Interface Problems-Fluid Structure Interaction: Description, Application and Review

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Abstract: - This paper presents a critical review of numerical methods for solving a wide variety of interface problems emphasizing the immersed finite element method (IFEM). It is found in the literature that most of the researchers considered the well-known methods with some modifications, however limited number of research articles proposed new algorithms. Apart from the algorithm, this study highlights the wide range of applications of interface problems specifically in biomedical, heat-transfer and turbo-machinery. Different numerical methods for interface problems with their major finding are listed in tabulated form at the end.

Key-Words: - Interface Problems, Algorithms, Immersed Finite Element Methods, Fluid-Structure Interactions, Numerical Models, Biomedical Engineering, Heat-Transfer, Turbo Machinery.

Received: August 25, 2023. Revised: February 11, 2024. Accepted: March 12, 2024. Published: May 23, 2024.

1 Introduction

In the last few decades, several numerical methods are discussed to study the interface problems. Nowadays, Bio-medical engineering is one of the emerging fields in which medical practitioners have focused on computational results for quick diagnosis and treatment. Most of the diseases are growing on the surface-wall (blood flow in artery, airflow in human airways/lungs) because of interaction between fluid (air, blood etc.,) and solid surface where interface phenomena exists between the two surfaces. The Fluid-Structure Interface (FSI) method is used for solving interface-problem in many bio-medical applications. In general, interface issues include differential equations, for which the solutions and input data are discontinuous across some interfaces. Construction and testing of any physical prototypes is expensive in terms of cost and time, and even unable to provide sufficient results. Mathematical methods, like the Immersed Finite Element Method, are more economical and effective for analysing intricate geometries that include the interface of two or more entities. It is found that in most of the real life problems, To solve the entire system, a huge number of linear systems must be used. with more efficient method because of various realistic needs such as curvature, random dimension, moving interface. In this reference, [1], have studied the bilinear and two-dimensional linear IFE solutions computed from the algebraic multi-grid solver for both moving and stationary interface problems.The study, [1], discussed several issues of immersed interface methods including development and behavior (immersed compressible and incompressible continuum). They have also discussed how the Peskin developed immersed boundary method for blood flow in human artery-valves, [2], [3]. Initially, In situations when an integral equation may not be accessible, the Immersed Interface Method (IIM) was designed as a second order accurate finite difference approach to solve elliptic and parabolic partial differential equations. Further, this method was expanded for two and three dimensional problems such as parabolic, elliptic, hyperbolic and mixed type equations, [4]. It is worth mentioning that at the initial stage of the development of interface methods; only three famous methods (Smoothing method, Both the Method of Harmonic Averaging and Peskin's Immersed Boundary Method (IBM) were applied. After that several additional boundary conditions (jump conditions, stationary & moving boundaries, etc.) have been incorporated that escort to nonsymmetrical matrix where standard numerical solvers fails, [5]. In [6], presented several points of Immersed Finite Element Method in their review paper, in immersed finite element method, Lagrangian dense grid shift on the topmost of Eulerian fluid grid that is expanded over the entire computational domain. In the end, modelling of both the fluid and solid domains is carried out, and continuity between the solid and fluid sub-domains is ensured by force distribution utilising the reproducing kernel particle technique (RKPM) delta function and velocity interpolation. When compared to the Immersed Boundary (IB) approach, it is discovered that the higher order replicating kernel particle method performs better at solving nonuniform spatial meshes with arbitrary geometries and boundary constraints. It is widely known, interface boundary value issues connected with different kinds of partial differential equations or systems may be used to represent a wide range of phenomena in biology, chemistry, engineering, and physics. As a result, the researchers have put up a number of strategies to address various real-world issues. The use of particular method varies from problem-to-problem and with their boundary conditions. Keeping in view, the requirement of new researchers, we described crux of the published papers with the intension to provide all important material related to partial differential equations with one or more interfaces for new researchers in a condense form. In the next section, a few real-life problems have been taken to elaborate the application of Immersed Finite Element Method.

