\{\begin{array}{l}\int_0^T\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma}+\Lambda H(s)X(s)\left(c(s)-b\right)\right]ds\\ +U_x\left(X(T)\right)\\ -\Lambda\left[H(T)X(T)-X(0)\right]\end{array}\right\}=0$$

i.e.,

$$\frac{\partial}{\partial X(T)}E\left\{U_x\left(X(T)\right)-\Lambda\left[H(T)X(T)-X(0)\right]\right\}=0$$

Interchanging derivative and expectations, we get:

$$\frac{\partial}{\partial X(T)}U_x\left(X(T)\right)-\frac{\partial}{\partial X(T)}\Lambda\left[H(T)X(T)-X(0)\right]=0$$

By setting $U_x\left(X(T)\right)=\dfrac{X(T)^{1-\gamma_x}}{1-\gamma_x}$, we get:

$$\frac{\partial}{\partial X(T)}\frac{X(T)^{1-\gamma_x}}{1-\gamma_x}-\frac{\partial}{\partial X(T)}\Lambda\left[H(T)X(T)-X(0)\right]=0$$

which yields:

$$X(T)^{-\gamma_x}-\Lambda H(T)=0$$

i.e., $X(T)^{-\gamma_x}=\Lambda H(T)$

Taking the ln of both sides, we get:

$$-\gamma_x\ln X(T)=\ln\Lambda H(T)$$

and $\ln X(T)=\left(\dfrac{-1}{\gamma_x}\right)\ln\Lambda H(T)$

Thus, $X(T)=\left(\Lambda\right)^{(-1/\gamma_x)}\left(H(T)\right)^{(-1/\gamma_x)}$ (A. 10)

We assume that this optimal value is also valid for all “t” [13].

<u>An SDE for X(t)</u>

Assume that the equation of X*(T) is partially valid for X*(t) i.e. the drift part is not correct while the diffusion part is correct. We know that $H(t)X(t)+\int_0^t H(s)X(s)\left[c(s)-b\right]ds$ is a martingale, and changing the probability measure does not change the diffusion part and only the drift part will change. And since it is a martingale, it only has a diffusion part. Thus, under the change of measure, the SDE for $H(t)X(t)+\int_0^t H(s)X(s)\left[c(s)-b\right]ds$ stays the same. This is why we work with only the diffusion [13].

We now derive an SDE for the optimal $0<X(t)\le T$. Since we are assuming that:

$$X(t)=\left(\Lambda\right)^{(-1/\gamma_x)}\left(H(t)\right)^{(-1/\gamma_x)},\ 0<t\le T\ ,$$

then, $dX(t)=\left(\Lambda\right)^{(-1/\gamma_x)}d\left(H(t)\right)^{(-1/\gamma_x)}\ \ 0<t\le T$

(A.11)

This is only for the diffusion part.
We recall that:

$$\frac{dH(t)}{H(t)}=\alpha_H dt+\beta_{Hb}dW_b(t)+\beta_{Hi}dW_i(t) \qquad \text{(A. 4)}$$

If we define $y=H^\alpha$, using Ito Lemma, we get:

$$dy=\frac{\partial H^\alpha}{\partial H}dH+\frac{1}{2}\frac{\partial^2 H^\alpha}{\partial H^2}\left(dH\right)^2$$

Substituting the expression for dH(t), we get:

$$dy=\alpha H^{\alpha-1}H\left(\alpha_H dt+\beta_{Hb}dW_b(t)+\beta_{Hi}dW_i(t)\right)$$
$$+\frac{1}{2}\alpha(\alpha-1)H^{\alpha-2}H^2\left(\alpha_H dt+\beta_{Hb}dW_b(t)+\beta_{Hi}dW_i(t)\right)^2$$

and

$$d\left(H(t)\right)^\alpha=(.)dt+\alpha\left(H(t)\right)^\alpha\left(\beta_{Hb}dW_b(t)+\beta_{Hi}dW_i(t)\right) \qquad \text{(A. 13)}$$

where $(.)$ has all the drift terms.
Setting $\alpha=\left(-1/\gamma_x\right)$, we get:

$$d\left(H(t)\right)^{(-1/\gamma_x)} = (.)dt + \left(-1/\gamma_x\right)\left(H(t)\right)^{(-1/\gamma_x)}$$
$$\left(\beta_{Hb}dW_b(t) + \beta_{Hi}dW_i(t)\right) \tag{A. 14}$$

Substituting "$d\left(H(t)\right)^{(-1/\gamma_x)}$" into the equation of dX(t) of (A. 11), we get:

$$dX(t) = \left(-1/\gamma_x\right)\left(\Lambda\right)^{(-1/\gamma_x)}\left(H(t)\right)^{(-1/\gamma_x)}$$
$$\left[(.)dt + \left(\beta_{Hb}dW_b(t) + \beta_{Hi}dW_i(t)\right)\right]$$

Since $X(t) = \left(\Lambda\right)^{(-1/\gamma_x)}\left(H(t)\right)^{(-1/\gamma_x)},\ 0 < t \le T$, then,

$$dX(t) = \left(-1/\gamma_x\right)X(t)\left[(.)dt + \left(\beta_{Hb}dW_b(t) + \beta_{Hi}dW_i(t)\right)\right] \tag{A.15}$$

Equation (A. 15) shows that the optimal net worth, X(t), follows a Geometric Brownian motion with linear trend. This equation is valid only for the diffusion part.

