Merdan-type Allee Effect on a Lotka-Volterra Commensal Symbiosis Model with Density-dependent Birth Rate

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Abstract: - A Lotka-Volterra commensal symbiosis model with a density dependent birth rate and a Merdan-type Allee effect on the second species has been proposed and examined. The global attractivity of system's equilibria is ensured by using the differential inequality theory. Our results show that the Allee effect has no effect on the existence or stability of the system's equilibrium point. However, both species take longer to approach extinction or a stable equilibrium state as the Allee effect increases.

Key-Words: -Commensalism; Global attractivity; Allee effect

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1 Introduction

The objective of this study is to analyze the dynamic patterns of a commensalism model that incorporates a density-dependent birth rate and a Merdan-type Allee effect. The model can be expressed as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}y\Big), \\ \frac{dy}{dt} &= y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\Big)\frac{y}{\beta+y}, \end{aligned} \quad (1)$$

The given equation involves positive constants b_{ij} for $i = 1, 2$ and $j = 1, 2, 3, 4$, as well as a_{11}, a_{12}, a_{22}, and β. The variables $x(t)$ and $y(t)$ represent the density of the first and second species, respectively, at a given time t. It is assumed that the second species is subject to the Allee effect, which is mathematically represented as $\frac{y}{\beta+y}$.

One of the fundamental relationships between species is commensalism, in which one species uses the resources of the other without causing harm or gaining anything in return. Commensalism occurs frequently in nature, but theoretical study of it has only just started in the last 20 years ([1]-[24]). Regarding this particular direction, the study's initial framework is based on the well-known Lotka-Volterra type commensalism model ([6, 7, 8, 9, 12]). Afterward, numerous scholars ([1, 2, 3, 4, 5, 10] argued that the functional response function must be included to describe how intense the commensalism between the species is. and they primarily concentrate on the system's stability property. Recently, researchers started looking into the Allee effect's impact on the commensalism model ([4], [11],[14]) and the nonlinear density dependent birth rate ([19],[21], [22]).

It is well known that food and other resources, and space are always limited. With the increase in population density, intraspecific competition intensifies and the number of natural enemies increases continuously. leading to a reduction in the birth rate of the population, This suggests that it is more realistic to consider density-dependent birth rates in population models. Recently, Chen et al [19] argued that the Beverton-Holt function provides a suitable framework for understanding this phenomenon. The model they considered takes the form:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}y\Big), \\ \frac{dy}{dt} &= y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\Big), \end{aligned} \quad (2)$$

The given constants $b_{ij}, i = 1, 2, j = 1, 2, 3, 4$, and a_{11}, a_{12} and a_{22} are all positive. These values play an important role in determining the density of the first and second species at time t, which is represented by $x(t)$ and $y(t)$ respectively. The system can have four nonnegative equilibria. The authors demonstrated that by using appropriate Lyapunov functions, all four equilibria can be globally asymptotically stable, given certain assumptions. This distinguishes this system from commensalism systems with constant birth rates and is a special characteristic.

On the other hand, with the overexploitation of natural resources by humans, more and more species have become endangered, and their populations have declined sharply, which will make it difficult for them to find mates and cooperate effectively, which will exacerbate further reductions in populations. The phenomenon is called the Allee effect. Merdan [13] proposed a predator-prey system described below:

$$\begin{aligned} \frac{dx}{dt} &= rx(1-x)\frac{x}{\beta+x} - axy, \\ \frac{dy}{dt} &= ay(x-y), \end{aligned} \quad (3)$$

where $\frac{x}{\beta + x}$ represents the Allee effect term, and β indicates the strength of the Allee effect. Merdan demonstrated the detrimental effects of the Allee effect on the species, the Allee effect delays the system's arrival at its steady-state solution and reduces the final densities of both species. Since then, many academics have suggested the ecosystem incorporating the Allee effect of the Merdan type ([4],[14, 15, 16, 17, 18]).

Lin[4] introduced a Lotka-Volterra commensalism model incorporating the Allee effect in relation to the first species.

$$\begin{aligned} \frac{dx}{dt} &= x(b_1 - a_{11}x)\frac{x}{\beta + x} + a_{12}xy, \\ \frac{dy}{dt} &= y(b_2 - a_{22}y). \end{aligned} \tag{4}$$

He demonstrated that the Allee effect increases the species' final densities. Such a finding is very different from Merdan's findings.

