Fuzzy System Reliability Analysis for Kumaraswamy Distribution: Bayesian and Non-Bayesian Estimation with Simulation and an Application on Cancer Data Set

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Abstract: - This paper proposes the fuzzy Bayesian (FB) estimation to get the best estimate of the unknown parameters of a two-parameter Kumaraswamy distribution from a frequentist point of view. These estimations of parameters are employed to estimate the fuzzy reliability function of the Kumaraswamy distribution and to select the best estimate of the parameters and fuzzy reliability function. To achieve this goal we investigate the efficiency of seven classical estimators and compare them with FB proposed estimation. Monte Carlo simulations and cancer data set applications are performed to compare the performances of the estimators for both small and large samples. Tierney and Kadane approximation is used to obtain FB estimates of traditional and fuzzy reliability for the Kumaraswamy distribution. The results showed that the fuzziness is better than the reality for all sample sizes and the fuzzy reliability at the estimates of the FB proposed estimated is better than other estimators, it gives the lowest Bias and root mean squared error.

Key-Words: - Fuzzy Bayesian estimation, Fuzzy information system, Kumaraswamy Distribution, Maximum likelihood estimator, Maximum product spacing estimator, Monte Carlo simulation, Reliability analysis, R software.

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1. Introduction

In probability and statistics, the Kumaraswamy double bounded distribution is a family of continuous probability distributions defined on the interval $[0,1]$. It is similar to the Beta distribution, but much simpler to use especially in simulation studies since its probability density function, cumulative distribution function and quantile functions can be expressed in closed form. The behavior of both distributions is governed by two shape and two boundary parameters. The relationships between the distributions possible shapes and the values of their shape parameters are qualitatively identical, and both distributions are special cases of McDonald's [1] generalized Beta of the first kind. Most importantly, these two distributions are very flexible and can take approximately the same shapes. This distribution was originally proposed by Poondi Kumaraswamy in 1980 for variables that are lower and upper bounded with a zero-inflation. This was extended to inflations at both extremes $[0,1]$.

Over the last few years, there has been a great interest in studying the Kumaraswamy distribution, and mixing it with other well-known probability models to achieve greater flexibility in modeling several types of real data exhibiting various patterns. Some of these recent developments in the Kumaraswamy distribution are: Barreto-Souza and Lemonte [2] introduced a bivariate Kumaraswamy distribution for which the marginal distributions are univariate Kumaraswamy laws. Ghosh and Nadarajah [3] studied in details, Bayesian inference for Kumaraswamy distribution based on censored samples. Al-Fattah et al. [4] introduced the inverted Kumaraswamy distribution with two positive shape parameters. Aly et al. [5] studied the estimation of the bivariate Kumaraswamy lifetime distribution under progressive Type-I censoring. Sagrillo et al. [6] discussed new modified Kumaraswamy distributions for double bounded hydro-environmental data. Finally, Mohammed et al. [7] introduced bivariate Kumaraswamy distribution based on conditional hazard functions.

Moreover, the maximum likelihood (ML) and method of moments (MM) estimation methods are known to be traditional estimation methods. Even though ML is efficient and has good theoretical features, there is evidence that it does not perform effectively, particularly for small samples. The MM approach is simple to use and frequently produces explicit forms for unknown parameter estimators. In other circumstances, however, the MM does not provide explicit estimators. As a result, different estimating approaches have been offered in the literature as alternatives to traditional methods. The L-moments (LM), ordinary least squares (OLS) and maximum Product Spacing (MPS) estimators are frequently recommended among them. These approaches, in general, do not have good theoretical properties, but they can yield better estimates of unknown parameters in specific instances than the ML and the MM when the data set does not contain extreme observations. These Traditional techniques are ineffective when the data set comprises extreme observations. The parameters must be estimated using a robust estimator. Many papers in several models explore many robust estimators; see e.g. [8-12].

On the other hand, the occurrence of fuzzy random variable makes the combination of randomness and fuzziness more persuasive, since the probability theory can be used to model uncertainty and the fuzzy sets theory can be used to model imprecision. In the fuzzy reliability analysis, the survivors are sometimes unable to be reported accurately due to unforeseen circumstances. For example, the item may not have failed fully during the test, or some failed items may have been recorded incorrectly due to human error. It claims that the survival probability cannot be calculated precisely. As a result, in this case, the survival probability should be treated as a fuzzy real number.