2 Some Real Life Applications on Immersed Finite Element Method

The various real life problems deal with Immersed Finite Element Method, few of them are listed as follows.

2.1 Immersed Finite Element Method for Biomedical Problems

It is found that there are several real-world problems dealing with interface-phenomena. One of the vital use of interface problem for solving the partial differential equations in biological systems (Human lungs, Heart, Skins etc.). Human lugs and heart (coronary artery) are flexible in nature where the interface phenomena exist on the wall. It is seen that inner injury in human lungs (Figure 1) occurs between the wall and air. Elastic interface problems have broad applications in continuum mechanics specially for the problems involves stress and strain, [7]. Wall shear stress is the main parameter that is used for the interface-problems (between wall & air, between wall & blood, between wall & fluids) in biological system. Nowadays, Finite element method and computational fluid dynamics software?s are useful for solving such interface problems where large numbers of nodes/control volumes occur.

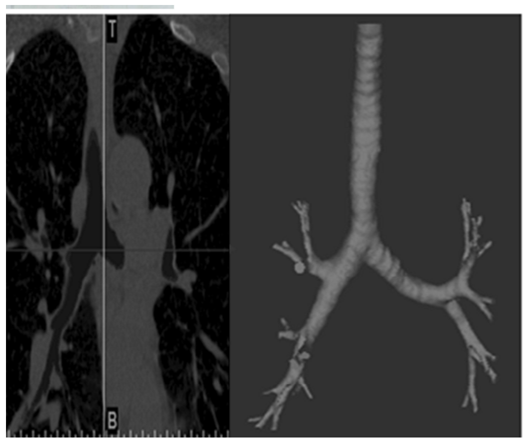

Figure 1: Interface problems in Human Lungs, [8]

Human heart (Figure 2) is another part of human body where researchers are trying to solve partial differential equations for interface problems between blood and arteries. In this context, [9], have discussed simulation of expandable stent using immersed finite element method, which was introduced for solving complex fluid-structure interaction (FSI) problems. Coronary stents is used to physically open the channel of blocked artery. They presented model and simulation of balloon elastic stent (Figure 3) cooperating with its nearby fluid that is a class of interface problem.

Numerical methods play the major role to solve these interfaces problems of biological system so that the medical practitioner can exactly diagnose and further prognosis the disease.

2.2 Immersed Finite Element Method for Turbo Machinery Problems

Turbo-machinery is another area of application of interface-problem. It is found that the inner wall of water-pump becomes erosion due to high pressure water/oil and that reduce the efficiency of pump. These interface problems are also used for cavitations in turbo-machinery. It is seen that turbo-machinery contains two important parts namely stationary and

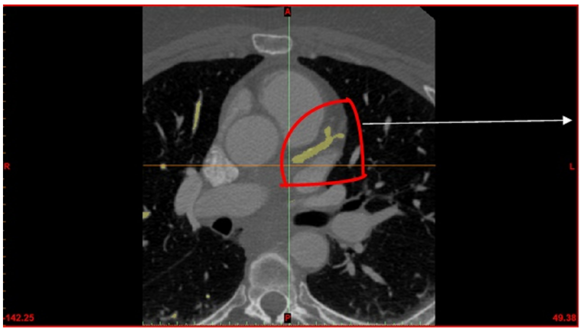

Figure 2: Interface problems in human heart, [10]

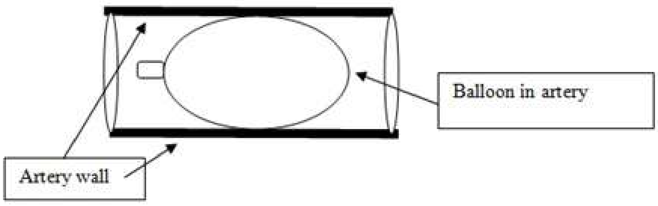

Figure 3: Stent in artery wall using IFEM

moving (rotating) components. Interface is occurred between stationary and moving components which pay a crucial role for increasing the efficiency of turbo-machinery (Figure 4). It is known that fluid and solid are an important component of the rotor system which influence each other and which creates sensations in some conditions, [11].