In her study, Wu[4] presented the following commensal symbiosis model:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1 + y^p}\Big), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u + y}, \end{aligned} \tag{5}$$

The findings of her study indicate that the system exhibits permanence and possesses a singular, globally stable positive equilibrium. Furthermore, the Allee effect does not appear to exert any discernible influence on the ultimate density of the species.

As we can see from systems (3)-(5), the Merdan type Allee effect affects each ecosystem in a different way. To the best of our knowledge, the Allee effect has not yet been included in the system (2). This inspired us to suggest the system (1).

Finding out the impact of the nonlinear density birth rate is a fascinating topic with regard to the system (1). **Is it possible that the system (1) admits some dynamic behaviors that are comparable to those of the system (2)? Could we provide a positive response regarding the Allee effect?**

The remainder of the paper is organized as follows. The purpose of this essay is to determine the resolution to the above problems. Section 2 discusses the existence of the equilibria; Section 3 presents the paper's main conclusion; and Section 4 establishes a new lemma and uses it and the differential inequality theory to strictly prove the paper's main conclusions. A brief discussion follows this essay's conclusion.

2 Equilibria

The equilibria of system (1) fulfill the equation

$$\begin{aligned} x\Big(\frac{b_{11}}{b_{12} + b_{13}x} - b_{14} - a_{11}x + a_{12}y\Big) &= 0, \\ y\Big(\frac{b_{21}}{b_{22} + b_{23}y} - b_{24} - a_{22}y\Big)\frac{y}{\beta + y} &= 0. \end{aligned} \tag{6}$$

There is always a boundary equilibrium $A_1(0, 0)$ in the system (1). If

$$\frac{b_{11}}{b_{12}} > b_{14} \tag{7}$$

holds, then the system represented by equation (1) exhibits a nonnegative boundary equilibrium denoted by $A_2(x^*, 0)$, where

$$x^* = \frac{-(b_{14}b_{13} + a_{11}b_{12}) + \sqrt{\Delta_1}}{2a_{11}b_{13}}. \tag{8}$$

here

$$\Delta_1 = (b_{14}b_{13} + a_{11}b_{12})^2 - 4a_{11}b_{13}(b_{14}b_{12} - b_{11}).$$

Assume that

$$\frac{b_{21}}{b_{22}} > b_{24} \tag{9}$$

holds, then it can be observed that the system represented by equation (1) possesses a boundary equilibrium denoted as $A_3(0, y^*)$, which is nonnegative, where

$$y^* = \frac{-(b_{24}b_{23} + a_{22}b_{22}) + \sqrt{\Delta_2}}{2a_{22}b_{23}}. \tag{10}$$

here

$$\Delta_2 = (b_{24}b_{23} + a_{22}b_{22})^2 - 4a_{22}b_{23}(b_{24}b_{22} - b_{21}).$$

Assume that (9) and

$$\frac{b_{11}}{b_{12}} + a_{12}y_1 > b_{14} \tag{11}$$

holds, the system represented by equation (1) possesses a unique positive equilibrium denoted as $A_4(x_1, y_1)$, where

$$y_1 = \frac{-(b_{24}b_{23} + a_{22}b_{22}) + \sqrt{\Delta_3}}{2a_{22}b_{23}}, \tag{12}$$

here

$$\Delta_3 = (b_{24}b_{23} + a_{22}b_{22})^2 - 4a_{22}b_{23}(b_{24}b_{22} - b_{21}).$$

the variable x_1 denotes the unique positive root of the following equation:

$$\frac{b_{11}}{b_{12} + b_{13}x} - b_{14} - a_{11}x + a_{12}y_1 = 0. \tag{13}$$

Remark 2.1. The observation of four equilibria indicates that the Allee effect does not exert any discernible impact on the existence of the equilibrium.