We know from the setup of the problem that X(t) has another SDE:

$$\frac{dX(t)}{X(t)} = \left[-c(t) - i\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t) + b\left(1 + l(t)\right)\right]dt$$
$$-\left(\frac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i dW_i(t) + \left(1 + l(t)\right)\sigma_b dW_b(t) \tag{A.1}$$

We recall that β_{Hb} and β_{Hi} could be positive or negative and their signs do not matter because they are multiplied by a Wiener process.

Equating both equations, only the diffusion terms, we get:

$\left(1 + l(t)\right)\sigma_b = \left(-1/\gamma_x\right)\beta_{Hb}$, or more accurately $\left(1 + l(t)\right)\sigma_b = \left(1/\gamma_x\right)\beta_{Hb},\ \beta_{Hb} > 0$

which yields:

$$l(t) = \left(1/\gamma_x\right)\frac{\beta_{Hb}}{\sigma_b} - 1 = \frac{\left(1/\gamma_x\right)\beta_{Hb} - \sigma_b}{\sigma_b}$$

$$= \frac{\left(1/\gamma_x\right)\dfrac{b - \xi i\left(\dfrac{p^*(t)e(t)}{p(t)}\right)}{\sigma_b} - \sigma_b}{\sigma_b}$$

$$= \left(1/\gamma_x\right)\frac{b - \xi i\left(\dfrac{p^*(t)e(t)}{p(t)}\right)}{\sigma_b^2} - 1 \tag{A. 16}$$

and $-\left(\dfrac{p^*(t)e(t)}{p(t)}\right)l(t)\sigma_i = \left(-1/\gamma_x\right)\beta_{Hi}$

i.e.,

$$\left(\frac{p^*(t)e(t)}{p(t)}\right) = \left(1/\gamma_x\right)\frac{\beta_{Hi}}{\sigma_i}\frac{1}{l(t)} = \left(1/\gamma_x\right)\frac{(1-\xi)i}{\sigma_i^2}\frac{1}{l(t)}$$

$$,\ \gamma_x > 0,\ \beta_{Hi} > 0 \tag{A. 17}$$

This result indicates that as the economy improves, the optimal value of l(t) increases, which in turn leads to decreases in the exchange rate. Thus, an increase in the l(t), reflecting an increase in the GDP performance, results in an exchange rate appreciation, i.e., e(t) is reduced.

Substituting l(t) of equation (A. 16) into equation (A. 17), we get:

$$\left(\frac{p^*(t)e(t)}{p(t)}\right) = \left(1/\gamma_x\right)\frac{(1-\xi)i}{\sigma_i^2}\frac{\gamma_x\sigma_b^2}{\left[b - \xi i\left(\dfrac{p^*(t)e(t)}{p(t)}\right) - \gamma_x\sigma_b^2\right]}$$

$$= \frac{(1-\xi)i}{\sigma_i^2\xi i}\frac{\sigma_b^2}{\left[\dfrac{b}{\xi i} - \left(\dfrac{p^*(t)e(t)}{p(t)}\right) - \dfrac{\gamma_x\sigma_b^2}{\xi i}\right]}$$

$$= \frac{(1-\xi)}{\sigma_i^2\xi}\frac{\sigma_b^2}{\left[\dfrac{b}{\xi i} - \left(\dfrac{p^*(t)e(t)}{p(t)}\right) - \dfrac{\gamma_x\sigma_b^2}{\xi i}\right]}$$

Or,

$$\left(\frac{p^*(t)e(t)}{p(t)}\right) = \frac{(1-\xi)i}{\sigma_i^2}\frac{\sigma_b^2}{b}\frac{1}{\left[1 - \xi\dfrac{i}{b}\left(\dfrac{p^*(t)e(t)}{p(t)}\right) - \gamma_x\dfrac{\sigma_b^2}{b}\right]} \tag{A. 18}$$

Assuming $\xi << 1$, such that $\left[1-\xi\frac{i}{b}\left(\frac{p^*(t)e(t)}{p(t)}\right)-\gamma_x\frac{\sigma_b^2}{b}\right]\approx\left(1-\gamma_x\frac{\sigma_b^2}{b}\right)$

then,

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)\approx\frac{i}{\sigma_i^2}\frac{\sigma_b^2}{b}\frac{1}{\left(1-\gamma_x\frac{\sigma_b^2}{b}\right)}$$

Assuming that $\xi << 1$ and $\gamma_x << 1$

Then,

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)\approx\frac{i}{\sigma_i^2}\frac{\sigma_b^2}{b}=\frac{i/\sigma_i^2}{b/\sigma_b^2} \qquad \text{(A. 19)}$$

This result indicates that as the mean rate of return on investments, b, increases relative to the US interest rate, *i*, the exchange rate e(t) will decrease, i.e., the local currency will appreciate.

