

On a dynamic system of five interacting species in a natural environment with Holling's Type I functional responses for some and Holling's Type II functional responses for others

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Abstract: We propose and analyse a prey-predator model with five interacting species in a natural environment. We assume that there exists ecosystems which can group together in a given place a number of species interacting with each other through different types of interaction. In our model we assume that in such environment it can exist many type of interaction between species which live there. In our model, we use a mixed functional response. Both Holling's Type I and Type II functional responses are considered. We carry out a mathematical analysis of the proposed model followed by a numerical analysis. Our aim is to examine the dynamics of different populations in an interaction where the super predator has a diverse food source in a multi-species ecosystem and through the study of this model, to highlight the impact of species diversity and interaction diversity on the dynamics of species living in an environment favorable to these types of interactions. The mathematical analysis first concerns the existence, boundedness, local and global stability of the solutions using the Routh-Hurwitz' criterion and the Lyapunov's principle. In addition, we look for conditions under which solutions persist or die out. Finally, numerical simulations are carried out to support the theoretical results. The research will contribute to a better understanding and prediction of the evolution of populations living in areas that bring together several species with multiple interactions.

Key- Words: Prey-predator model, population dynamics, stability, persistence, extinction, diverse food source.

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1 Introduction

In nature, species are found in vast networks of ecological interactions that generally represent a set of environments, fragmented or compact, comprising a sufficient number of interconnected physical and biological elements to enable the renewal of the life cycle of the species living there. These interactions refer to the various complex relationships that exist between living beings in ecological networks, [1], [2]. Much research has been devoted to analyze and understand these complex relationships, [3], [4]. All these investigations demonstrate the essential role played by mathematics in the analysis, understanding and prediction of many natural phenomena. In the field of ecology in particular, mathematics offers the possibility of modeling complex systems that govern the relationships between living beings and their environment, [3], [4], [5], [6]. Mathematics uses equations to describe ecological systems. This

helps us to better understand certain ecological phenomena and develop decision-making policies aimed at protecting our planet and its biodiversity. This approach is important given the quality and quantity of the resources and services that biodiversity and ecosystems provide us with. They support many economic activities such as agriculture, fishing and tourism, etc., which are essential to our well-being, [3], [7].

Several studies have highlighted the different types of relationships that exist between species. Indeed, when different populations coexist in the same natural environment, they develop several types of interactions: predation, parasitism, mutualism, competition, neutralism, etc., which are likely to disrupt their survival, which can change their dynamics over time. These interactions can affect reproduction, induce high mortality rates, lead to emigration and immigration and thus impact species evolution,

[8], [9], [10], [11].

In their reproductive process, plants benefit from immeasurable help from animals. The work carried out by insects such as bees, butterflies, bumblebees, etc., in the process of pollinating and dispersing plant seeds in environments favorable to their germination is crucial to the reproduction, evolution and perpetuation of plants. In fact, these disseminators disperse seeds in the environment and enable plants to exist. On the other hand, these plants are essential to the survival of these animals, insofar as they constitute a principal and essential food resource in various forms. Animals use flower nectar, fruit pulp, leaves and buds, etc., to store energy and increase their fitness and population size. This type of relationship between different species is the phenomenon of mutualism, [12]. It should be noted that another form of biological interaction called interspecific commensalism is a form of coexistence between two species in which only one of the two species benefits from the relationship. The second species, which shelters and feeds the other, does not benefit from the relationship and is not endangered by it, [13]. The relationship between herons and oxen is a case in point. However, there is another type of interaction that is advantageous for one of the species in the relationship and dangerous for the other: parasitism. In this interaction, one of the species, the parasite, lives at the expense of the other species, called the host, which shelters and feeds it, [13].

Competition between two species is interpreted as a rivalry, a fight, a confrontation between these species in pursuit of the same goal. The aim is to monopolize the limited resources of a given environment by dominating competitors. This type of relationship has been extensively studied in [14]. These studies have highlighted a number of unexpected behaviors in the coevolution of interacting species. An example is the model described in references, [3], [4].

We should also mention the predation relationship between species. This relationship, first studied by [15], [16], hides several secrets of species coevolution. Prey switching, which is when a predator preferentially consumes the most abundant type of prey, and natural selection, which is when nature eliminates the least fit individuals, [17], are some examples. Indeed, predation is essential for regulating the evolution of different species in many environments. The predator exploits its prey by

feeding on it to gain energy, increase its fitness, and increase its population density; this is a relationship that is beneficial for the predator and harmful for the prey, [12].

This relationship will be further highlighted in this study. Indeed, in a given environment, it is possible to encounter several types of interactions listed above simultaneously. It therefore seems important to take into account the diversity of species living in the same environment, as well as the diversity of intraspecific and interspecific relationships between these species, to predict the various changes they are likely to undergo over time. The work carried out in [14], clearly shows notable changes in the evolution of interacting populations. The number of equilibrium points, the nature of these equilibria, and their dynamics are all elements that fluctuate considerably when the number of species and the nature of the interactions are taken into account. The interest is therefore to learn more about the dynamics by providing a more in-depth study in this regard.

This approach would certainly make it possible to establish or provide some insight into the relationship between the number of interacting populations and changes in quality or quantity within populations, thereby enabling a certain type of resilience. These changes can be structural, behavioral, organizational, or adaptive.

In this paper, we propose and analyze a new predator-prey interaction model in which five species interact. Our model incorporates generalist or polyphagous predators, whose distinctive feature is their highly diverse food source. They can use these resources alternately to improve their fitness and increase their population density. A second community consists of two predator populations, considered oligophagous. The resources they feed on to maintain their fitness and increase their density are in limited quantities. The final group consists of two prey populations that feed on resources available in limited quantities in nature: herbaceous plants, vegetables, conifers, mushrooms, etc.

Our work is organized as follows. In Section 2, we present our population evolution model for five species in a food-limited environment; then, we describe the various interactions that occur between the species, as well as their functional responses.

Next, in section 3, we study the boundedness of the solutions of the proposed system. In

sections 4 and 5, we analyze the various equilibrium points of the system and provide their conditions of existence, then we present an analysis of the global stability of the positive equilibrium point. The sections 6 and 7 are devoted to the study of species permanence and extinction. We provide the conditions under which species survive in the long period or, rather, become extinct after a certain period. Furthermore, in section 8, we perform numerical simulations to support our theoretical results using Matlab version R2014b. Several types of predator-prey dynamics are highlighted.

Finally, section 9 concludes. In this section, we summarize the various results obtained during the study. The main points of our work and the tools used are listed.

2 Description of model interactions

In this section, we propose a prey-predator model that highlights a food chain of five species. The model includes a community of two populations of prey of densities u and v competing with respect to the resource available in nature in limited quantities. Two other populations of predators of densities x and y are in competition but with respect to the habitat and a last community, where there is a super predator of density z . Predator x consumes prey u and predator y has prey v as resource. Predator z , more generalist than the two first, can attack species of the two other communities. The functional response of predators x and y to their respective prey will be of the Holling's type II. Furthermore, their growth is linked to the availability of food : the prey u and v . The functional response of the generalist predator to its prey is Holling's type I functional response for its prey u , v and x whose attack is systematic. But Holling's type II functional response for its prey y which is considered a little more difficult to capture.

Indeed, the quantity of prey that a predator can ingest per day and per unit of time is defined as the functional response of the predator to variations in the quantity of prey in the environment. This functional response is called Holling's Type I when the quantity of prey ingested by the predator evolves regularly over time. This linear evolution stipulates that the predator systematically captures any prey it encounters, based on its hunting efficiency and prey availability. It is described by the function $f(u) = \alpha u$, where α is the predation rate and u

is the prey density. Regarding the Holling's Type II functional response, it takes into account the predator's efficiency and satiety. On the one hand, the predator's efficiency plays an important role in the time it takes to capture its prey. The less efficient the predator, the longer it takes to find, reach, and manipulate its prey. On the other hand, when the predator reaches satiety, it is reasonable that the prey consumption rate decreases. This is represented by the function $f(u) = \frac{\beta u}{u+h}$, where β is the predation rate per individual, h the environmental protection rate, and u the prey density.



Figure 1: Example of a food chain, interaction of five species in nature. Source : [18].

An example of a terrestrial trophic structure that could reflect these prey-predator relationships is the golden eagle, the super predator, the marten and weasel, the intermediate predators, and the vole and squirrel, the prey of other predators (Figure 1). In this relationship, the golden eagle, capable of feeding on certain mammals such as foxes, cats, etc., reptiles such as snakes, lizards, etc., mustelids such as weasels, martens, stone martens, etc., rodents such as rats, squirrels, voles, etc., and even birds of prey, is an effective predator of both prey species mentioned above.

Mustelids (martens, weasels, etc.) may consume rodents such as rats, squirrels, voles, mice, hares, etc., amphibians, bird eggs, etc., so martens and weasels compete for resources and even habitat. In the latter community, rodents (squirrels, voles, etc.) feed on herbaceous plants, vegetables, conifers, mushrooms, etc., and it is assumed that the resources available to rodents in the environment are limited in quantity.

The above-mentioned interactions are summarized in the diagram in Figure 2 below.

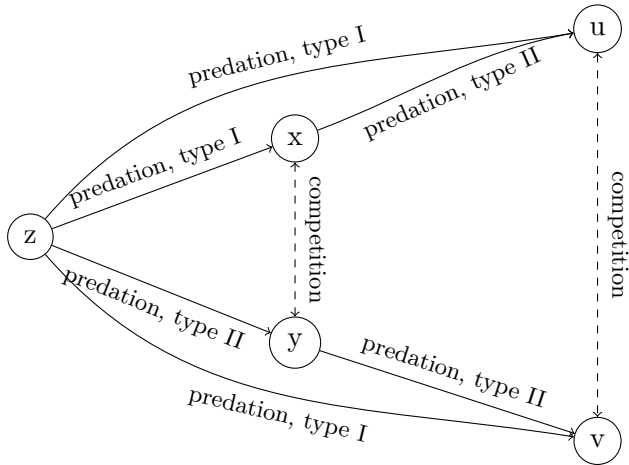


Figure 2: Diagram of interactions of five species. Source : Created by the authors.

Each population evolves over time according to a process of birth and death in a limited capacity environment. An intrinsic growth rate and a mortality rate then describe the evolution of populations outside of any other interaction. The local interactions thus defined are of the modified Leslie Gower type with Holling's type I and II functional responses. The model that results from these local interactions is then described by the following system of differential equations :

$$\begin{cases} \frac{du}{dt} = \left(r_u - \frac{r_u}{k}u - \frac{a_1}{u+h_u}x - a_2v - a_3z \right)u, \\ \frac{dv}{dt} = \left(r_v - \frac{r_v}{k}v - \frac{b_1}{v+h_v}y - b_2u - b_3z \right)v, \\ \frac{dx}{dt} = \left(r_x - \frac{c_1}{u+h_x}x - c_2y - c_3z \right)x, \\ \frac{dy}{dt} = \left(r_y - \frac{d_1}{v+h_y}y - d_2x - \frac{d_3}{y+h_y}z \right)y, \\ \frac{dz}{dt} = \left(r_z - \frac{e_1}{y+h_z}z + e_2u + e_3v + e_4x - m \right)z, \end{cases} \quad (1)$$

with,

$$u(0) \geq 0, \quad v(0) \geq 0, \quad x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0. \quad (2)$$

The parameters u, v, x, y and z are the densities of the interacting species. All parameters are strictly positive and different. In this model, r_u, r_v, r_x, r_y and r_z are the growth rates of the species under consideration; a_1, a_3, b_1, b_3, c_3 and d_3 are the predation rates of predators x, y and w on their respective preys. a_2, b_2, c_2 and d_2 are the competitive pressures. The parameters c_1, d_1 and e_1 are the maximum mortality rates of predators $x, y,$ and $z,$ respectively. The parameters e_2, e_3 and e_4 represent the preys assimilation coefficients. Parameter m is the mortality rate of species $z.$ Parameters h_u, h_v, h_x, h_y and h_z measure the environmental protection that species benefit from respectively. Finally, k denotes the capacity of the environment to support the species' populations.

We perform a mathematical analysis of the model resulting from these interactions. We study the qualitative behavior of the solutions. We examine the existence and boundedness of the solutions, then their local and global stability. We also verify the conditions under which the system could be permanent or, on the contrary, become extinct.