3 Main results

The preceding section entailed a discourse on the presence of equilibria. With respect to the stability of those equilibrium points, the following outcomes have been observed:

Theorem 3.1

(1) Assume that

$$\frac{b_{11}}{b_{12}} < b_{14} \tag{14}$$

and

$$\frac{b_{21}}{b_{22}} < b_{24} \tag{15}$$

holds, the global attractivity of the boundary equilibrium $A_1(0,0)$ holds true;

(2) Assume that

$$\frac{b_{11}}{b_{12}} > b_{14} \tag{16}$$

and

$$\frac{b_{21}}{b_{22}} < b_{24} \tag{17}$$

holds, then the boundary equilibrium $A_2(x^*,0)$ is globally attractive;

(3) Assume that

$$\frac{b_{21}}{b_{22}} > b_{24} \tag{18}$$

and

$$\frac{b_{11}}{b_{12}} + a_{12}y^* < b_{14} \tag{19}$$

holds, it can be inferred that the boundary equilibrium denoted as $A_3(0,y^*)$ exhibits global attractivity, where y^* is defined by (10);

(4) Assume that

$$\frac{b_{21}}{b_{22}} > b_{24} \tag{20}$$

and

$$\frac{b_{11}}{b_{12}} + a_{12}y_1 > b_{14} \tag{21}$$

then the system (1) exists a unique positive equilibrium $A_4(x_1,y_1)$, which is globally attractive, where y_1 is defined by (12).

Remark 3.1. Equations (20) and (21) represent the necessary conditions for the presence of a positive equilibrium point. The global attractivity of the positive equilibrium point is guaranteed if and only if (20) and (21) holds.

4 Proof of the main result

The following Lemma is useful in the demonstration of Theorem 3.1.

Lemma 4.1 *Consider the following equation*

$$\frac{dy}{dt} = y\Big(\frac{a}{b+cy} - d - ey\Big)\frac{y}{\beta+y}. \tag{22}$$

Assuming positive constants a, b, c, d, and e, and a non-negative constant β, we can conclude the following:

(a) If the inequality $a > bd$ is satisfied, then the system represented by equation (22) possesses a single positive equilibrium y^* that is globally asymptotically stable.

(b) Conversely, if $a < bd$, then the equilibrium $y = 0$ of the same system is globally asymptotically stable.

Proof. The approach employed to prove Lemma 4.1 has a similarity to that of the proof of Lemma 2.1 as presented in Chen et al.'s work [19]. For the sake of brevity, we shall not analyze into the specifics of these proofs in this context.

Proof of Theorem 4.1.

(1) For enough small $\varepsilon > 0$, without loss of generality, one many assume that

$$\varepsilon < \frac{1}{2}(b_{14} - \frac{b_{11}}{b_{12}}),$$

then

$$\frac{b_{11}}{b_{12}} - b_{14} + \varepsilon < -\varepsilon. \tag{23}$$

holds. Consider the equation

$$\frac{dy}{dt} = y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\Big)\frac{y}{\beta+y}, \tag{24}$$

the global attractiveness of the equilibrium $y = 0$ in system (24) can be deduced from (14) and Lemma 4.1.

$$\lim_{t\to+\infty} y(t) = 0. \tag{25}$$

From (25), there exists a $T_1 > 0$ such that

$$y(t) < \frac{\varepsilon}{a_{12}} \text{ for all } t \geq T_1. \tag{26}$$

Subsequently, it can be deduced from the first equation of system (1) and (23) that

$$\begin{aligned}\frac{dx}{dt} &= x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}y\Big)\\ &\leq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + \varepsilon\Big)\\ &\leq x\Big(\frac{b_{11}}{b_{12}} - b_{14} + \varepsilon\Big)\\ &\leq -\varepsilon x,\end{aligned} \tag{27}$$

consequently

$$x(t) \leq x(T_1)\exp\Big\{-\varepsilon(t-T_1)\Big\} \to 0 \ \text{ as } \ t \to +\infty. \tag{28}$$

(25) and (28) show that $A_1(0,0)$ is globally attractive.