2.3 Immersed Finite Element Method for Heat Problems

Heat-transfer problems are another important real application where an interface phenomenon occurs near the moving boundaries, [12]. The details of such problems are discussed in the elliptic interface section 3.1.

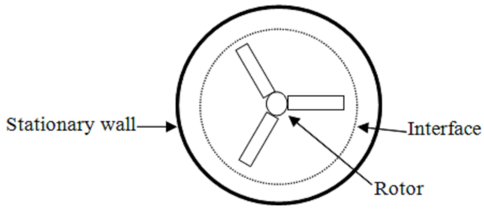

Figure 4: Interface problem in Turbo-Machinery

3 Classification of Interface Finite Element Methods

Because of several variability conditions in interface problems (fluid-solid interaction, fluid-fluid interaction, solid-solid interaction etc.), it is very difficult to used exact interface method for explaining the partial differential equations. The following classifications are done on the basis of their application/popularity in different areas of sciences and engineering (Figure 5).

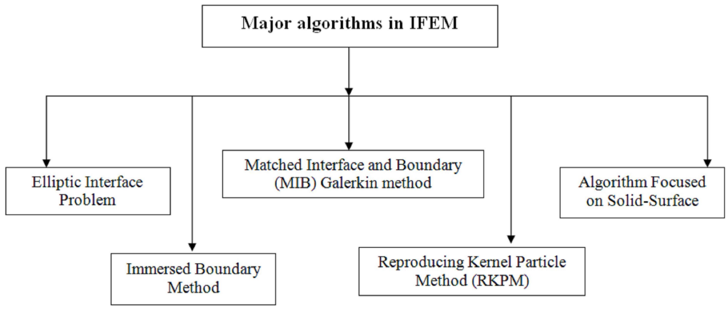

Figure 5: Classification of IFEM

These above methods are further classified in the literature depending upon the problems and their boundary conditions.

3.1 Elliptic Interface Problem

It is seen that elliptic interface phenomena comes from the elliptic boundary value problems having discontinuity in their coefficients or solution around one or more interfaces of the flow domain. In such circumstances, it is very difficult to find approximate results using standard finite difference method due to irregularity or non-smooth near the interface. There are numerous numerical methods that use Cartesian grids for evaluating elliptic interface problems. Finite difference method has been modified such as immersed interface method to provide most accurate results but still there is need to develop new methods for refining the solution or exact solution, [13]. The elliptic interface equation of second order is

$$\begin{cases} [-\nabla.\{\beta\nabla u\}] = f(x) & X \in \Omega \\ [u(X)] = g(X) & X \in \partial\Omega \end{cases}$$

Jump conditions

$$|u|_{\Gamma} = 0$$
$$|\beta\frac{\partial u}{\partial n}|_{\Gamma} = 0$$

where,

$\overrightarrow{u}$ is the fluid velocity,

$\overrightarrow{X}(s,t)$ is the Lagrangian representation of the immersed moving boundary,

υ is the constant fluid viscosity,

$\overrightarrow{f}(s,t)$is the force strength along the interface,

ρ is the fluid pressure,

Ω = open rectangular domain,

Γ = interface curve,

$\beta(X)$ = positive piecewise constant.

$$\beta(X) = \begin{cases} [\beta^- & X \in \Omega^- \\ \beta^+ & X \in \Omega^+ \end{cases}$$

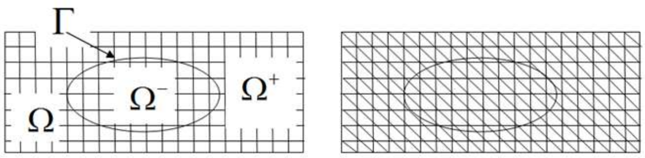

Figure 6: The domain Ω has divided into two subdomain and representation of Grid generation

Numerical methods such as finite element methods (FEM) and finite difference methods (FDM) can be used to tackle the interface problem mentioned above. Finding the precise answer to the curved-interface problem is a challenging endeavor, [14].