3 Analysis of the boundness of the model solutions

In this section, we adopt the following notations

$$\mathbb{R}_+^5 = \left\{ (u, v, x, y, z) \in \mathbb{R}^5 \mid \begin{array}{l} u \geq 0, v \geq 0, \\ x \geq 0, y \geq 0, \\ z \geq 0 \end{array} \right\},$$

and the interior of this set will be noted,

$$Int(\mathbb{R}_+^5) = \left\{ (u, v, x, y, z) \in \mathbb{R}^5 \mid \begin{array}{l} u > 0, v > 0, \\ x > 0, y > 0, \\ z > 0 \end{array} \right\},$$

let's put $E(t) = (u(t), v(t), x(t), y(t), z(t))^T,$

$$E(t_0) = (u(t_0), v(t_0), x(t_0), y(t_0), z(t_0))^T,$$

$$h(t, E(t)) = \begin{pmatrix} h_1(u, v, x, y, z) \\ h_2(u, v, x, y, z) \\ h_3(u, v, x, y, z) \\ h_4(u, v, x, y, z) \\ h_5(u, v, x, y, z) \end{pmatrix}$$

$$= \begin{pmatrix} \left(r_u - \frac{r_u}{k}u - \frac{a_1}{u+h_u}x - a_2v - a_3z \right)u \\ \left(r_v - \frac{r_v}{k}v - \frac{b_1}{v+h_v}y - b_2u - b_3z \right)v \\ \left(r_x - \frac{c_1}{u+h_x}x - c_2y - c_3z \right)x \\ \left(r_y - \frac{d_1}{v+h_{y1}}y - d_2x - \frac{d_3}{y+h_{y2}}z \right)y \\ \left(r_z - \frac{e_1}{y+h_z}z + e_2u + e_3v + e_4x - m_z \right)z \end{pmatrix},$$

thus, we define a function, $h(t, E(t)) = \frac{dE(t)}{dt}$, such as $h : [t_0, +\infty[\times \mathbb{R}_+^5 \rightarrow \mathbb{R}_+^5$ which is continuous and locally Lipschitzian on \mathbb{R}_+^5 . We first recall the following comparison lemma.

Lemma 3.1 [19] *Let λ and μ be two positive real numbers. We have the following assertions.*

1. *If $\lambda > 0$, $\mu > 0$ and $\frac{du}{dt} \geq u(\lambda - \mu u)$, when $t \geq 0$ and $u(0) > 0$, then, we have*

$$\liminf_{t \rightarrow +\infty} u(t) \geq \frac{\lambda}{\mu}. \quad (3)$$

2. *If $\lambda > 0$, $\mu > 0$ and $\frac{du}{dt} \leq u(\lambda - \mu u)$, when $t \geq 0$ and $u(0) > 0$, then, we have*

$$\limsup_{t \rightarrow +\infty} u(t) \leq \frac{\lambda}{\mu}. \quad (4)$$

Lemma 3.2 *The open orthant $Int(\mathbb{R}_+^5)$ is positively invariant under the system (1).*

Proof: We consider the following function $h : \mathbb{R}_+^* \times \mathbb{R}_+^5 \rightarrow \mathbb{R}_+^5$ defined by the following expression : $\frac{dE(t)}{dt} = h(t, E(t))$ i.e $\frac{dE(t)}{dt} = (h_1(t), h_2(t), h_3(t), h_4(t), h_5(t))$. Then, h is continuous and locally Lipschitzian on \mathbb{R}_+^5 , then the system (1) exists and is unique on the compact $[0, \epsilon]$ where, $0 < \epsilon \leq +\infty$ with the initial conditions $u(0) > 0$, $v(0) > 0$, $x(0) > 0$, $y(0) > 0$, and $z(0) > 0$. Furthermore, it is noted that

$$\frac{du(t)}{u} = \left[r_u - \frac{r_u}{k}u - \frac{a_1}{u+h_u}x - a_2v - a_3z \right] dt, \\ \frac{du(t)}{u} = h_1(u, v, x, z)dt, \quad (5)$$

and an integration of the equation (5) in the region $[0, t]$ leads to

$$u(t) = u(0)\exp\left(\int_0^t h_1(u, v, x, z)dt\right) > 0, \quad \forall t, \quad (6)$$

Moreover,

$$\frac{dv(t)}{v} = \left[r_v - \frac{r_v}{k}v - \frac{b_1}{v+h_v}y - b_2u - b_3z \right] dt, \\ \frac{dv(t)}{v} = h_2(u, v, y, z)dt, \quad (7)$$

and an integration of the equation (7) in the region $[0, t]$ leads to

$$v(t) = v(0)\exp\left(\int_0^t h_2(u, v, y, z)dt\right) > 0, \quad \forall t, \quad (8)$$

by reasoning in this way, we obtain the inequalities below

$$x(t) = u(0)\exp\left(\int_0^t h_3(u, x, y, z)dt\right) > 0, \quad \forall t, \\ y(t) = y(0)\exp\left(\int_0^t h_4(v, x, y, z)dt\right) > 0, \quad \forall t, \\ z(t) = z(0)\exp\left(\int_0^t h_5(u, v, x, y, z)dt\right) > 0, \quad \forall t,$$

then the densities $u(t)$, $v(t)$, $x(t)$, $y(t)$ et $z(t)$ are strictly positive for all $t > 0$. ■

Theorem 3.1 *Let the set Φ be defined by :*

$$\Phi = \left\{ E \in \mathbb{R}_+^5 \mid \begin{array}{l} 0 \leq u \leq k, 0 \leq v \leq k, \\ 0 \leq x \leq \alpha_x, 0 \leq y \leq \alpha_y, \\ 0 \leq z \leq \alpha_z \end{array} \right\},$$

where,

$$\alpha_x = \frac{r_x(k+h_x)}{c_1}, \quad \alpha_y = \frac{r_y(k+h_{y1})}{d_1} \text{ and} \\ \alpha_z = \frac{(c_1(r_z+ke_2+ke_3)+r_xe_4(k+h_x))}{c_1d_1e_1} \times \\ (r_y(k+h_y)+d_1h_z).$$

- i) *The set Φ is positively invariant under the system (1).*

- ii) *All solutions of (1) from \mathbb{R}_+^5 are bounded in \mathbb{R}_+^5 and enter the region of attraction Φ .*

Proof: Consider the initial solution $E(0)$ in the set Φ . Then, lemma 3.2 ensures that any solution $E(t)$, $t > 0$, of the system (1) is strictly positive. We study each equation of the system (1). Thus, according to (1)₁ we obtain :

$$\begin{cases} \frac{du}{dt} \leq (r_u - \frac{r_u}{k}u)u, \\ u(0) = u_0 \geq 0, \end{cases} \quad (9)$$

and by lemma 3.1 we have,

$$\limsup_{t \rightarrow \infty} u(t) \leq k, \quad (10)$$

thus, for all $\varepsilon_1 > 0$, there exists $T > 0$ such that

$$u(t) \leq k + \varepsilon_1 \text{ for all } t \geq T, \quad (11)$$

consequently in an analogous way, for all $\varepsilon_2 > 0$ there exists $T > 0$ such that :

$$v(t) \leq k + \varepsilon_2 \text{ for all } t \geq T, \quad (12)$$

Moreover, from (1)₃ we have

$$\begin{cases} \frac{dx}{dt} \leq \left(r_x - \frac{c_1}{u+h_x}x\right)x \\ x(0) = x_0 \geq 0 \end{cases} \\ \implies \begin{cases} \frac{dx}{dt} \leq \left(r_x - \frac{c_1}{k+h_x}x\right)x, \\ x(0) = x_0 \geq 0, \end{cases} \quad (13)$$

and according to the comparison lemma 3.1 we have,

$$\limsup_{t \rightarrow \infty} x(t) \leq \frac{k + h_x}{c_1}, \quad (14)$$

thus, for all $\varepsilon_3 > 0$, there exists $T > 0$ such that

$$x(t) \leq \frac{r_x(k + h_x)}{c_1} + \varepsilon_3 \text{ for all } t \geq T, \quad (15)$$

in a similar way, for all $\varepsilon_4 > 0$ there exists $T > 0$ such that :

$$y(t) \leq \frac{r_y(k + h_y)}{d_1} + \varepsilon_4 \text{ for all } t \geq T, \quad (16)$$

consequently, we have

$$\begin{cases} \frac{dz}{dt} \leq \left(r_z + e_2u + e_3v + e_4x - \frac{e_1}{y+h_z}z\right)z, \\ z(0) = z_0 \geq 0, \end{cases} \implies$$

$$\begin{cases} \frac{dz}{dt} \leq \left(\frac{c_1(r_z + ke_2 + ke_3) + r_x e_4(k + h_x)}{c_1} - \frac{d_1 e_1}{r_y(k + h_y) + d_1 h_z}z\right)z \\ z(0) = z_0 \geq 0 \end{cases}$$

$$\implies \begin{cases} \frac{dz}{dt} \leq (\alpha_1 - \alpha_2 z)z, \\ z(0) = z_0 \geq 0, \end{cases} \quad (17)$$

and using the comparison lemma 3.1 again we have,

$$\limsup_{t \rightarrow \infty} z(t) \leq \alpha_z, \quad (18)$$

thus, for all $\varepsilon_5 > 0$, there exists $T > 0$ such that

$$z(t) \leq \alpha_z + \varepsilon_5 \text{ for all } t \geq T, \quad (19)$$

with,

$$\alpha_1 = \frac{c_1(r_z + ke_2 + ke_3) + r_x e_4(k + h_x)}{c_1},$$

$$\alpha_2 = \frac{d_1 e_1}{r_y(k + h_y) + d_1 h_z} \text{ and } \alpha_z = \frac{\alpha_1}{\alpha_2}.$$

Furthermore, for all $t_0 \geq 0$, let $E(t_0) = (u(0), v(0), x(0), y(0), z(0))^T$ with positive components and let

$$\phi(t) = u(t) + v(t) + x(t) + y(t) + z(t). \quad (20)$$

Then, by deriving along the solutions of the system (1), we have $\forall \lambda > 0$,

$$\begin{aligned} \frac{d\phi(t)}{dt} + \lambda\phi(t) &= \frac{du(t)}{dt} + \frac{dv(t)}{dt} + \frac{dx(t)}{dt} + \frac{dy(t)}{dt} \\ &+ \frac{dz(t)}{dt} + \lambda(u(t) + v(t) + x(t) \\ &+ y(t) + z(t)), \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{d\phi(t)}{dt} + \lambda\phi(t) &\leq (q_1 - s_1 x(t))u(t) \\ &+ (q_2 - s_2 v(t))v(t) \\ &+ (q_3 - s_3 y(t))x(t) \\ &+ (q_4 - s_4 y(t))y(t) \\ &+ (q_5 - s_5 z(t))z(t), \end{aligned} \quad (22)$$

by using following equality :

$$\forall \alpha > 0, \beta > 0 \max_{x \geq 0} (\alpha - \beta x)x = \frac{\alpha^2}{4\beta} \text{ in}$$

[20, page 50] we have

$$\frac{d\phi(t)}{dt} + \lambda\phi(t) \leq \sum_{i=1}^5 \frac{q_i^2}{4s_i} (\approx \nu), \quad (23)$$

where

$$\begin{aligned} s_1 &= \frac{r_u}{k}, \quad s_2 = \frac{r_v}{k}, \quad s_3 = \frac{c_1}{k + h_x}, \\ s_4 &= \frac{d_1}{k + h_y}, \quad s_5 = \frac{e_1}{\alpha_y + h_z}, \quad q_1 = r_u + \lambda, \\ q_2 &= r_v + \lambda, \quad q_3 = r_x + \lambda, \quad q_4 = r_y + \lambda, \\ q_5 &= r_z + k(e_2 + e_3) + e_4 \alpha_x, \end{aligned}$$

applying Gronwall's inequality, we obtain

$$0 < \phi(E(t)) \leq \frac{\nu}{\lambda} (1 - e^{-\lambda t}) + \sigma(E(t_0))e^{-\lambda t}, \quad (24)$$

afterwards, when $t \rightarrow +\infty$, we have

$$0 < \phi(E(t)) \leq \frac{\nu}{\lambda}. \quad (25)$$

Therefore, any solution $E(t)$ of the system (1) is uniformly bounded for any initial condition such that $E(t_0) \geq 0$. Thus, the set Φ defines the region in which all solutions of the system (1) are attracted into \mathbb{R}_+^5 , so they are bounded. ■

4 Determination of trivial and semi-trivial equilibrium points and their conditions of existence

The equilibrium points are the solutions of the following equations :

$$\frac{du}{dt} = 0, \quad \frac{dv}{dt} = 0, \quad \frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0. \quad (26)$$

A large number of equilibrium points are obtained for the system (1), but the emphasis is on the interior equilibrium point, which is of vital importance because it provides an overview of the behavior of all species in the system. The system (1) admits a trivial equilibrium point $(0, 0, 0, 0, 0)$ whose stability will be determined in the next section. The various semi-trivial equilibrium points are determined below.