(2) Similarly to the analysis of (24)-(26), under the assumption (17) holds, from the second equation of system (1) it can be inferred that

$$\lim_{t\to+\infty} y(t) = 0. \tag{29}$$

From (29), there exists a $T_2 > 0$ such that for $\varepsilon > 0$ small enough, the following inequality holds.

$$y(t) < \frac{\varepsilon}{a_{12}} \ \text{ for all } \ t \geq T_2. \tag{30}$$

Subsequently, it can be deduced from the first equation of the system (1) that

$$\frac{dx}{dt} \ \leq \ x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + \varepsilon\Big). \tag{31}$$

Consider the equation

$$\frac{du}{dt} \ = \ u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u + \varepsilon\Big). \tag{32}$$

Obviously, condition (16) implies that

$$\frac{b_{11}}{b_{12}} > b_{14} - \varepsilon. \tag{33}$$

It follows from Lemma 4.1 that system (32) admit a unique positive equilibrium $u_1(\varepsilon)$, which is globally attractive, indeed,

$$u_1(\varepsilon) = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \tag{34}$$

here

$$\begin{aligned} A_1 &= a_{11}b_{13}, \\ A_2 &= a_{11}b_{12} + b_{13}b_{14} - b_{13}\varepsilon, \\ A_3 &= b_{12}b_{14} - b_{12}\varepsilon - b_{11}. \end{aligned} \tag{35}$$

Now, from (31) and (32), by using comparison theorem of differential equation, we have

$$\limsup_{t\to+\infty} x(t) \leq u_1(\varepsilon) + \varepsilon. \tag{36}$$

At the same time, it can be inferred from the first equation of system (1) that

$$\frac{dx}{dt} \ \geq \ x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x\Big) \tag{37}$$

Consider the equation

$$\frac{dv}{dt} \ = \ v\Big(\frac{b_{11}}{b_{12}+b_{13}v} - b_{14} - a_{11}v\Big) \tag{38}$$

It follows from Lemma 4.1 that under the assumption (16) holds, system (38) admit a unique positive equilibrium x^*, where

$$x^* = \frac{-(b_{14}b_{13} + a_{11}b_{12}) + \sqrt{\Delta_1}}{2a_{11}b_{13}}, \tag{39}$$

which is globally attractive, here

$$\Delta_1 = (b_{14}b_{13} + a_{11}b_{12})^2 - 4a_{11}b_{13}(b_{14}b_{12} - b_{11}).$$

Now, from (37) and (38), by using comparison theorem of differential equation, we have

$$\liminf_{t\to+\infty} x(t) \geq x^* - \varepsilon. \tag{40}$$

(36) together with (40) leads to

$$x^* - \varepsilon \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq u_1(\varepsilon) + \varepsilon. \tag{41}$$

Since ε are any small positive constants, and noting that $u_1(\varepsilon) \to x^*$ as $\varepsilon \to 0$. setting $\varepsilon \to 0$ in (41) leads to

$$x^* \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq x^*. \tag{42}$$

Therefore,

$$\lim_{t\to+\infty} x(t) = x^*. \tag{43}$$

(29) and (43) show that $A_2(x^*, 0)$ is globally attractive.

(3) For enough small $\varepsilon > 0$, from (19), one could see that

$$\frac{b_{11}}{b_{12}} + a_{12}(y^* + \varepsilon) < b_{14} - \varepsilon. \tag{44}$$

holds. Consider the equation

$$\frac{dy}{dt} = y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\Big)\frac{y}{\beta + y}, \tag{45}$$

it follows from (18) and Lemma 4.1 that the equilibrium $y = y^*$ of system (45) is globally attractive, i. e.,

$$\lim_{t\to+\infty} y(t) = y^*. \tag{46}$$

where

$$y^* = \frac{-(b_{24}b_{23} + a_{22}b_{22}) + \sqrt{\Delta_2}}{2a_{22}b_{23}}, \tag{47}$$

here

$$\Delta_2 = (b_{24}b_{23} + a_{22}b_{22})^2 - 4a_{22}b_{23}(b_{24}b_{22} - b_{21}).$$

Hence, for above $\varepsilon > 0$, there exists a $T_3 > 0$ such that

$$y^* - \varepsilon < y(t) < y^* + \varepsilon \ \text{ for all } \ t \geq T_3. \tag{48}$$

For $t > T_3$, from the first equation of system (1) and (48), we have

$$\begin{aligned} \frac{dx}{dt} &\leq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}(y^*+\varepsilon)\Big) \\ &\leq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} + a_{12}(y^*+\varepsilon)\Big) \\ &\leq -\varepsilon x. \end{aligned} \tag{49}$$

Hence

$$x(t) \leq x(T_3)\exp\Big\{-\varepsilon(t-T_3)\Big\} \to 0 \ \text{ as } \ t \to +\infty. \tag{50}$$

(46) and (50) shows that $A_3(0, y^*)$ is globally attractive.