There are many types of interrelated problems depending on geographic area and environment (Figure 6). In this context, [4], discussed four types of elliptic interface problems in their review article, [4]. These are: (i) The coefficients of the differential equation are constant (ii) The coefficients of the differential equation are not constant. (iii) The interface is fixed (iv) the interface is movable, [4]. Seyidmamedovy and Özbilgey discussed the transportation problems of the crisis in the state in the media. In this study, two different types of jumps of the propagation method are presented, and the local error for each case is estimated on the non-uniform grid, especially using the standard deviation, [15]. Mathematical methods are described in this reference by [3]. It includes different types of problems including: discussed the computational problems of the immersed finite element method.

In [16], author is discussed three types of computational schemes involving non-conforming rotated Immersed Finite Element functions for evaluating the elliptic interface problems, [16].
(I). Finite Element Galerkin Immersed Methods.
(II). Partially Penalized Galerkin (PPG) Immersed Finite Element Methods.
(III). Interior Penalty Discontinuous Galerkin (IPDG) Schemes.
An and Chen (2014) studied a partially penalty immersed interface finite element method for solving the anisotropic elliptic interface problems. The proposed method is based on linear immersed interface finite elements and used not-continuous Galerkin formulation nearby the boundary, [17]. Further, higher degree immersed finite element methods for second order elliptic interface problem was discussed by [18].

[16], has mentioned several real life applications of elliptic interface problems such as plasma particle simulations in ion thruster optics. It is known that ion thruster is an electronic propulsion device that releases a high-energy ion beam to drive a spacecraft. It is also used in projection methods to solve Navier-Stokes problems in multiphase-flow. Another, application of elliptic interface problem is used for the topology-optimization in heat-conduction problems.

Several methods are discussed by the researchers in which immersed interface method is very popular because of method efficiency. In immersed interface method, jump conditions are included in standard finite difference formula across the interface to obtain minimum error. It is found that discontinuity in general solution and its derivative occurs because of discontinuity in the coefficient of differential equation. After that Immersed Interface method was further implemented by the researchers and given decomposed immersed interface method for solving elliptic equations that provides non-smooth and discontinuous solution. Later on, fourth order accurate finite difference was given for solving elliptic equations having embedded interface of discontinuity, [4].

3.2 Galerkin Method of Matching Interfaces and Bounds (MIB)

It is well known that interfaces exist in many real-world models and devices that help solve relevant problems and support the development of new algorithms. Due to the different properties of various natural materials and models, the part of the equation with constant coefficients and positive terms needs to be solved. A partial equation is difficult to solve due to interactions and interactions between two or more interacting objects. The integrated and boundary layer approach was adopted by [19], Solving multicomponent interface problems. This method involves extending interface-based solutions from various perspectives on both sides of the interface, [20]. Matching interfaces and boundary Galerkin techniques were introduced by [21], in a specific study solved two-dimensional partial equations with complex interfaces, geometric singularities, and rare solutions. Since higher order methods are important for many problems involving high-frequency waves, they proposed further development of three- and four-dimensional matching interface and boundary methods . Furthermore, for three-dimensional elliptic interface issues, [22], establish the Galerkin formulation of the MIB technique. Boundary Galerkin technique and the suggested three-dimensional Matched Interface are verified for correctness and stability across three different types of elliptic interfaces problems. The first type of interface is analytically defined by level set functions that admits geometrical singularities. The second types of Protein surfaces define interfaces. and last one is generated from multi-protein complexes.

Thus, Numerous issues in the real world may be solved with the renowned Matched Interface and Boundary (MIB) Galerkin approach.