(i) We have semi-trivial equilibrium points $(\frac{r_u}{k}, 0, 0, 0, 0)$, $(0, \frac{r_v}{k}, 0, 0, 0)$, $(0, 0, \frac{h_x}{c_1}, 0, 0)$, $(0, 0, 0, \frac{h_{y1}}{d_1}, 0)$ and $(0, 0, 0, 0, \frac{h_z}{e_1})$ obtained by retaining only one of the prey or predator populations. The others are considered extinct. We obtain the same type of equation for each case. When we consider species u , for example, we have the following equation:

$$\frac{du}{dt} = \left(r_u - \frac{r_u}{k}u\right)u, \quad (27)$$

which is a logistic equation, and the behavior of solutions to such an equation has been extensively studied (see, [12], for example).

(ii) Several equilibrium points are obtained by retaining only two species, the others being considered extinct. In the uv and xy planes, we obtain the same type of model, a model of two

competing species with logistic growth, whose equation in the uv plane is as follows :

$$\begin{cases} \frac{du}{dt} = \left(r_u - \frac{r_u}{k}u\right)u - a_2 uv, \\ \frac{dv}{dt} = \left(r_v - \frac{r_v}{k}v\right)v - b_2 uv. \end{cases} \quad (28)$$

The non-trivial equilibrium point of this system in the uv plane is $\left(\frac{k^2 a_2 r_v - k r_u r_v}{k^2 a_2 b_2 - r_u r_v}, \frac{k^2 b_2 r_u - k r_u r_v}{k^2 a_2 b_2 - r_u r_v}\right)$ and in the xy plane we obtain $\left(\frac{d_1 r_x h_x - c_2 r_y h_y h_x}{c_1 d_1 - c_2 d_2 h_x h_y}, \frac{c_1 r_y h_y - d_2 r_x h_y h_x}{c_1 d_1 - c_2 d_2 h_x h_y}\right)$. These equilibrium points are respectively the restrictions of the equilibrium points $\left(\frac{k^2 a_2 r_v - k r_u r_v}{k^2 a_2 b_2 - r_u r_v}, \frac{k^2 b_2 r_u - k r_u r_v}{k^2 a_2 b_2 - r_u r_v}, 0, 0, 0\right)$ and $\left(\frac{d_1 r_x h_x - c_2 r_y h_y h_x}{c_1 d_1 - c_2 d_2 h_x h_y}, \frac{c_1 r_y h_y - d_2 r_x h_y h_x}{c_1 d_1 - c_2 d_2 h_x h_y}, 0, 0, 0\right)$ of the model (1). Moreover, the behaviors and conditions of existence of these equilibrium points have been studied in [12], for example.

Furthermore, in the ux , vy , and yz planes, we have the same type of model, a Leslie Gower model with a Holling's type II functional response whose representative equation obtained in the ux plane is :

$$\begin{cases} \frac{du}{dt} = \left(r_u - \frac{r_u}{k}u\right)u - \frac{a_1}{u + h_u} ux, \\ \frac{dx}{dt} = \left(r_x - \frac{c_1 x}{u + h_x}\right)x, \end{cases} \quad (29)$$

the non-trivial equilibrium of this system in the ux plane is $(u^*, x^*) = \left(u_+, \frac{r_x(h_x + u_+)}{c_1}\right)$ where $u^* = u_+$ is the positive solution of the following second-degree equation :

$$\frac{r_u}{k}u^2 + \left(\frac{r_u h_u}{k} + \frac{r_x a_1}{c_1} - r_u\right)u + \frac{r_x a_1 h_x}{c_1} - r_u h_u = 0, \quad (30)$$

and whose discriminant is,

$$\Delta = \left(\frac{r_u h_u}{k} + \frac{r_x a_1}{c_1} - r_u\right)^2 - 4 \frac{r_u}{k} \left(\frac{r_x a_1 h_x}{c_1} - r_u h_u\right), \quad (31)$$

therefore,

$$u_+ = \frac{k}{2r_u} \left(\Delta^{\frac{1}{2}} - \left(\frac{r_u h_u}{k} + \frac{r_x a_1}{c_1} - r_u\right)\right). \quad (32)$$

and (u^*, x^*) represents the restriction of the equilibrium point $(u^*, x^*, 0, 0, 0)$ of the model (1). Furthermore, in the uy and vx planes we have the same type of model, a logistic growth

model of two populations evolving independently. It should be noted that there are two other equilibrium point similar to this one that we obtain in the vy and yz planes. Consequently, in the uz , vz and xz planes we obtain the same type of model, a model of two interacting populations with a Holling's type I functional response and logistic growth, of which a system in the uz plane is :

$$\begin{cases} \frac{du}{dt} = \left(r_u - \frac{r_u}{k}u\right)u - a_3uz, \\ \frac{dz}{dt} = \left(r_z - \frac{e_1z}{h_z}\right)z + e_2uz - m_zz, \end{cases} \quad (33)$$

and the non-trivial equilibrium point of this system in the uz plane representing the restriction of the equilibrium $(u^*, 0, 0, 0, z^*)$ of the model (1) is given by the following expression : $(u^*, z^*) = \left(\frac{r_u e_1 k + a_3 m_z h_z k - a_3 r_z h_z k}{r_u e_1 + a_3 e_2 h_z k}, \frac{r_u e_2 h_z k^2 + r_u r_z h_z k - r_u m_z h_z}{r_u e_1 + a_3 e_2 h_z k}\right)$. The other equilibrium point in the vz and xz planes are of this form.

(iii) Several other equilibrium points are obtained by considering two extinct species. In the uvx and uvy planes, the system reduces to a modified Leslie Gower model of three interacting species with two competing prey and one predator. Furthermore, in the uxy , uxz , and vxy planes, we again obtain a modified Leslie Gower model of three interacting species, but with one competing prey and two competing predators. Consequently, in the uyz and vyz planes, we again obtain a modified Leslie Gower model of three interacting species, this time with one prey, one predator, and one superpredator. A projection onto the uvz and vxz planes yields a model of three interacting species with two competing prey and one predator, and a Holling's I functional response. This restriction of equation (1) to the uvz plane is as follows :

$$\begin{cases} \frac{du}{dt} = \left(r_u - \frac{r_u}{k}u\right)u - a_2uv - a_3uz, \\ \frac{dv}{dt} = \left(r_v - \frac{r_v}{k}v\right)v - b_2uv - b_3vz, \\ \frac{dz}{dt} = \left(r_z - \frac{e_1z}{h_z}\right)z + e_2uz + e_3vz - m_zz, \end{cases} \quad (34)$$

and the non-trivial equilibrium of this system in the uvz plane representing the equilibrium restriction $(u^*, v^*, 0, 0, z^*)$ of the model (1) is $(u^*, v^*, z^*) = (u^*, \mu_1 + \mu_2 u^*, \mu_3 + \mu_4 u^*)$ where $\mu_1, \mu_2, \mu_3, \mu_4$ and u^* are given by :

$$\begin{cases} \mu_1 = \frac{kr_v a_3 - kr_u b_3}{r_v a_3 - ka_2 b_3}, \\ \mu_2 = \frac{r_u b_3 - ka_3 b_3}{r_v a_3 - ka_2 b_3}, \\ \mu_3 = \frac{r_u}{a_3} + \frac{kr_u a_2 b_3 - kr_v a_2 a_3}{r_v a_3^2 - ka_2 a_3 b_3}, \\ \mu_4 = \frac{ka_2 a_3 b_2 - r_u a_2 b_3}{r_v a_3^2 - ka_2 a_3 b_3} - \frac{r_u}{ka_3}, \\ u^* = \frac{m_z h_z + e_1 \mu_3 - r_z h_z - e_3 \mu_1}{e_2 h_z + e_3 h_z \mu_2 - e_1 \mu_4}, \end{cases} \quad (35)$$

(iv) When we consider an extinct prey or predator population, projection onto the other axes highlights three models of four interacting species. Further investigation of these models would provide more information. Nevertheless, it should be noted that in the $uxyz$ and $vxyz$ planes we have a model of four interacting species with one prey, two competing predators and one super-predator. Then, in the $uvxy$ plane, we have a model of four species interacting with two competing prey and two competing predators and a Holling's type II functional response. Finally, we have a model of four interacting species with two competing prey, one predator and one super-predator in the $uvxz$ and $uvyz$ planes. Note also that for each of the above semi-trivial equilibrium points, it is necessary for all its components to be positive to ensure its existence.

5 Study of the global stability of trivial equilibrium point and interior equilibrium point

In this section, we limit our study to the analysis of the stability of trivial and interior equilibrium points, given the computational complexity for semi-trivial equilibrium points. It should be noted that the analysis of semi-trivial equilibrium points is limited to the study of interactions between fewer than five species, which are studied in depth in articles, [12], [14], [20].

5.1 Study of trivial equilibrium point stability

Proposition 5.1 *The trivial equilibrium point $(0, 0, 0, 0, 0)$ of the system (1) is unstable.*

Proof: The Jacobian matrix of the system (1) is given by the following expression :

$$Jac(k)(\tilde{E}) = (b_{ij})_{\substack{1 \leq i \leq 5 \\ 1 \leq j \leq 5}}$$

where the coefficients of the matrix are :

$$b_{11} = r_u - \frac{2r_u}{k}u - a_2 v - a_3 z - \frac{a_1 x h_u}{(u + h_u)^2},$$

$$b_{12} = -a_2 u, \quad b_{13} = -\frac{a_1 u}{u + h_u},$$

$$b_{22} = r_v - \frac{2r_v}{k}v - b_2 u - b_3 z - \frac{b_1 y h_v}{(v + h_v)^2},$$

$$b_{21} = -b_2 v, \quad b_{24} = -\frac{b_1 v}{v + h_v},$$

$$b_{33} = r_x - c_2 y - c_3 z - \frac{2c_1 x}{u + h_x},$$

$$b_{31} = \frac{c_1 x^2}{(u + h_x)^2}, \quad b_{34} = -c_2 x,$$

$$b_{44} = r_y - d_2 x - \frac{2d_1 y}{v + h_{y1}} + \frac{d_3 z h_{y2}}{(y + h_{y2})^2},$$

$$b_{42} = \frac{d_1 y^2}{(v + h_{y1})^2}, \quad b_{43} = -d_2 y,$$

$$b_{55} = r_z - \frac{2e_1 z}{y + h_z} + e_2 u + e_3 v + e_4 x - m_z,$$

$$b_{51} = e_2 z, \quad b_{52} = e_3 z, \quad b_{53} = e_4 z,$$

$$b_{14} = 0, \quad b_{23} = 0, \quad b_{32} = 0, \quad b_{41} = 0,$$

$$b_{15} = -a_3 u, \quad b_{25} = -b_3 v, \quad b_{35} = -c_3 x,$$

$$b_{45} = -\frac{d_3 y}{y + h_{y2}}, \quad b_{54} = \frac{e_1 z^2}{(y + h_z)^2},$$

The eigenvalues of this matrix are : r_u, r_v, r_x, r_y and $r_z - m_z$. Eigenvalues r_u, r_v, r_x and r_y are positive. Consequently, the trivial equilibrium point $(0, 0, 0, 0, 0)$ is unstable. ■

5.2 Existence of interior equilibrium point

The following theorem shows that the model (1) has a unique interior equilibrium point

$\tilde{E} = (\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y}, \tilde{z})$ with, $\tilde{u} > 0, \tilde{v} > 0, \tilde{x} > 0, \tilde{y} > 0, \tilde{z} > 0$.