(4) For enough small $\varepsilon > 0$, from (21) one could see that

$$\frac{b_{11}}{b_{12}} + a_{12}(y_1+\varepsilon) > b_{14} \tag{51}$$

holds. Consider the equation

$$\frac{dy}{dt} = y\Big(\frac{b_{21}}{b_{22}+b_{23}y} - b_{24} - a_{22}y\Big), \tag{52}$$

it follows from (20) and Lemma 4.1 that the positive equilibrium $y = y_1$ of system (52) is globally attractive, i. e.,

$$\lim_{t\to+\infty} y(t) = y_1. \tag{53}$$

where

$$y_1 = \frac{-(b_{24}b_{23} + a_{22}b_{22}) + \sqrt{\Delta_3}}{2a_{22}b_{23}}, \tag{54}$$

here

$$\Delta_3 = (b_{24}b_{23} + a_{22}b_{22})^2 - 4a_{22}b_{23}(b_{24}b_{22} - b_{21}).$$

Hence, for above $\varepsilon > 0$, there exists a $T_4 > 0$ such that

$$y_1 - \varepsilon < y(t) < y_1 + \varepsilon \ \text{ for all } \ t \geq T_4. \tag{55}$$

For $t > T_4$, from the first equation of system (1) and (55), we have

$$\frac{dx}{dt} \leq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}(y_1+\varepsilon)\Big). \tag{56}$$

Now let's consider the equation

$$\frac{du}{dt} = u\Big(\frac{b_{11}}{b_{12}+b_{13}u} - b_{14} - a_{11}u + a_{12}(y_1+\varepsilon)\Big). \tag{57}$$

Noting that from (51) we have

$$\frac{b_{11}}{b_{12}} > b_{14} - a_{12}(y_1+\varepsilon). \tag{58}$$

Therefore, it can be deduced from Lemma 4.1 that the system represented by equation (58) possesses a globally attractive positive equilibrium denoted as $u^*(\varepsilon)$. Indeed,

$$u^*(\varepsilon) = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}, \tag{59}$$

here

$$\begin{aligned} B_1 &= a_{11}b_{13}, \\ B_2 &= a_{11}b_{12} + b_{13}b_{14} - b_{13}a_{12}(y_1+\varepsilon), \\ B_3 &= b_{12}b_{14} - b_{12}a_{12}(y_1+\varepsilon) - b_{11}. \end{aligned} \tag{60}$$

Now, from (56) and (57), by using comparison theorem of differential equation, we have

$$\limsup_{t\to+\infty} x(t) \leq u^*(\varepsilon) + \varepsilon. \tag{61}$$

On the other hand, for enough small $\varepsilon > 0$, from (**??**) one could see that

$$\frac{b_{11}}{b_{12}} + a_{12}(y_1-\varepsilon) > b_{14} \tag{62}$$

holds. For $t > T_4$, again, by utilizing the first equation of system (1) in conjunction with (55), it follows that

$$\frac{dx}{dt} \geq x\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}x + a_{12}(y_1-\varepsilon)\Big). \tag{63}$$

Consider the equation

$$\frac{dv}{dt} = v\Big(\frac{b_{11}}{b_{12}+b_{13}x} - b_{14} - a_{11}v + a_{12}(y_1-\varepsilon)\Big). \tag{64}$$

Noting that from (62) we have

$$\frac{b_{11}}{b_{12}} > b_{14} - a_{12}(y_1-\varepsilon). \tag{65}$$

Therefore, it can be deduced from Lemma 4.1 that the system represented by equation (64) possesses a globally attractive positive equilibrium, denoted as $v^*(\varepsilon)$. Indeed,

$$v^*(\varepsilon) = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1}, \tag{66}$$

here

$$\begin{aligned} C_1 &= a_{11}b_{13}, \\ C_2 &= a_{11}b_{12} + b_{13}b_{14} - b_{13}a_{12}(y_1 - \varepsilon), \\ C_3 &= b_{12}b_{14} - b_{12}a_{12}(y_1 - \varepsilon) - b_{11}. \end{aligned} \quad (67)$$