3.3 Immersed Boundary Method

The Immersed Boundary (IB) method was developed by [2], to solve the problem of interaction between blood and the vessel wall. The main advantage of this method is that it has a network update algorithm between the liquid and the walls. This method is based on the difference method by dividing line, and the submergence is represented by the fiber network, [3], [6]. Fluid interaction is achieved by dividing the node forces and interacting between the Eulerian and Lagrangian fields using the Dirac delta function. However, one of the limitations of this method is that it provides an immersed fibrous structure that has mass but does not retain volume in the liquid, [6]. In [3], authors proposed the continuous expansion method for boundary conditions (EIBM). Instead of using small particles in the elastic barrier, they use material in the elastic material that holds the small volume inside the liquid. In the proposed method, the meshless Regenerative Kernel Particle Technique (RKPM) kernel function is used by the discretized delta function. They achieved better results with these modifications, [23], [6] It briefly introduces the immersed finite element method for solving complex fluid and solid problems. Lagrangian objects and Eulerian fluid networks control all calculations in the immersed finite element method. In solving non-uniform spatial networks, the Kernel Particle Propagation Method (RKPM) outperforms the immersed boundary method. They eliminated the disadvantages of the submerged boundary layer and incorporated the extended finite element method (EFEM) concept of [23], into their design. The computational fluid dynamics (CFD) fails to solve many problems such as elastic boundary moves fluid and eventually fluid reverse back against it. These issues lead to the interface problem with a solitary source and discontinuous coefficient, [4].

The Stoke's equation:

$\nabla p = \nu \Delta \overrightarrow{u} + \overrightarrow{F}(\overrightarrow{x}, t)$

$\nabla . \overrightarrow{u} = 0$

where, $\overrightarrow{F}(\overrightarrow{x}, t)$is boundary force and it can be defined as:

$\overrightarrow{F}(\overrightarrow{x}, t) = \int_{\Gamma(s,t)} \overrightarrow{f}(s,t)\delta_2(\overrightarrow{x} - \overrightarrow{X}(s,t))ds.$

The discrete delta function was used in the immersed boundary approach to distribute the singular source to the closest grid, and a collection of lagrangian points discretizes the immersed boundary. Additionally, a discrete sum is used in place of the integral function, while a discrete approximation with length support is used in place of the delta function. This approach is popular because it can be implemented easily and therefore, used in many real life application, [4].

3.4 Reproducing Kernel Particle Method (RKPM)

[6], suggested an immersed finite element approach for solving the fluid-structure interaction issue. To ensure continuity between the fluid and solid subdomains, they employed the replicating kernel particle technique (RKPM). They have also incorporated the concept of the extended immersed boundary technique (EIBM). The fluid-solid interaction is used to model the flow vibration in vocal folds during phonation. [24], described a linked sharp-interface immersed boundary-finite element approach for interaction problems involving human phonation. They investigated the dynamics of the glottal jet in a three-