Theorem 5.1 Suppose the function $h(t, E(t))$, $h : [t_0, +\infty[\times \mathbb{R}_+^5 \rightarrow \mathbb{R}_+^5$ defining the system (1),

$$\begin{cases} \frac{dE(t)}{dt} = h(t, E(t)), & t \in [t_0, +\infty[\\ E(t_0) > 0, & t_0 \geq 0, \end{cases} \quad (36)$$

is continuous and k -Lipschitzian with $k \in [0, 1]$, then, there is a unique solution $\tilde{E}(t)$ for the system (1) in the set Φ_1 defined by the following expression :

$$\Phi_1 = \left\{ \begin{array}{l} E(t), t \in [t_0, +\infty[, t_0 \geq 0 : \\ \max\{|u|; |v|; |x|; |y|; |z|\} \leq \rho, \\ \rho > 0. \end{array} \right\},$$

Proof: Let $E(t)$ and $E_1(t) \in \Phi_1$. A solution $E(t)$ of the system (1) with initial condition $E(t_0)$ is defined as follows, [21]

$$E(t) = E(t_0) + \int_{t_0}^t h(s, E(s)) ds. \quad (37)$$

Let ϕ an application defined as follows

$$\phi(E(t)) = E(t_0) + \int_{t_0}^t h(s, E(s)) ds,$$

then, we have :

$$|\phi(E(t)) - \phi(E_1(t))| \leq \int_{t_0}^t |(h(s, E(s)) - h(s, E_1(s)))| ds \Rightarrow$$

$$\begin{aligned} & \int_{t_0}^t |(h(s, E(s)) - h(s, E_1(s)))| ds \\ & \leq \tau_0 \max[\rho_u \rho_v \rho_x \rho_y \rho_z]^T, \end{aligned}$$

with $\tau_0 \in [t_0, t]$ and T the transpose of the matrix, we have :

$$\begin{aligned} \rho_u &= (\rho(2 + a_2 + a_3 + \frac{a_1 h_u}{(\rho + h_u)^2}) + 1) |u - u_1| \\ &+ \frac{a_1 \rho}{\rho + h_u} |x - x_1| + a_2 \rho |v - v_1| + a_3 \rho |z - z_1|, \\ \rho_v &= (\rho(2 + b_2 + b_3 + \frac{b_1 h_v}{(\rho + h_v)^2}) + 1) |v - v_1| \\ &+ \frac{b_1 \rho}{\rho + h_v} |y - y_1| + b_2 \rho |u - u_1| + b_3 \rho |z - z_1|, \\ \rho_x &= (\rho(c_2 + c_3) + \frac{2c_1 h_x}{(\rho + h_x)^2} + 1) |x - x_1| \\ &+ \frac{2\rho^3}{(\rho + h_x)^2} + c_2 \rho |y - y_1| + c_3 \rho |z - z_1|, \end{aligned}$$

$$\begin{aligned} \rho_y &= \left(\rho(d_2 + \frac{2d_1 h_{y1}}{(\rho + h_{y1})^2} + \frac{d_3 h_{y2}}{(\rho + h_{y2})^2} + 1) \right. \\ &\times |y - y_1| + \frac{2d_1 \rho^2}{(\rho + h_{y1})^2} \\ &+ d_2 \rho |x - x_1| + \frac{d_3 \rho}{\rho + h_{y2}} |z - z_1|, \\ \rho_z &= \left(\rho(e_2 + e_3 + e_4 + \frac{2e_1 h_z}{(\rho + h_z)^2}) + m_z + 1 \right) \\ &\times |z - z_1| + \frac{2e_1 \rho^3}{(\rho + h_z)^2} + e_2 \rho |u - u_1| \\ &+ e_3 \rho |v - v_1| + e_3 \rho |x - x_1|, \end{aligned}$$

The supremum for a continuous function $g(t) \in \mathcal{C}([t_0, +\infty[))$ is given by :

$\|g\| = \sup_{t \in [t_0, +\infty[} \|g(t)\|$ and the matrix norm for a matrix $\mathcal{M} \in M_{m,n}(\mathbb{C})$, $\mathcal{M} = (m_{ij}(t))$ is given by : $\|\mathcal{M}\| = \sum_{i,j} \sup_{t \in [t_0, +\infty[} |m_{ij}(t)|$. Applying these norms to the operator ϕ , we obtain,

$$\|\phi(E(t)) - \phi(E_1(t))\| = \tau_0 \sup \{ \rho_u + \rho_v + \rho_x + \rho_y + \rho_z \},$$

$$\implies \|\phi(E(t)) - \phi(E_1(t))\| \leq \tau_0 \max \{ \bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_x, \bar{\rho}_y, \bar{\rho}_z \} \|E - E_1\|,$$

$$\implies \|\phi(E(t)) - \phi(E_1(t))\| \leq k \|E - E_1\|, \quad (38)$$

with $k = \tau_0 \max \{ \bar{\rho}_u, \bar{\rho}_v, \bar{\rho}_x, \bar{\rho}_y, \bar{\rho}_z \}$ and

$$\begin{aligned} \bar{\rho}_u &= \rho \left(\rho(1 + a_2 + a_3) + \frac{a_1 h_u}{(\rho + h_u)^2} \right) + \frac{a_1 \rho}{\rho + h_u}, \\ \bar{\rho}_v &= \rho \left(\rho(1 + b_2 + b_3) + \frac{b_1 h_v}{(\rho + h_v)^2} \right) + \frac{b_1 \rho}{\rho + h_v}, \\ \bar{\rho}_x &= \rho \left(2\rho(c_2 + c_3) + \frac{2c_1 h_x}{(\rho + h_x)^2} + \frac{2\rho^3}{(\rho + h_x)^2} + 1 \right), \end{aligned}$$

$$\begin{aligned} \bar{\rho}_y &= \rho \left(\rho \left(d_2 + \frac{2d_1 h_{y1}}{(\rho + h_{y1})^2} + \frac{d_3 h_{y2}}{(\rho + h_{y2})^2} \right) \right. \\ &+ \frac{2d_1 \rho^2}{(\rho + h_{y1})^2} + \frac{d_3 \rho}{\rho + h_{y2}} + d_2 \rho + 1 \Big), \\ \bar{\rho}_z &= \rho \left(2\rho(e_2 + e_3 + e_4) + \frac{2e_1 h_z}{(\rho + h_z)^2} \right) \\ &+ \frac{2e_1 \rho^3}{(\rho + h_z)^2} + m_z + 1 \Big), \end{aligned}$$

According to the inequality (38), the operator ϕ is a contraction if $k \in [0, 1]$. By the contraction principle, ϕ has a unique

fixed point that satisfies (37). Consequently, the system (1) has a unique solution. This unique solution will be denoted $\tilde{E}(t)$. ■

5.3 Global stability of interior equilibrium point

In this section we study the global stability of the equilibrium point $\tilde{E} = (\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y}, \tilde{z})$ of the system (1). We exploit a Lyapunov function associated with this equilibrium point. First, we recall the definition of an invariant set and LaSalle's invariance theorem.

Definition 5.1 [14] Consider a dynamic system defined by $\dot{u} = f(u)$. The set \mathcal{I} of initial conditions u_0 such that the solution $\varphi(u_0, t)$ of the system remains in \mathcal{I} for all $t \geq 0$, is called the invariant set for the above dynamical system. We have

$$\mathcal{I} = \{ u | u_0 \in \mathcal{I}, \varphi(u_0, t) \in \mathcal{I} \forall t \geq 0 \}. \quad (39)$$

Theorem 5.2 [22] Let ψ and \mathcal{V} be subsets of \mathbb{R}^n . Consider the following sets : \mathcal{I} a compact and positively invariant subset of \mathcal{V} and

$$\Omega = \{ u \in \mathcal{I}, \dot{g}(u) = 0 \} \quad (40)$$

such that ψ is the largest invariant subset of Ω . Consider $f : \psi \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathcal{V} \subset \mathbb{R}^n \rightarrow \mathbb{R}^+$ functions of class \mathcal{C}^1 and suppose $\dot{g}(u) \leq 0$, for all $u \in \mathcal{V}$. Then all solutions of the system (1) bounded for $t \geq 0$, converge to ψ when $t \rightarrow \infty$.

Theorem 5.3 Let's assume the following hypotheses are satisfied,

$$\begin{aligned} \frac{a_1 \tilde{x}}{(u + h_u)(\tilde{u} + h_u)} + \frac{1}{2} \mu_2 &< \frac{r_u}{k}, \\ \frac{b_1 \tilde{y}}{(v + h_v)(\tilde{v} + h_v)} + \frac{1}{2 p_2} \mu_2 &< \frac{r_v}{k}, \\ \frac{d_3 \tilde{z}}{(y + h_y)(\tilde{y} + h_y)} + \frac{1}{2 p_4} \mu_6 &< \frac{d_1}{k + h_y}, \\ \text{and } \frac{1}{2} \mu_6 &< \frac{p_3 c_1}{k + h_x}, \end{aligned} \quad (41)$$

then, the positive equilibrium point \tilde{E} , of the system (1) is globally asymptotically stable.

Proof: Let the following Lyapunov function $L : \mathbb{R}^5 \rightarrow \mathbb{R}$ be defined by,

$$L(E) = L_1(E) + L_2(E) + L_3(E) + L_4(E) + L_5(E). \quad (42)$$

with,

$$L_1(E) = p_1 \left((u - \tilde{u}) - \tilde{u} \ln\left(\frac{u}{\tilde{u}}\right) \right), \quad (43)$$

$$L_2(E) = p_2 \left((v - \tilde{v}) - \tilde{v} \ln\left(\frac{v}{\tilde{v}}\right) \right), \quad (44)$$

$$L_3(E) = p_3 \left((x - \tilde{x}) - \tilde{x} \ln\left(\frac{x}{\tilde{x}}\right) \right), \quad (45)$$

$$L_4(E) = p_4 \left((y - \tilde{y}) - \tilde{y} \ln\left(\frac{y}{\tilde{y}}\right) \right), \quad (46)$$

$$L_5(E) = p_5 \left((z - \tilde{z}) - \tilde{z} \ln\left(\frac{z}{\tilde{z}}\right) \right). \quad (47)$$

where $p_i, i \in \{1, \dots, 5\}$ are positive constants to be determined in the further steps. The function $L(E)$ is defined and continuous on $Int(\mathbb{R}_+^5)$, furthermore, note that

$$\begin{cases} L(\tilde{E}) = 0, \\ L(E) > 0 \text{ for all } E \in \Theta \setminus \tilde{E}. \end{cases} \quad (48)$$

The point \tilde{E} is therefore a global minimum of L . Furthermore, all solutions of the system are bounded and converge to the set Φ , for t sufficiently large, we can restrict the study to this to this set. We then calculate the orbital derivative of L .

$$\frac{dL}{dt} = \frac{dL_1}{dt} + \frac{dL_2}{dt} + \frac{dL_3}{dt} + \frac{dL_4}{dt} + \frac{dL_5}{dt}, \quad (49)$$

we obtain ,

$$\frac{dL_1}{dt} = p_1 (u - \tilde{u}) \times \left(r_u - \frac{r_u}{k} u - \frac{a_1}{u + h_u} x - a_2 v - a_3 z \right),$$

$$\frac{dL_2}{dt} = p_2 (v - \tilde{v}) \times \left(r_v - \frac{r_v}{k} v - \frac{b_1}{v + h_v} y - b_2 u - b_3 z \right),$$

$$\frac{dL_3}{dt} = p_3 (x - \tilde{x}) \left(r_x - \frac{c_1}{u + h_x} x - c_2 y - c_3 z \right),$$

$$\frac{dL_4}{dt} = p_4 (y - \tilde{y}) \times \left(r_y - \frac{d_1}{v + h_y} y - d_2 x - \frac{d_3}{y + h_y} z \right),$$

$$\frac{dL_5}{dt} = p_5 (z - \tilde{z}) \times \left(r_z - \frac{e_1}{y + h_z} z + e_2 u + e_3 v + e_4 x - m_z \right),$$