Now, from (63) and (64), by using comparison theorem of differential equation, we have

$$\liminf_{t\to+\infty} x(t) \geq v^*(\varepsilon) - \varepsilon. \quad (68)$$

(61) together with (68) leads to

$$v^*(\varepsilon) - \varepsilon \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq u^*(\varepsilon) + \varepsilon. \quad (69)$$

Since ε are any small positive constants, and noting that $u^*(\varepsilon) \to x_1$ as $\varepsilon \to 0$, $v^*(\varepsilon) \to x_1$ as $\varepsilon \to 0$, setting $\varepsilon \to 0$ in (69) leads to

$$x_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq x_1. \quad (70)$$

Therefore,

$$\lim_{t\to+\infty} x(t) = x_1. \quad (71)$$

(53) and (71) provide evidence that the equilibrium $A_4(x_1, y_1)$ exhibits global attractivity. The demonstration of Theorem 4.1 has been concluded.

5 Numeric simulations

As was shown in Section 2, 3 and 4, it has been determined that the Allee effect does not exert any impact on the presence or stability of the equilibria. It seems interesting to give some more insight to the influence of Allee effect, let's consider some numeric simulations. Following we will use Maple 2021 to draw the numeric simulations.

Example 5.1 In system (1), let's choose $b_{13} = b_{14} = a_{11} = a_{12} = b_{24} = a_{22} = b_{23} = 1$.

(A) Take $b_{11} = b_{21} = 1, b_{12} = b_{22} = 2$, than the global asymptotic stability of the boundary equilibrium $A_1(0, 0)$ can be deduced from Theorem 4.1.

Figures 1 and 2 present numerical simulations of $x(t)$ and $y(t)$, respectively, for values of β equal to 0, 2, 5, and 10. Figures 1 and 2 demonstrate that as the Allee effect of the second species increases, both species x and y require a longer time to dying out. Such a phenomenon is quite different to the knowledge about the Allee effect, since generally speaking, the species suffer to Allee effect become more endangered and the chance for it to be driven to extinction is increasing. However, numeric simulations shows that it is possible for human being to take some suitable method to avoid the extinction of the species. In a sense, it seems that the Allee effect is an effective way to slow down the rate of population extinction.

(b) Take $b_{11} = b_{21} = 2, b_{12} = b_{22} = 1$. According to Theorem 4.1, it can be deduced that the positive equilibrium $A_4(0.6364, 0.4142)$ exhibits global asymptotic stability. Figures 3 and 4 demonstrate that as the Allee effect of the second species increases, both species x and y require a longer time to reach their final density.

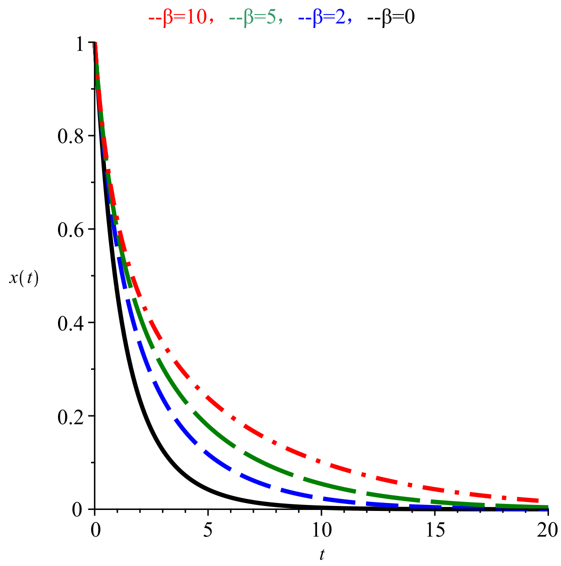

Figure 1: The dynamic behaviors exhibited by the first species in Case (A), the initial condition $(x(0), y(0)) = (1, 1)$, $\beta = 0, 2, 5$ and 10, respectively.