dimensional model, as well as the two-dimensional laryngeal model. [25], developed an immersed finite element approach with integral equation correction. They portrayed the physical domain border as a domain to aid in the mapping of finite element and integral equation solutions. [4], discussed numerous numerical approaches for handling elliptic interface issues. In their survey piece, they focused on newly discovered numerical approaches and their benefits and drawbacks. They determined that the Finite Difference Method (FDM) is easier to adopt than the Finite Element Method (FEM), which has less regularity. It is observed that numerical techniques using grids are more costly than meshless methods. [26], introduced immersed finite element approaches for elliptic interface issues with non-homogeneous jumps. This approach was proposed to solve second-order elliptic problems with discontinuous jump conditions. [27], addressed the Immersed smoothed finite element method for simulating fluid-structure interactions in aortic valves. They addressed the solid domain using a smoothed finite element approach, which is appropriate for handling hyperelastic materials with substantial deformation. [28], reported an immersed finite volume approach for simulating unsteady three-dimensional heat transfers and turbulent flows in an industrial furnace with three conducting solid bodies. Heltaia and Costanzo (2012) proposed the generalised finite element immersed boundary technique (FEIBM), in which an incompressible Newtonian fluid interacts with a generic hyperelastic solid. The suggested technique is based on independent discretization for both the fluid and solid domains.The fluid and solid domains are modelled by the Eulerian and Lagrangian models, [29]. [30], investigated the immersed smooth finite element approach for two-dimensional fluid-structure interaction issues. They employed a characteristic-based split technique to compute fluid-structure interaction forces. In this approach, a fictional fluid mesh is employed to calculate the fluid-structure interaction force acting on the solid. [31], has used adaptive interface finite element method for elastic interface. He has established a residual based posteriori error estimate constant for refining the finite element mesh. This posteriori error was further tested and is found competitive nature of adaptive method. [32], explored the immersed molecular finite element technique (IMFEM). It was discovered that the interaction between submerged objects and the fluctuating fluid is caused by hydrodynamic forces, which gave rise to the fluid-structure interaction term. It has been stated that a molecular type force field paired with a coulomb potential between submerged objects enables the immersed molecular finite element method to provide a thorough description of nanoscale structure.

3.5 Algorithm Focused on Solid-Surface

The literature on related topics seems to be limited to researchers focusing solely on the material, so this chapter introduces many aspects of the material. [33], studied the modified immersed finite element method for fluid structure assembly. The proposed system focuses on material dynamics rather than fluid dynamics. This approach is especially important in situations where strong forces play an important role in fluid interactions. Captures material behavior better than standard interface finite element method algorithms, especially at high Reynolds numbers.

[34], discussed the use of the immersed finite element method to describe tissues interacting with fluids. They found three biological applications using the interfacial finite element method. A special method for combining finite element technical formulas is required for stable and accurate calculations of liquids and solids. [35], applied the solution method to fluid interface problems. They found that the applicability of the method is general and independent of the specific choice of finite field in waste and continuous water.

[36], describes the immersed finite element method to solve symmetric and similar problems. The proposed method uses a non-uniform finite element method, which is symmetric and follows the second truth. [37], discussed the regular Galerkin immersed finite volume element method for anisotropic flow model in porous media. This method was developed by mapping the orbital function space into the immersed finite element space. [38], proposed a nonlinear finite element method for nonlinear material mechanics. Therefore, the authors used this method to provide the most important solution for the competition.

4 Conclusions

Many variables and boundary conditions must be considered while developing an appropriate numerical approach for fluid-structure interaction issues. It is seen that a lot of work has been done on the fluid-solid interaction problem; still it is far away to predict the accurate result. Use of correct numerical method is one of the challenging task as well as important parameter which have been discussed in the present study. Based on the review of recent literature as discussed above, the point wise conclusions and suggestions are furnished herewith:

(i) Immersed boundary method is one of the popular methods that have been used/implemented by several researchers because of finite difference based method with uniform grid distribution.

(ii) Most of the methods used finite difference method for the development of new-methods such as immersed finite element method.

(iii) It is seen that immersed boundary (IB) method is based on finite difference with uniform grid distribution and having mesh updating algorithms between fluid and solid wall.
(iv) The reproducing kernel particle approach is effective for addressing non-uniform spatial meshes. This approach used a discretized delta function in a mesh-free distribution, and greater order precision improved the interaction between fluid and solid.
(v) In the present paper, real-life problems have been presented to make out the application of interface finite element method.
(vi) The different types of numerical methods presented in this study can be applied in several real life problems depending upon the interface conditions.

Acknowledgment:
The authors thank the reviewers, editor, and editing department for giving valuable inputs and suggestions to get the current form of the manuscript.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Dr. V.K. Srivastav is involved in the formation and derivation of the mathematical calculations. Dr. S. Thota, M. Kumar, and A.R. Anand are involved in suggestions, revisions, and verification of the mathematical calculations. All authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.