Let's put,

$$r_u - \frac{r_u}{k} \tilde{u} = \frac{a_1 \tilde{x}}{\tilde{u} + h_u} + a_2 \tilde{v} + a_3 \tilde{z},$$

$$r_v - \frac{r_v}{k} \tilde{v} = \frac{b_1 \tilde{y}}{\tilde{v} + h_v} + b_2 \tilde{u} + b_3 \tilde{z},$$

$$r_x - c_2 \tilde{y} = \frac{c_1 \tilde{x}}{\tilde{u} + h_x} + c_3 \tilde{z},$$

$$r_y - d_2 \tilde{x} = \frac{d_1 \tilde{y}}{\tilde{v} + h_y} + \frac{d_3 \tilde{z}}{\tilde{y} + h_y},$$

$$m_z - r_z = -\frac{e_1 \tilde{z}}{\tilde{y} + h_z} + e_2 \tilde{u} + e_2 \tilde{v} + e_3 \tilde{x},$$

the following results are obtained

$$\begin{aligned} \frac{dL_1}{dt} &= p_1 (u - \tilde{u}) \left(-\frac{r_u}{k} (u - \tilde{u}) - a_2 (v - \tilde{v}) \right. \\ &\quad \left. - a_3 (z - \tilde{z}) + \frac{a_1 \tilde{x}}{\tilde{u} + h_u} - \frac{a_1 x}{u + h_u} \right), \end{aligned}$$

$$\begin{aligned} \frac{dL_1}{dt} &= p_1 (u - \tilde{u}) \left(-\frac{r_u}{k} (u - \tilde{u}) - a_2 (v - \tilde{v}) \right. \\ &\quad \left. - a_3 (z - \tilde{z}) - \frac{a_1 (x - \tilde{x})}{(u + h_u)} + \frac{a_1 \tilde{x} (u - \tilde{u})}{(u + h_u)(\tilde{u} + h_u)} \right), \end{aligned}$$

$$\begin{aligned} \frac{dL_1}{dt} &= p_1 \left(\frac{a_1 \tilde{x}}{(u + h_u)(\tilde{u} + h_u)} - \frac{r_u}{k} \right) (u - \tilde{u})^2 \\ &\quad - \left(\frac{p_1 a_1}{u + h_u} \right) (u - \tilde{u})(x - \tilde{x}) \\ &\quad - p_1 a_2 (u - \tilde{u})(v - \tilde{v}) - p_1 a_3 (u - \tilde{u})(z - \tilde{z}), \end{aligned}$$

similarly, the calculation of $\frac{dL_2}{dt}$ leads to,

$$\begin{aligned} \frac{dL_2}{dt} &= p_2 \left(\frac{b_1 \tilde{y}}{(v + h_v)(\tilde{v} + h_v)} - \frac{r_v}{k} \right) (v - \tilde{v})^2 \\ &\quad - \left(\frac{p_2 b_1}{v + h_v} \right) (v - \tilde{v})(y - \tilde{y}) \\ &\quad - p_2 b_2 (u - \tilde{u})(v - \tilde{v}) - p_2 b_3 (v - \tilde{v})(z - \tilde{z}), \end{aligned} \quad (50)$$

we calculate $\frac{dL_3}{dt}$ and obtain :

$$\begin{aligned} \frac{dL_3}{dt} &= -p_3 c_2 (x - \tilde{x})(y - \tilde{y}) - p_3 c_3 (x - \tilde{x})(z - \tilde{z}) \\ &\quad - \frac{p_3 c_1 (x - \tilde{x})^2}{(u + h_x)} + \frac{p_3 c_1 \tilde{x} (u - \tilde{u})(x - \tilde{x})}{(u + h_x)(\tilde{u} + h_x)}, \end{aligned} \quad (51)$$

after calculating $\frac{dL_4}{dt}$ gives,

$$\begin{aligned} \frac{dL_4}{dt} &= -p_4 d_2 (x - \tilde{x})(y - \tilde{y}) - \frac{p_4 d_1 (y - \tilde{y})^2}{(v + h_y)} \\ &\quad + \frac{p_4 d_1 \tilde{y} (v - \tilde{v})(y - \tilde{y})}{(v + h_y)(\tilde{v} + h_y)} - \frac{p_4 d_3 (y - \tilde{y})(z - \tilde{z})}{(y + h_y)} \\ &\quad + \frac{p_4 d_3 \tilde{z} (y - \tilde{y})^2}{(y + h_y)(\tilde{y} + h_y)}, \end{aligned} \quad (52)$$

then for $\frac{dL_5}{dt}$ we obtain the following expression :

$$\begin{aligned} \frac{dL_5}{dt} = & p_5 e_2(u - \tilde{u})(z - \tilde{z}) + p_5 e_3(v - \tilde{v})(z - \tilde{z}) \\ & + p_5 e_4(x - \tilde{x})(z - \tilde{z}) - \frac{p_5 e_1(z - \tilde{z})^2}{(y + h_z)} \\ & + \frac{p_5 e_1 \tilde{z}(y - \tilde{y})(z - \tilde{z})}{(y + h_z)(\tilde{y} + h_z)}, \end{aligned} \quad (53)$$

it follows that,

$$\begin{aligned} \frac{dL}{dt} = & p_1 \left(\frac{a_1 \tilde{x}}{(u + h_u)(\tilde{u} + h_u)} - \frac{r_u}{k} \right) (u - \tilde{u})^2 \\ & + p_2 \left(\frac{b_1 \tilde{y}}{(v + h_v)(\tilde{v} + h_v)} - \frac{r_v}{k} \right) (v - \tilde{v})^2 \\ & + p_4 \left(\frac{d_3 \tilde{z}}{(y + h_y)(\tilde{y} + h_y)} - \frac{d_1}{(v + h_y)} \right) (y - \tilde{y})^2 \\ & - \frac{p_5 e_1(z - \tilde{z})^2}{(y + h_z)} - \frac{p_3 c_1(x - \tilde{x})^2}{(u + h_x)} \\ & + \mu_1 (u - \tilde{u})(x - \tilde{x}) - \mu_2 (u - \tilde{u})(v - \tilde{v}) \\ & + \mu_3 (u - \tilde{u})(z - \tilde{z}) + \mu_4 (v - \tilde{v})(y - \tilde{y}) \\ & + \mu_5 (v - \tilde{v})(z - \tilde{z}) - \mu_6 (x - \tilde{x})(y - \tilde{y}) \\ & + \mu_7 (x - \tilde{x})(z - \tilde{z}) + \mu_8 (y - \tilde{y})(z - \tilde{z}), \end{aligned}$$

where,

$$\begin{aligned} \mu_1 &= \frac{p_3 c_1 \tilde{x}}{(u + h_x)(\tilde{u} + h_x)} - \frac{p_1 a_1}{u + h_u}, \\ \mu_2 &= (p_2 b_2 + p_1 a_2), \quad \mu_3 = p_5 e_2 - p_1 a_3, \\ \mu_4 &= \frac{p_4 d_1 \tilde{y}}{(v + h_y)(\tilde{v} + h_y)} - \frac{p_2 b_1}{v + h_v}, \\ \mu_5 &= (p_5 e_3 - p_2 b_3), \quad \mu_6 = (p_4 d_2 + p_3 c_2), \\ \mu_7 &= p_5 e_4 - p_3 c_3, \\ \mu_8 &= \frac{p_5 e_1 \tilde{z}}{(y + h_z)(\tilde{y} + h_z)} - \frac{p_4 d_3}{(y + h_y)}, \end{aligned}$$

using,

$$\begin{aligned} p_2 &= \frac{a_3 e_3}{e_2 b_3}, \quad p_3 = \frac{a_1}{c_1} \times \frac{(u + h_x)(\tilde{u} + h_x)}{\tilde{x}(u + h_u)}, \\ p_1 &= 1, \quad p_5 = \frac{a_3}{e_2}, \text{ and } p_4 \text{ such that} \\ \frac{e_3 b_1 d_3}{e_1 b_2 d_1} &= \frac{\tilde{y} \tilde{z}(y + h_y)(v + h_v)}{(y + h_z)(\tilde{y} + h_z)(v + h_y)(\tilde{v} + h_y)}, \end{aligned}$$

we obtain,

$$\begin{aligned} \frac{dL}{dt} \leq & p_1 \left(\frac{a_1 \tilde{x}}{(u + h_u)(\tilde{u} + h_u)} - \frac{r_u}{k} \right) (u - \tilde{u})^2 \\ & + p_2 \left(\frac{b_1 \tilde{y}}{(v + h_v)(\tilde{v} + h_v)} - \frac{r_v}{k} \right) (v - \tilde{v})^2 \\ & + p_4 \left(\frac{d_3 \tilde{z}}{(y + h_y)(\tilde{y} + h_y)} - \frac{d_1}{(1 + h_y)} \right) (y - \tilde{y})^2 \\ & - \frac{p_3 c_1(x - \tilde{x})^2}{(1 + h_x)} - \frac{p_5 e_1 d_1(z - \tilde{z})^2}{(1 + h_y + d_1 h_z)} \\ & + \frac{1}{2} \mu_2 \left((u - \tilde{u})^2 + (v - \tilde{v})^2 \right) \\ & + \frac{1}{2} \mu_6 \left((x - \tilde{x})^2 + (y - \tilde{y})^2 \right), \end{aligned} \quad (54)$$

we conclude that,

$$\begin{aligned} \frac{dL}{dt} \leq & \left(\frac{a_1 \tilde{x}}{(u + h_u)(\tilde{u} + h_u)} + \frac{1}{2} \mu_2 - \frac{r_u}{k} \right) (u - \tilde{u})^2 \\ & + p_2 \left(\frac{b_1 \tilde{y}}{(v + h_v)(\tilde{v} + h_v)} + \frac{1}{2 p_2} \mu_2 - \frac{r_v}{k} \right) (v - \tilde{v})^2 \\ & + p_4 \left(\frac{d_3 \tilde{z}}{(y + h_y)(\tilde{y} + h_y)} + \frac{1}{2} \frac{\mu_6}{p_4} - \frac{d_1}{(1 + h_y)} \right) \\ & \times (y - \tilde{y})^2 + \left(\frac{1}{2} \mu_6 - \frac{p_3 c_1}{(1 + h_x)} \right) (x - \tilde{x})^2 \\ & - \frac{p_5 e_1 d_1}{(1 + h_y + d_1 h_z)} (z - \tilde{z})^2, \end{aligned} \quad (55)$$

Thus, under the assumptions (41), L is a strict Lyapunov function and, by the LaSalle invariance theorem, it follows that \tilde{E} is globally asymptotically stable on Φ . ■

6 System permanence

In this section, we examine the conditions under which the system (1) persists over time. The persistence or permanence of a population refers to its ability to continue its demographic evolution over time in a given environment, while coping with the impact of biotic factors, i.e. the action of other species sharing its environment, as well as pressures linked to environmental change, which constitute the abiotic factors of the ecosystem. We define permanence in the mathematical sense.

Definition 6.1 [14] *Let m_i and M_i , $i \in \{1, \dots, 5\}$ be positive constants. Let $E(t)$ be the positive solution of the system (1). If for each component of $E(t)$ we have the following inequalities :*

$$\begin{aligned} m_1 &\leq \liminf_{t \rightarrow +\infty} u(t) \leq \limsup_{t \rightarrow +\infty} u(t) \leq M_1, \\ m_2 &\leq \liminf_{t \rightarrow +\infty} v(t) \leq \limsup_{t \rightarrow +\infty} v(t) \leq M_2, \\ m_3 &\leq \liminf_{t \rightarrow +\infty} x(t) \leq \limsup_{t \rightarrow +\infty} x(t) \leq M_3, \\ m_4 &\leq \liminf_{t \rightarrow +\infty} y(t) \leq \limsup_{t \rightarrow +\infty} y(t) \leq M_4, \\ m_5 &\leq \liminf_{t \rightarrow +\infty} z(t) \leq \limsup_{t \rightarrow +\infty} z(t) \leq M_5, \end{aligned}$$

then the system (1) is said to be permanent.

Lemma 6.1 [14] Let $N_i > 0$ and $n_i > 0$, $i \in \{1, \dots, 5\}$ be positive constants. Let $E(t)$ be a positive solution of the system (1). If the following inequality holds,

$$0 < n_i \leq \liminf_{t \rightarrow +\infty} \|E(t)\| \leq \limsup_{t \rightarrow +\infty} \|E(t)\| \leq N_i,$$

for all $i \in \{1, \dots, 5\}$, then the system (1) is said to be uniformly permanent.

The following assumptions are defined,

$$r_u > \vartheta_1, \quad r_v > \vartheta_2, \quad \vartheta_3 > 0, \quad \vartheta_4 > 0, \quad \vartheta_5 > 0. \quad (56)$$

Theorem 6.1 For any positive solution $E(t)$, the system (1) is permanent under the assumptions (56).