6 Summary and discussion

Our research shows that: Allee effect will not affect the existence and stability of the equilibrium point of the system, but with the increase of the Allee effect, the solution of the system needs more time to tend to the boundary equilibrium point or positive equilibrium point. That is, it takes more time to go extinct or to stabilize population densities.

As we all know, when the amount of species is small or the generations are distinct, it is more appropriate to use the difference equation to describe the dynamic behavior of the population. Recently, scholars such as Zhou et al [25] discussed the discrete amensalism system with Merdan-type Allee effect. Their research shows that, the Allee effect will substantive change the dynamic behaviors of the system, and the system will produce various bifurcation phenomena. An interesting question: what about the dynamic behavior of the discrete system correspond-

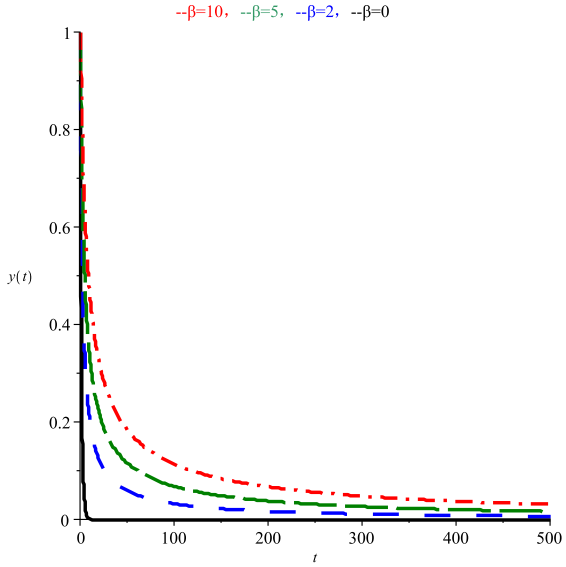

Figure 2: The dynamic behaviors exhibited by the second species in Case (A), the initial condition $(x(0), y(0)) = (1, 1)$, corresponding to $\beta = 0, 2, 5$ and 10, respectively.

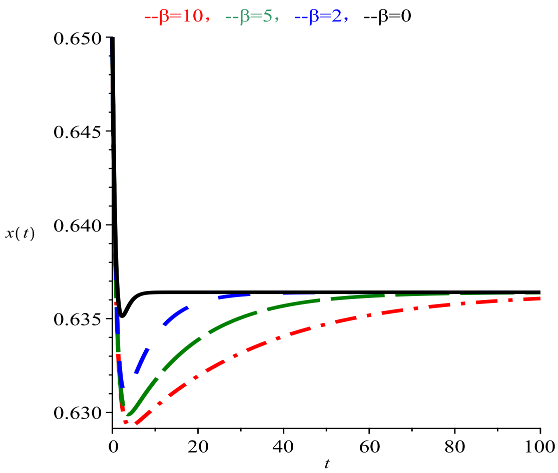

Figure 3: The dynamic behaviors exhibited by the first species in Case (B), the initial condition $(x(0), y(0)) = (1, 1)$, corresponding to $\beta = 0, 2, 5$ and 10, respectively.

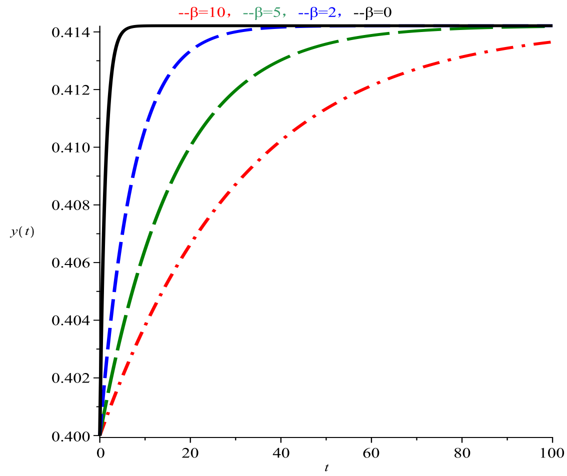

Figure 4: The dynamic behaviors exhibited by the second species in Case (B), the initial condition $(x(0), y(0)) = (1, 1)$, corresponding to $\beta = 0, 2, 5$ and 10, respectively.

ing to the system (1.1)? We will conduct further research in subsequent articles.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.