An exact expression for the optimal exchange rate could be obtained if we use eqn. (A. 18) without approximations Viz;

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)\left[\left(\frac{b}{\xi i}-\left(\frac{p^*(t)e(t)}{p(t)}\right)\right)+\gamma_x\sigma_b^2\right]$$
$$=\frac{\sigma_b^2}{\sigma_i^2}\gamma_x\frac{(1-\xi)}{\xi}$$

i.e

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)^2-\left(\frac{p^*(t)e(t)}{p(t)}\right)\left(\frac{b}{\xi i}+\gamma_x\sigma_b^2\right)+\frac{\sigma_b^2}{\sigma_i^2}\gamma_x\frac{(1-\xi)}{\xi}=0 \qquad \text{(A. 20)}$$

Solving the quadratic equation we get:

$$\left(\frac{p^*(t)e(t)}{p(t)}\right)=\frac{\left(\frac{b}{\xi i}+\gamma_x\sigma_b^2\right)}{2} \pm\frac{1}{2}\sqrt{\left(\frac{b}{\xi i}+\gamma_x\sigma_b^2\right)^2-4\frac{\sigma_b^2}{\sigma_i^2}\gamma_x\frac{(1-\xi)}{\xi}}\ ,\ 0<\xi<1,$$
$$\gamma_x>0 \qquad \text{(A. 21)}$$

This is the exact expression for the optimal exchange rate.

References:

[1] Fleming, W. H., and J. L. Stein, “Stochastic Optimal Control, International Finance and Debt,” Journal of Banking and Finance, Vol. 28(5), pp. 979–996, 2004.
[2] Fischer, C., “Equilibrium Real Exchange Rate Estimates Across Time and Space,” Federal Reserve Bank of Dalas, Globalization Institute Working Paper No. 362, April, Dallas: Federal Reserve Bank of Dallas, 2019
[3] Brixiova, Z., B. Égert, and T. Hadj Amor Essid, “The Real Exchange Rate and External Competitiveness in Egypt, Morocco and Tunisia”, AFRICAN DEVELOPMENT BANK GROUP, Working Paper No. 187, 2013.
[4] Bussiere, M., M. Zorzi, A. Chudik, and A. Dieppe, “Methodological Advances in the Assessment of Equilibrium Exchange Rates,” European Central Bank, Working Paper No. 1151, 2010.
[5] Montiel, Peter J., “The Long-Run Equilibrium Real Exchange Rate: Theory and Measurement,” in Mohsin S. Khan, Saleh M. Nsouli, and Chorng-Huey Wong (eds), Macroeconomic Management: Programs and Policies, Washington: International Monetary Fund, 2002.
[6] MacDonald, R., “Concepts to Calculate Equilibrium Exchange Rates: An Overview”, Discussion Paper 3/00, Economic Research Group, Deutsche Bundesbank, 2000.
[7] Hansen, J., and W. Roeger, “Estimation of Real Equilibrium Exchange Rates,” European Commission, Directorate-General for Economic and Financial Affairs, Economic Papers No. 144, September (ECFIN/534/00-EN), Brussels: European Communities, 2000.

[8] Abutaleb, A., and M. G. Hamad, 2012, "Optimal Foreign Debt for Egypt: A Stochastic Control Approach," Economic Modeling, Vol. 29(3), pp. 544-556, 2012.

[9] Karatzas, Ioannis, John P. Lehoczky, and Steven E. Shreve, 1991, "Equilibrium Models with Singular Asset Prices," Mathematical Finance, Vol. 1, No. 3 (July), pp. 11-29, 1991.

[10] Dumas, B., 1992, "Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World," The Review of Financial Studies, Vol. 5 (2), pp. 153-180.

[11] Zhang A., Stochastic Optimization in Finance and Life Insurance: Applications of the Martingale Method, Ph. D dissertation, Department of Mathematics, University of Kaiserslautern, Germany, 2007.

[12] Abutaleb, A., "The estimation of the instantaneous amplitudes of sum of sinusoids with unknown frequencies and phases: The martingale approach", Signal Processing, Vol. 93, No 4, 2013.

[13] Karlsson, E., Optimal Portfolio Allocation by the Martingale Method in an Incomplete and Partially Observable Market, M. Sc., kth Royal Institute of Technology, School of Engineering Sciences, Sweden, 2016.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Dr. Abutaleb is responsible for the technical derivations and writings.

Dr. Papaiouannou is responsible for the conceptual framework and the conclusions.