Proof: According to Theorem 3.1, there exists a positive constant $\bar{\varrho} = \max\{M, \tilde{N}\}$, $\bar{\varrho} < +\infty$, not depending on the initial condition such that, for any positive solution $E(t)$ of (1), the following inequality holds.

$$\max \left\{ \begin{array}{l} \limsup_{t \rightarrow +\infty} u(t), \\ \limsup_{t \rightarrow +\infty} v(t), \\ \limsup_{t \rightarrow +\infty} x(t), \\ \limsup_{t \rightarrow +\infty} y(t), \\ \limsup_{t \rightarrow +\infty} z(t) \end{array} \right\} \leq \bar{\varrho},$$

Furthermore, considering equation (1)₁ we have :

$$\begin{aligned} \frac{du}{dt} &= \left(r_u - \frac{r_u}{k} u - \frac{a_1}{u + h_u} x - a_2 v - a_3 z \right) u, \\ \frac{du}{dt} &\geq \left(r_u - \frac{r_u}{k} u - \frac{a_1 \bar{\varrho}}{h_u} - a_2 \bar{\varrho} - a_3 \bar{\varrho} \right) u, \\ \frac{du}{dt} &\geq \left(r_u - \vartheta_1 - \frac{r_u}{k} u \right) u, \end{aligned}$$

using the lemma 3.1 we get that

$$\liminf_{t \rightarrow +\infty} u(t) \geq \frac{k(r_u - \vartheta_1)}{r_u} := \tilde{\xi}_1,$$

so, for all $\epsilon_1 > 0$, there exists $T_1 > 0$ such that

$$u(t) > \tilde{\xi}_1 - \epsilon_1 \text{ for all } t \geq T_1.$$

with $\vartheta_1 = \frac{a_1 + a_2 h_u + a_3 h_u}{h_u} \bar{\varrho}$.

by the same process, considering the equation (1)₂ we get,

$$\liminf_{t \rightarrow +\infty} v(t) \geq \frac{k(r_v - \vartheta_2)}{r_v} := \tilde{\xi}_2;$$

so, for all $\epsilon_2 > 0$, there exists $T_2 > 0$ such that

$$v(t) > \tilde{\xi}_2 - \epsilon_2 \text{ for all } t \geq T_2,$$

with $\vartheta_2 = \frac{b_1 + b_2 h_v + b_3 h_v}{h_v} \bar{\varrho}$,

therefore, considering the equation (1)₃ we have :

$$\begin{aligned} \frac{dx}{dt} &= \left(r_x - \frac{c_1}{u + h_x} x - c_2 y - c_3 z \right) x, \\ \frac{dx}{dt} &\geq \left(r_x - \frac{c_1}{h_x} x - c_2 \bar{\varrho} - c_3 \bar{\varrho} \right) x, \\ \frac{dx}{dt} &\geq \left(\vartheta_3 - \frac{c_1}{h_x} x \right) x, \end{aligned}$$

then,

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{\vartheta_3 h_x}{c_1} := \tilde{\xi}_3,$$

thus, for all $\epsilon_3 > 0$, there exists $T_3 > 0$ such that

$$x(t) > \tilde{\xi}_3 - \epsilon_3 \text{ for all } t \geq T_3,$$

with, $\vartheta_3 = r_x - (c_2 + c_3) \bar{\varrho}$,

in an analogous way,

$$\liminf_{t \rightarrow +\infty} y(t) \geq \frac{\vartheta_4 h_{y1}}{d_1} := \tilde{\xi}_4,$$

then, for all $\epsilon_4 > 0$, there exists $T_4 > 0$ such that

$$y(t) > \tilde{\xi}_4 - \epsilon_4 \text{ for all } t \geq T_4,$$

with, $\vartheta_4 = r_y - (d_2 + \frac{d_3}{h_{y2}}) \bar{\varrho}$.

Finally, considering equation (1)₅, we obtain that

$$\begin{aligned} \frac{dz}{dt} &= \left(r_z - \frac{e_1}{y + h_z} z + e_2 u + e_3 v + e_4 x - m_z \right) z, \\ \frac{dz}{dt} &\geq \left(r_z + e_2 \tilde{\xi}_1 + e_3 \tilde{\xi}_2 + e_4 \tilde{\xi}_3 - m_z - \frac{e_1}{h_z} z \right) z, \\ \frac{dz}{dt} &\geq \left(\vartheta_5 - \frac{e_1}{h_z} z \right) z, \end{aligned}$$

then, by applying lemma 3.1, we obtain that

$$\liminf_{t \rightarrow +\infty} z(t) \geq \frac{\vartheta_5 h_z}{e_1} := \tilde{\xi}_5,$$

we have for all $\epsilon_5 > 0$, there exists $T_5 > 0$ such that

$$z(t) > \tilde{\xi}_5 - \epsilon_5 \text{ for all } t \geq T_5,$$

with,

$$\vartheta_5 = r_z + e_2 \tilde{\xi}_1 + e_3 \tilde{\xi}_2 + e_4 \tilde{\xi}_3 - m_z.$$

Let $\varrho = \min_i \tilde{\xi}_i$, for all $i \in \{1, \dots, 5\}$, according to the theorem 6.1, for any positive solution $E(t)$, of the system (1) we have the following inequality,

$$\varrho \leq \liminf_{t \rightarrow +\infty} E(t),$$

which completes the proof of the theorem. ■

7 Extinction of species

The extinction of a system is the annihilation of all the species that interact in that system. The extinction of a population is the death of all specimens of the species as a result of ecological interactions such as competition, predation, etc., and human action and its many consequences, such as the reduction of biodiversity, the disruption of ecosystems and the disruption of species reproductive cycles. In this subsection, we outline the conditions under which the system (1) can become extinct over time. First, we give a mathematical definition of system extinction.

Definition 7.1 [23] *System (1) turns off, if*

$$\liminf_{t \rightarrow +\infty} \|E(t)\| = 0. \quad (57)$$

Lemma 7.1 *The system (1) turns off, if for any positive solution $E(t)$ we have the following equalities :*

$$\liminf_{t \rightarrow +\infty} u(t) = \liminf_{t \rightarrow +\infty} v(t) = \liminf_{t \rightarrow +\infty} x(t) =$$

$$\liminf_{t \rightarrow +\infty} y(t) = \liminf_{t \rightarrow +\infty} z(t) = 0. \quad (58)$$

Theorem 7.1 *If the inequalities below hold, then the system (1) turn off, i.e. all populations disappear.*

$$i) \quad r_u < \frac{r_u}{k} \tilde{\xi}_1 + \frac{a_1 \tilde{\xi}_3}{\varrho + h_u} + a_2 \tilde{\xi}_2 + a_3 \tilde{\xi}_5,$$

$$ii) \quad r_v < \frac{r_v}{k} \tilde{\xi}_2 + \frac{b_1 \tilde{\xi}_4}{\varrho + h_v} + b_2 \tilde{\xi}_1 + b_3 \tilde{\xi}_5,$$

$$iii) \quad \int_{t_0}^t (\tilde{\kappa}_1 - \frac{c_1}{\varrho + h_x} x(s)) ds < 0,$$

$$iv) \quad \int_{t_0}^t (\tilde{\kappa}_2 - \frac{d_1}{\varrho + h_{y1}} y(s) - \frac{d_3 \tilde{\xi}_5}{y(s) + h_{y2}}) ds < 0,$$

$$v) \quad \int_{t_0}^t (\tilde{\kappa}_3 - \frac{e_1}{\varrho + h_z} z(s)) ds < 0,$$

Proof: Let's look at each equation in the system (1). It follows from equation (1)₁ that,

$$\begin{aligned} \frac{du}{dt} &\leq u(t) \left(r_u - \frac{r_u}{k} \tilde{\xi}_1 - \frac{a_1 \tilde{\xi}_3}{\varrho + h_u} - a_2 \tilde{\xi}_2 - a_3 \tilde{\xi}_5 \right), \\ u(t) &\leq u(0) e^{\left\{ \left(r_u - \frac{r_u}{k} \tilde{\xi}_1 - \frac{a_1 \tilde{\xi}_3}{\varrho + h_u} - a_2 \tilde{\xi}_2 - a_3 \tilde{\xi}_5 \right) t \right\}}, \end{aligned}$$

hence, according to i), $u(t)$ tends to 0 when t tends to $+\infty$. There is extinction of population u . Similarly, from equation (1)₂ we obtain that,

$$v(t) \leq v(0) e^{\left\{ \left(r_v - \frac{r_v}{k} \tilde{\xi}_2 - \frac{b_1 \tilde{\xi}_4}{\varrho + h_v} - b_2 \tilde{\xi}_1 - b_3 \tilde{\xi}_5 \right) t \right\}},$$

and if condition ii) is satisfied $v(t)$ tends to 0 when t tends to $+\infty$. There is extinction of the species v . Now, consider equation (1)₃ we have :

$$\begin{aligned} \frac{dx}{dt} &\leq x \left(r_x - \frac{c_1}{\varrho + h_x} x(s) - c_2 \tilde{\xi}_4 - c_3 \tilde{\xi}_5 \right), \\ x(t) &\leq x(0) \exp \left\{ \int_{t_0}^t \left(\tilde{\kappa}_1 - \frac{c_1}{\varrho + h_x} x(s) \right) ds \right\}, \end{aligned}$$

thus according to iii), $x(t)$ tends towards 0 when t tends towards $+\infty$: the predators x become extinct. Also, the equation (1)₄ gives,

$$\begin{aligned} \frac{dy}{dt} &\leq y \left(r_y - \frac{d_1}{\varrho + h_{y1}} y(s) - d_2 \tilde{\xi}_3 - \frac{d_3 \tilde{\xi}_5}{y(s) + h_{y2}} \right), \\ y(t) &\leq y(0) e^{\left\{ \int_{t_0}^t \left(\tilde{\kappa}_2 - \frac{d_1}{\varrho + h_{y1}} y(s) - \frac{d_3 \tilde{\xi}_5}{y(s) + h_{y2}} \right) ds \right\}}, \end{aligned}$$

hence, according to iv), $y(t)$ tends to 0 when t tends to $+\infty$: the predators y become extinct. Finally, equation (1)₅ leads to,

$$\begin{aligned} \frac{dz}{dt} &\leq z \left(r_z - \frac{e_1}{\varrho + h_z} z(s) + e_2 \tilde{\xi}_1 + e_3 \tilde{\xi}_2 + e_4 \tilde{\xi}_3 - m_z \right), \\ z(t) &\leq z(0) \exp \left\{ \int_{t_0}^t \left(\tilde{\kappa}_3 - \frac{e_1}{\varrho + h_z} z(s) \right) ds \right\}, \end{aligned}$$

thus, according to v), $z(t)$ tends towards 0 when t tends towards $+\infty$: the predators z become extinct where,
 $\tilde{\kappa}_1 = r_x - c_2 \tilde{\xi}_4 - c_3 \tilde{\xi}_5$, $\tilde{\kappa}_2 = r_y - d_2 \tilde{\xi}_3$,
 $\tilde{\kappa}_3 = r_z + e_2 \tilde{\xi}_1 + e_3 \tilde{\xi}_2 + e_4 \tilde{\xi}_3 - m_z$ which completes the proof of the theorem. ■

8 Numerical simulations of a five interacting species model

In this section, we numerically analyze the model (1) that we have proposed and studied in the present paper. All numerical simulations were carried out using Matlab software version R2014b and the parameters used to produce the different evolution curves and phase portraits are listed in tables. This analysis reveals different behaviors and different types of dynamics developed by the interacting populations.

8.1 Some dynamics of evolution and extinction of interacting population density

The values of the various parameters used to obtain the figures are assigned in Table 1 and Table 2 below. It should be noted that each parameter has a non-negligible impact on the dynamics of the interacting species.

In general, densities obtained in the literature vary from 390 to 450 for golden eagles, from 160,000 to 2.5 million for rodents such as squirrels, and from 10,000 to 14,000 for mustelids such as weasels and martens. Birth rates vary from 0.77 to 2.7 for birds of prey such as golden eagles, and slightly more for mustelids and rodents.

We observe several forms of behavior that are stable or converge towards a stable state during interactions. Here are just a few of them (Figure 5). After an aperiodic fluctuation in the different densities of interacting species (Figure 3a), we observe that they converge towards a stable state from $t = 50$. The initial densities used are ($u_0 = 1500$, $v_0 = 1400$, $x_0 = 160$, $y_0 = 80$, $Z_0 = 170$) and the growth rates are ($r_u = 2.75$, $r_v = 2.78$, $r_x = 2.09$, $r_y = 1.28$, $r_Z = 0.76$). Note that species v and x are practically on the brink of extinction, while the other species in the system adapt and survive over a long period.

For Figure 3b, we observe a particular form in the evolution of the species : crenelated evolution with initial densities ($u_0 = 1200$, $v_0 = 1500$, $x_0 = 70$, $y_0 = 80$, $Z_0 = 40$) and growth

Table 1: Parameters values used for numerical simulations of Figure 3.
 Source : Created by the authors.