However, one of the most essential and successful strategies for evaluating the work of systems or units is reliability. It is the function that determines the likelihood of a unit or vehicle operating without failure for a given period of time. In its traditional form, many approaches and models in reliability theory assume that all of the parameters of the life-time probability function are crisp. In real-world applications, it is required to generalize traditional real-number statistical estimation methods to fuzzy numbers. This is because, due to flaws in experience, personal judgment, estimation, or unanticipated conditions, the parameters of a probability distribution can occasionally be inaccurately reported. The parameters in the life distributions are then ambiguous. As a result, dealing with the function of traditional dependability may prove problematic for the system of reliability. As a result, we can deal with a broader word than the standard concept of dependability. Zadeh [13] introduced fuzzy logic in 1968, when he used the phrase “fuzzy variables” to describe approximate or erroneous linguistic expressions and language. This is the first book to lay the groundwork for the theory of fuzzy sets. A fuzzy set is a collection of objects or elements with varying degrees of membership. The function of membership to each object in the set distinguishes them. The degree of membership is usually somewhere between zero and one.

In recent years, numerous papers on generalization of classical statistical methods to analysis of fuzzy data have appeared in the literature. Wu [14] calculated fuzzy dependability in the fuzzy environment using the Bayes approach, assuming a fuzzy treatment of fuzzy variables with preceding fuzzy distributions. By incorporating the notion of resolution identity and determining the degree of membership to any Bayes estimate of reliability, the classic Bayes estimation method was applied to develop the fuzzy estimator of reliability. In 2006, Huang and Zuo [15] looked on the basis reliability of fuzzy life data. By assuming a new approach to identify the function of membership estimation and the reliability function of multi-parameter life distributions, the Bayes method was used to estimate fuzzy reliability based on the size of a small sample.

Pak [16] used the Lindley distribution with one parameter when the data was available in a fuzzy data format, using ML estimation and Bayes estimation by EM-algorithm to determine the MLE of the parameter and establish confidence limits using the maximum potential estimator's asymptotic normality. The researcher determined from the Monte Carlo investigation that Bayes estimates based on prior non-informative as well as maximum likelihood estimates provided identical estimation outcomes. The Bayes estimate has the lowest average error squares in the case of previous information.

The main objective of this paper is to propose the fuzzy Bayesian (FB) estimation method to get the best estimate of the unknown two-parameter Kumaraswamy distribution and reliability function. To achieve this goal we investigate the efficiency of seven classical estimators and compare them with FB proposed estimation. Simulations are used to compare the performance as it is not possible to compare all estimators theoretically and also to determine which method is more efficient according to the Bias and root mean square error (RMSE). The uniqueness of this study comes from the fact that thus far no attempt has been made to compare all these estimators for the two-parameter Kumaraswamy distribution.

The rest of this paper is organized as follows: After this introduction, Section 2 presents the Kumaraswamy distribution, definition and properties. While Section 3, provides fuzzy systems, basic definitions and fuzzy reliability analysis. In Section 4, we discuss the seven classical point estimation methods for the unknown parameters. The fuzzy Bayesian estimation analyses are provided in Section 5. A Monte Carlo simulation study and caner real data application are presented in Section 6, which provides a comparison of all estimation procedures developed in this paper. Finally, conclusions appear in Section 7.

2. The Kumaraswamy Distribution: Definition and Properties

Kumaraswamy [17] introduced a two parameter absolutely continuous distribution which compares extremely favorably, in terms of simplicity, with the beta distribution. In its general form, the probability density function of the continuous part of the distribution Kumaraswamy introduced in his 1980 article can be written as;

$$f_Z(z) = \frac{1}{(d-c)}\theta\beta\left(\frac{z-c}{d-c}\right)^{\theta-1}\left[1-\left(\frac{z-c}{d-c}\right)^{\theta}\right]^{\beta-1} \quad ; \; c < z < d \quad (1)$$

with shape parameters $\theta > 0$ and $\beta > 0$, and boundary parameters c and d. The general form of the distribution will be denoted by $K(\theta, \beta, c, d)$. Making the transformation $X = \frac{Z-c}{d-c}$ and using the change of variable theorem, we obtain the standard form of the Kumaraswamy density function. The cumulative distribution function (CDF) and the corresponding probability density function (PDF) can be expressed as;

$$F(x; \theta, \beta) = 1 - \left(1 - x^{\theta}\right)^{\beta}, \quad 0 < x < 1 \; ; \; \theta, \beta > 0 \quad (2)$$

$$f(x; \theta, \beta) = \theta\beta x^{\theta-1}\left(1 - x^{\theta}\right)^{\beta-1}, \quad 0 < x < 1 \; ; \; \theta, \beta > 0 \quad (3)$$

Which will be denoted by $K(\theta, \beta) \equiv K(\theta, \beta, 0, 1)$. In what follows the standard form of the distribution will be employed unless otherwise indicated.