Figure3a	Figure3b	Figure3c
$a_1 = 0.0210$	$a_1 = 0.0210$	$a_1 = 0.2100$
$a_2 = 0.0295$	$a_2 = 0.0295$	$a_2 = 0.0295$
$a_3 = 0.0730$	$a_3 = 0.0100$	$a_3 = 0.0150$
$b_1 = 0.0193$	$b_1 = 0.0300$	$b_1 = 0.0193$
$b_2 = 0.00295$	$b_2 = 0.0020$	$b_2 = 0.00295$
$b_3 = 0.0730$	$b_3 = 0.0300$	$b_3 = 0.0730$
$c_1 = 0.1500$	$c_1 = 0.1500$	$c_1 = 0.1500$
$c_2 = 0.0450$	$c_2 = 0.0245$	$c_2 = 0.0450$
$c_3 = 0.0700$	$c_3 = 0.0700$	$c_3 = 0.0700$
$d_1 = 0.0540$	$d_1 = 0.0400$	$d_1 = 0.0540$
$d_2 = 0.0020$	$d_2 = 0.0200$	$d_2 = 0.0020$
$d_3 = 0.0074$	$d_3 = 0.4000$	$d_3 = 0.0074$
$e_1 = 0.2500$	$e_1 = 5.2500$	$e_1 = 0.2500$
$e_2 = 0.0450$	$e_2 = 0.0450$	$e_2 = 0.0450$
$e_3 = 0.0750$	$e_3 = 0.0750$	$e_3 = 0.0750$
$e_4 = 0.0400$	$e_4 = 0.0400$	$e_4 = 0.0400$
$h_1 = 0.0040$	$h_1 = 0.4000$	$h_1 = 0.0400$
$h_2 = 0.0320$	$h_2 = 0.3200$	$h_2 = 0.3200$
$h_3 = 0.0376$	$h_3 = 0.3760$	$h_3 = 0.3760$
$h_4 = 0.0400$	$h_4 = 0.4000$	$h_4 = 0.4000$
$h_5 = 0.0530$	$h_5 = 0.3200$	$h_5 = 0.5300$
$h_6 = 0.0320$	$h_6 = 0.0320$	$h_6 = 0.3200$
$\delta = 2.97000$	$\delta = 2.10125$	$\delta = 2.97000$
$k = 8000.00$	$k = 90.0000$	$k = 800.000$

Table 2: Parameters values used for numerical simulations of Figure 4.
 Source : Created by the authors.

Figure4a	Figure4b	Figure4c
$a_1 = 0.2100$	$a_1 = 0.0210$	$a_1 = 0.0721$
$a_2 = 0.0295$	$a_2 = 0.0295$	$a_2 = 0.0295$
$a_3 = 0.0150$	$a_3 = 0.0150$	$a_3 = 0.0150$
$b_1 = 0.0193$	$b_1 = 0.0193$	$b_1 = 0.0193$
$b_2 = 0.00295$	$b_2 = 0.00295$	$b_2 = 0.00295$
$b_3 = 0.0730$	$b_3 = 0.0730$	$b_3 = 0.0730$
$c_1 = 0.1500$	$c_1 = 0.1500$	$c_1 = 0.0150$
$c_2 = 0.4500$	$c_2 = 0.4500$	$c_2 = 0.0450$
$c_3 = 0.0700$	$c_3 = 0.0700$	$c_3 = 0.0070$
$d_1 = 0.0054$	$d_1 = 0.0540$	$d_1 = 0.0540$
$d_2 = 0.0020$	$d_2 = 0.0020$	$d_2 = 0.0200$
$d_3 = 0.0074$	$d_3 = 0.0074$	$d_3 = 0.0740$
$e_1 = 0.2500$	$e_1 = 0.2500$	$e_1 = 0.2600$
$e_2 = 0.0450$	$e_2 = 0.0450$	$e_2 = 0.0450$
$e_3 = 0.0750$	$e_3 = 0.0750$	$e_3 = 0.0750$
$e_4 = 0.0400$	$e_4 = 0.0400$	$e_4 = 0.0400$
$h_1 = 0.0040$	$h_1 = 0.0040$	$h_1 = 100.004$
$h_2 = 0.0320$	$h_2 = 0.0320$	$h_2 = 60.0320$
$h_3 = 0.0376$	$h_3 = 0.0376$	$h_3 = 500.0376$
$h_4 = 0.0400$	$h_4 = 0.0400$	$h_4 = 50.0400$
$h_5 = 0.0530$	$h_5 = 0.0530$	$h_5 = 70.0530$
$h_6 = 0.0320$	$h_6 = 0.0320$	$h_6 = 90.320$
$\delta = 2.97000$	$\delta = 2.97000$	$\delta = 2.97000$
$k = 8000.00$	$k = 800.000$	$k = 800.000$

rates ($r_u = 2.5$, $r_v = 2.78$, $r_x = 1.85$, $r_y = 1.1$, $r_Z = 0.86$). The species are able to rapidly increase their density, which they keep stable for

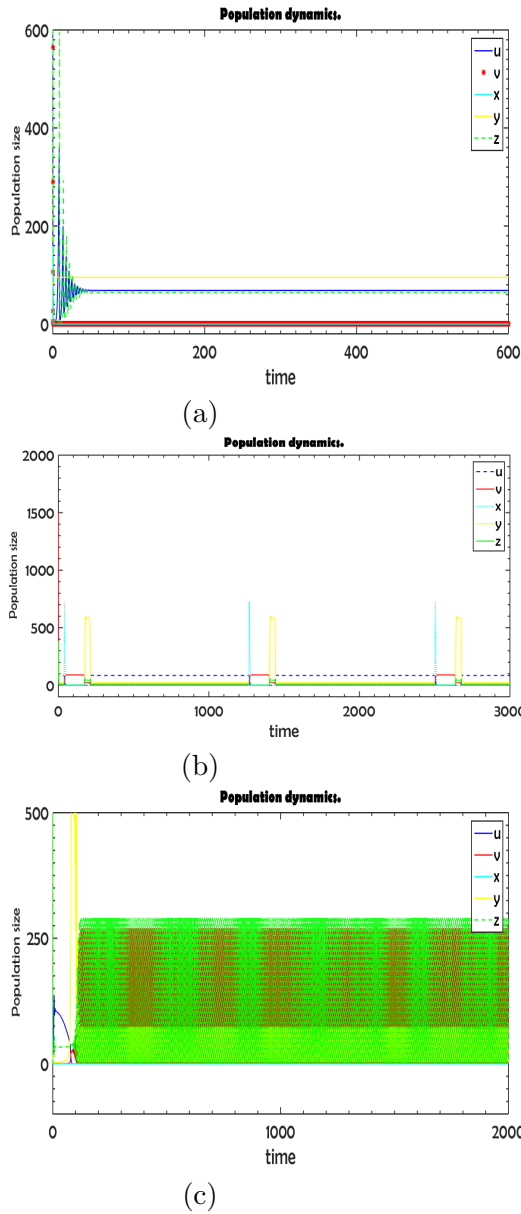


Figure 3: Different types of population density evolution, model of five interacting species.
 Source : Created by the authors.

a certain period. The number of species then declines rapidly to a certain level, where it is held stable for some time. This form of evolution in prey can be seen as a survival strategy. In fact, prey are able to avoid the presence of their predators for a certain time and increase their numbers. This strategy gives them a certain resilience. Figure 3c shows a case where some species benefit more than others from than others with initial densities ($u_0 = 1200, v_0 = 1500, x_0 = 70, y_0 = 80, Z_0 = 40$) and growth rates ($r_u = 2.5, r_v = 2.78, r_x =$

$1.85, r_y = 1.1, r_z = 0.86$) some maintain their high density compared to other species.

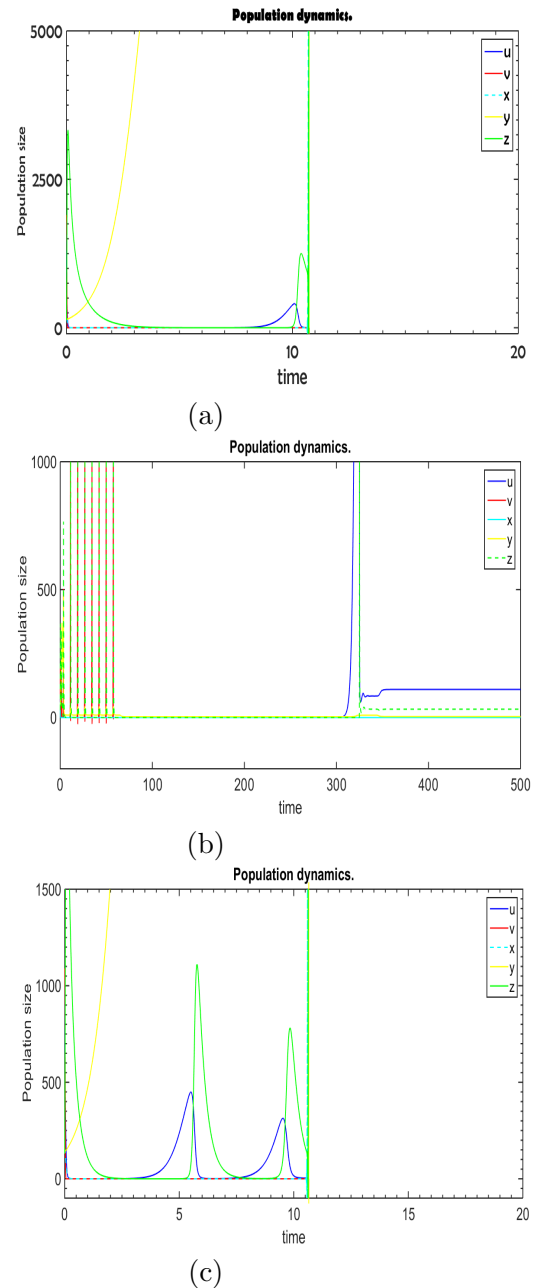


Figure 4: Some forms of population extinction for the five interacting species model.
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Figure 4 shows population extinction patterns. For initial densities ($u_0 = 1800, v_0 = 1900, x_0 = 160, y_0 = 140, Z_0 = 130$) and growth rates ($r_u = 2.50, r_v = 2.18, r_x = 1.75, r_y = 1.21, r_z = 1.60$) for Figure 4a and initial densities ($u_0 = 1080, v_0 = 1090, x_0 = 600, y_0 = 400, Z_0 = 380$) and growth rates ($r_u = 2.50, r_v = 2.1, r_x = 1.7, r_y = 1.2, r_z = 2.06$) for

the Figure 4c we have a total extinction : all the species interacting in the system (1) disappear. As for Figure 4b, with initial densities ($u_0 = 2000, v_0 = 1000, x_0 = 860, y_0 = 800, Z_0 = 280$) and growth rates ($r_u = 2.5, r_v = 1.7, r_x = 1.5, r_y = 1.2, r_Z = 1.06$) we are witnessing a partial extinction : some populations disappear while others adapt to the circumstances.

8.2 Phase portraits

The chronicles in Figure 5 represent different phase portraits. They provide the population dynamics in certain cases, based on the values listed in Table 3. In the different phase portraits presented below, Figures 5a and 5b, we observe a certain permanence of the system, a coexistence of different species. We note that all populations can cohabit under certain conditions and over a long period. Thanks to the action of predators, which act as density regulators, explosions in prey population density are rare. Furthermore, in Figures 5c and 5d, the numerical resolution of the system (1) leads to the extinction of prey u , which can be interpreted as an effect of excessive predation on the population of this species.

Table 3: Parameters values used for numerical simulations of Figure 5.
 Source : Created by the authors.

Figure5a	Figure5b	Figure5c	Figure5d
$a_1 = 0.021$	$a_1 = 0.021$	$a_1 = 0.0210$	$a_1 = 0.0210$
$a_2 = 0.0295$	$a_2 = 0.295$	$a_2 = 0.0295$	$a_2 = 0.0295$
$a_3 = 0.010$	$a_3 = 0.010$	$a_3 = 0.0150$	$a_3 = 0.0150$
$b_1 = 0.030$	$b_1 = 0.030$	$b_1 = 0.0193$	$b_1 = 0.0193$
$b_2 = 0.020$	$b_2 = 0.020$	$b_2 = 0.00295$	$b_2 = 0.00295$
$b_3 = 0.030$	$b_3 = 0.030$	$b_3 = 0.0730$	$b_3 = 0.0730$
$c_1 = 0.150$	$c_1 = 0.150$	$c_1 = 0.1500$	$c_1 = 0.1500$
$c_2 = 0.450$	$c_2 = 0.0245$	$c_2 = 0.4500$	$c_2 = 0.4500$
$c_3 = 0.070$	$c_3 = 0.070$	$c_3 = 0.0700$	$c_3 = 0.0700$
$d_1 = 0.040$	$d_1 = 0.040$	$d_1 = 0.0054$	$d_1 = 0.0054$
$d_2 = 0.020$	$d_2 = 0.020$	$d_2 = 0.0020$	$d_2 = 0.0020$
$d_3 = 0.400$	$d_3 = 0.400$	$d_3 = 0.0074$	$d_3 = 0.0074$
$e_1 = 0.250$	$e_1 = 5.250$	$e_1 = 0.0250$	$e_1 = 0.0250$
$e_2 = 0.045$	$e_2 = 0.045$	$e_2 = 0.0450$	$e_2 = 0.0450$
$e_3 = 0.075$	$e_3 = 0.075$	$e_3 = 0.0750$	$e_3 = 0.0750$
$e_4 = 0.040$	$e_4 = 0.040$	$e_4 = 0.0400$	$e_4 = 0.0400$
$h_1 = 0.400$	$h_1 = 0.400$	$h_1 = 0.0400$	$h_1 = 0.0400$
$h_2 = 0.320$	$h_2 = 0.320$	$h_2 = 0.3200$	$h_2 = 0.3200$
$h_3 = 0.376$	$h_3 = 0.376$	$h_3 = 0.3760$	$h_3 = 0.3760$
$h_4 = 0.400$	$h_4 = 0.400$	$h_4 = 0.4000$	$h_4 = 0.4000$
$h_5 = 0.320$	$h_5 = 0.320$	$h_5 = 0.5300$	$h_5 = 0.5300$
$h_6 = 0.320$	$h_6 = 0.320$	$h_6 = 0.3200$	$h_6 = 0.3200$
$\delta = 0.0125$	$\delta = 2.10125$	$\delta = 2.97000$	$\delta = 2.97000$
$k = 0.9000$	$k = 90.000$	$k = 800.000$	$k = 8000.00$

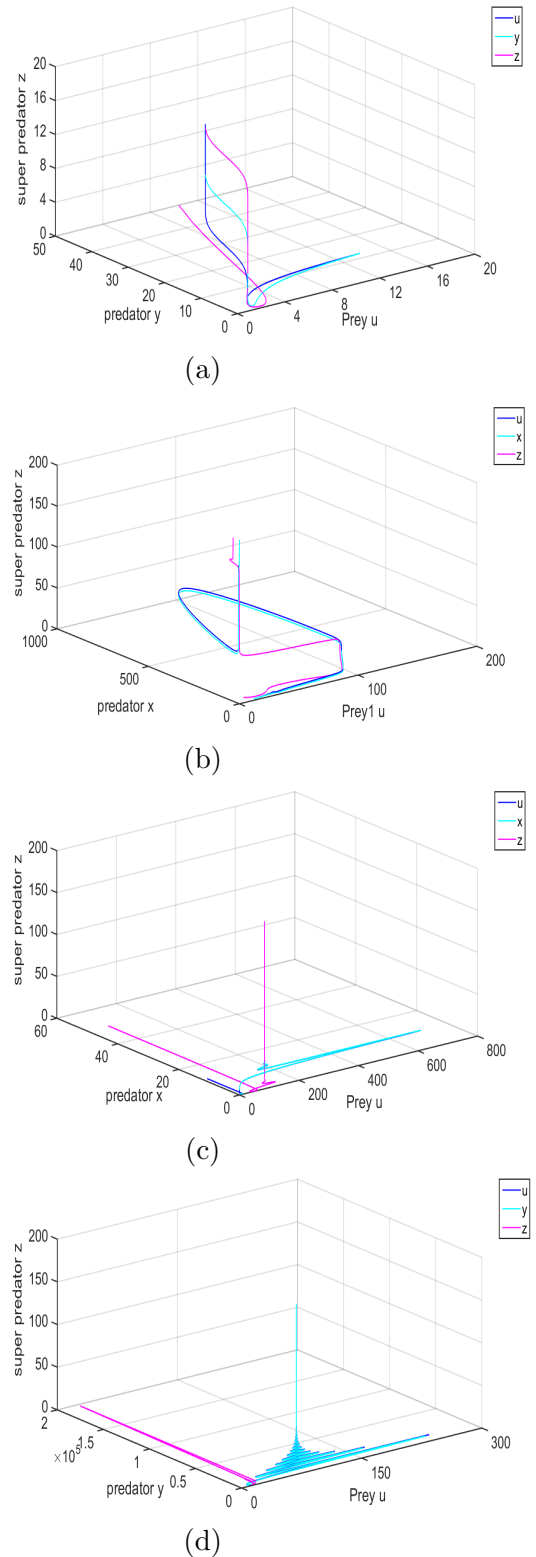


Figure 5: Phase portraits, model of five interacting species.
 Source : Created by the authors.

8.3 Perspectives and applications of the model

Mathematical models are very useful in many fields. For example, our model can be useful in agriculture. Plants are generally valued for their roots, stems and leaves by many predators: fungi, viruses, parasites, insects, rodents, carnivorous plants and so on. Each of these predators can attack the plant according to its own specificity. An adaptation of our model can be used to describe the relationship between plants and their predators, highlighting the physical, biological and chemical processes that link them, with a view to improving pest control and optimizing production of the plants or crops studied, while respecting the environment. Our model can find applications in the field of medicine, for example. It could be useful for studying the transmission of multifactorial diseases (diseases whose onset is linked to several genetic and environmental factors). The model could also be useful in the neurosciences, to describe the chemical functioning of brain neurons.

There are several possible perspectives for this model. We could, for example, incorporate other hypotheses relating to biological invasion phenomena, enabling us to study a model with diffusion. The system can also be studied using other tools such as fractional-order differential equations. Also, if we take into account the fact that animals develop at different stages, we can incorporate a delay factor into the model and obtain a system with delay. Furthermore, assumptions can be incorporated to obtain a mathematical model describing the dynamics of different interacting species, whose populations are structured into age classes or categories where the individuals in each category have common characteristics (survival probabilities, reproduction rate, energy quantity, predation rate).

9 Conclusion

The model presented in this study aims to clarify the consequences of a multiplicity of interaction types coupled with a multiplicity of competitors in relationships between preys and their predators. Predatory, competitive and neutral relationships between five species are represented with Holling's type I and II functional responses. A large number of derived models are obtained.

The theoretical study of the proposed model reveals the existence of several biologically

reasonable equilibrium points. However, particular attention has been paid to the trivial equilibrium point and the interior equilibrium point. The trivial equilibrium point is unstable, while the interior equilibrium point is globally stable. We have also obtained conditions for which solutions are bounded or permanent, as well as conditions leading to the extinction of species.

It emerges that the diversity of species living in the same environment induces a density-dependency phenomenon. Predators play the role of dependent factors, regulating species densities in the environment when they themselves do not have a high mortality rate (increased mortality is often due to disease or infection). The implementation of certain control strategies could be linked to the diversity of types of interaction in the same environment.

The various numerical tests carried out enabled us to corroborate the mathematical analyses, and also to discover a particular form in the evolution of species : crenelated evolution. This form of evolution reveals a strategy of struggle for survival among prey, a strategy that gives them a certain capacity for resilience.

We plan to extend this study by using several other methods and applying it to different domains. Indeed, we plan to re-investigate the proposed system in the context of biological invasion, using systems with diffusion, and then fractional order differential equations in a second step.

The target fields are medicine, for the control of bacterial contamination and the study of multi-vector transmission diseases in human, animal and plant populations, then agriculture, for questions relating to the invasion of species harmful to crops, as well as in the management of animal and fishery resources.

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References:

- [1] C. Clauzel. Les réseaux écologiques, une stratégie de conservation pour concilier fonctionnalités écologiques et aménagement du territoire. *Géocroniques*, 2022. fffalshs-03692583ff.

- [2] G. Delvoye. Modélisation de la dynamique d'une métacommunauté forestière. Modélisation et simulation. Université de Picardie Jules Verne, 2021. Français. ffnnt : 2021AMIE0030ff. fftel-03828759ff.
- [3] M. Bakhsh, M. H. Ghazali, M. K. Yar, A. Channo, The role of fish in global food and nutrition security: current aspects and future prospects, University of Sindh Journal of Animal Sciences, (december 2023).
- [4] A. Whitney Scheffel, L. Heck Jr Kenneth and P. Lawrence Rozas, Effect of Habitat Complexity on Predator-Prey Relationships: Implications for Black Mangrove Range Expansion into Northern Gulf of Mexico Salt Marshes. *Journal of Shellfish Research*, 36(1) (2019) 181-188.
- [5] A. Alexi, A. Rosenfeld, T. Lazebnik, Multi-species prey-predator dynamics during a multi-strain pandemic, *Chaos* 33, 073106 (2023).
- [6] N.W. Barendregt, and P.J. Thomas, Heteroclinic cycling and extinction in May-Leonard models with demographic stochasticity. *Journal of Mathematical Biology*, 86(2), 30. (2023). 10.1007/s00285-022-01859-4.
- [7] M. Troell, M. Jonell and B. Crona, The role of seafood for sustainable and healthy diets, The EAT-Lancet commission report through a blue lens Technical Report, (January 2019).
- [8] D. Ghosh, P.K. Santra, G.S. Mahapatra, A three-component prey-predator system with interval number, *Mathematical Modelling and Numerical Simulation with Applications*, 2023.
- [9] A. Kumar, B. Dubey, Stability and Bifurcation of a Prey-Predator System with Additional Food and Two Discrete Delays, *Computer Modeling in Tech Science Press Engineering and Sciences*, 2020.
- [10] V.P. Saxena, Mathematical modelling of biological populations with and without dispersion, *Research in Statistics*, 1:1, 2215638, (2023) DOI: 10.1080/27684520.2023.2215638.
- [11] V.P. Saxena, L. Dhurve, Mathematical Modelling of Prey and Predator Species with Two Age Group of Prey Population, *International Journal on Emerging Technologies* 13(2): 06-11(2022).
- [12] T. B. B. Lagui, M. Dosso, G. Sitionon, An analysis of some models of prey-predator interaction. *wseas transactions on biology and biomedicine*, Volume 21, (2024).
- [13] J. Paris, *Commensalisme et Parasitisme chez les Annélides Polychètes, Vie et Milieu*, 1955, pp.525-536. fhal-02636254f.
- [14] T. B. B. Lagui, M. Dosso, S. Traoré, On Models of Population Evolution of Three Interacting Species. *International Journal of Applied Mathematics, Computational Science and Systems Engineering*, 2024, p. 193-223.
- [15] A.J. Lotka, Analytical note on certain rhythmic relations in organic systems, *Proc. Natl. Acad. Sci.* 6, (1920) 410-415.
- [16] V. Volterra, Fluctuations in the abundance of a species considered mathematically. *Nature* 118, 558-560 (1926). Reprinted in L.A. Real, J.H. Brown (eds.) *Foundations of Ecology*, pp. 283-285. University of Chicago Press (1991).
- [17] J.R. Reimer, H. Brown, E. Beltaos-Kerr, G. de Vries, Evidence of intraspecific prey switching: stage-structured predation of polar bears on ringed seals, *Oecologia* (2018).
- [18] Freepik, Wild animals Images - Free Download. (Find & Download Free Graphic Resources for Wild animals Vectors, Stock Photos & PSD files.) [Online]. <https://freepik.com/photos/animals>. (Accessed Date: December 11, 2024).
- [19] F. Chen, Z. Li, and Y. Huang, Note on the permanence of a competitive system with infinite delay and feedback controls, *Nonlinear Analysis: Real World Applications* 8 (2007), 680-687.
- [20] A. T. N'guessan, Dynamique des populations et ses motivations économiques : cas des modèles de type Kaldor modifié. Ph.D. Thesis, Université Félix Houphouët-Boigny de Cocody-Abidjan, laboratoire de Mathématiques et Applications, UFR de Mathématiques et Informatique, (2019), 49-50.
- [21] L. Perko, *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics 3 Springer Verlag, Heidelberg, 1991.

- [22] M. Daher Okiye and M.A. Aziz-Alaoui. On the dynamics of a predator-prey model with the Holling-Tanner functional response. pages 270–278. Miriam Editions, Proc. ESMTB conf., 2002.
- [23] L. Zhao, F. Chen, S. Song and G. Xuan, The Extinction of a Non-Autonomous Allelopathic Phytoplankton Model with Non-linear Inter-Inhibition Terms and Feedback Controls, *Mathematics* 2020, 8(2), 173; <https://doi.org/10.3390/math8020173>.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

All authors have contributed equally to creation of this article. Indeed, Thierry Bi Boua Lagui carried out the digital simulation and the interpretation of the results of the curves of the different models. Thierry Bi Boua Lagui and Mouhamadou Dosso wrote the article after checking the simulation results. Thierry Bi Boua Lagui, Mouhamadou Dosso and Soma Ouattara have made corrections to the English translation of the article.

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