For simplicity, we denote Kumaraswamy distribution with two positive parameters θ and β as $K(\theta, \beta)$. Based on varying values of θ and β, there are similar shape properties between Kumaraswamy distribution and Beta distribution. However, the former is superior to the latter in some respects: there are not any special functions involved in $K(\theta, \beta)$ and its quantile function; the generation of random variables is simple, as L-moments and moments of order statistics for $K(\theta, \beta)$ have simple formulars. For the PDF of $K(\theta, \beta)$, as shown on the Figure 1 , when $\theta > 1$ and $\beta > 1$, it is unimodal; when $\theta > 1$ and $\beta \leq 1$, it is increasing; when $\theta \leq 1$ and $\beta > 1$, it is decreasing; when $\theta < 1$ and $\beta < 1$, it is uniantimodal; when $\theta = \beta = 1$, it is constant. The CDF of $K(\theta, \beta)$ as shown on the Figure 2, has an explicit expression, while the CDF of the Beta distribution appears in an integral form. Therefore, Kumaraswamy distribution is considered as a substitutive model for Beta distribution in practical terms.

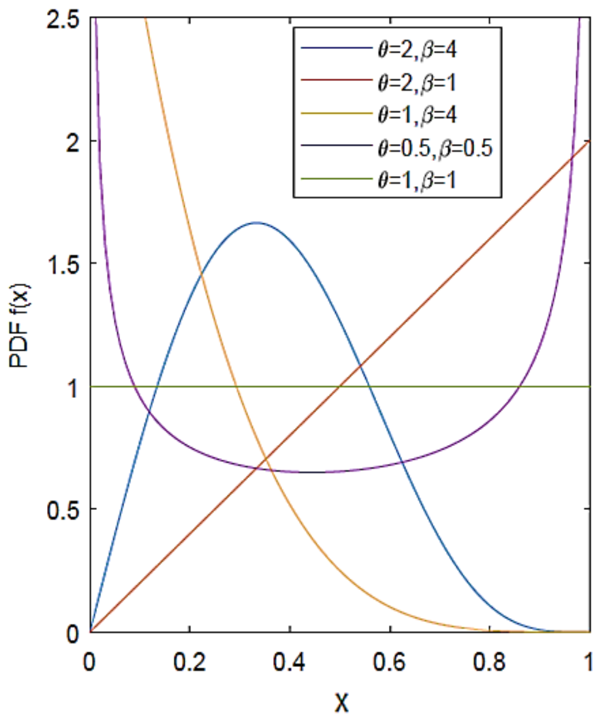

Fig. 1 The PDF plots of the Kumaraswamy distribution for different parameter values.

From Eq. (2), it immediately follows that the quantile function $F^{-1}(P)$ is also available in closed-form:

$$Q(P; \theta, \beta) = \left(1 - (1-P)^{\frac{1}{\beta}}\right)^{\frac{1}{\theta}}, 0 < P < 1 \quad (4)$$

In particular, the median of the Kumaraswamy distribution can be written as;

$$md(x) = \omega = (1 - 0.5^{\frac{1}{\beta}})^{\frac{1}{\theta}} \quad (5)$$

Therefore, the reliability and hazard functions at an arbitrary time t for the Kumaraswamy distribution are given by;

$$R(t) = \left(1 - x^{\theta}\right)^{\beta}, \qquad (6)$$

and:

$$H(t) = \frac{\theta\beta t^{\theta-1}}{1-t^{\theta}}, \quad 0 < t < 1 \qquad (7)$$

respectively.

If the random variable X is distributed $K(\theta,\beta)$ its moments around zero can be expressed as;

$$\mu_r'(X) = \beta B\left(1 + \frac{r}{\theta}, \beta\right) \qquad (8)$$

where $B(\theta,\beta) = \int_0^1 s^{\theta-1}(1-s)^{\beta-1}ds = \frac{\Gamma(\theta)\Gamma(\beta)}{\Gamma(\theta+\beta)}$ is the Beta function and $\Gamma(v) = \int_0^{\infty} t^{v-1}e^{-t}dt$ is the Gamma function. Thus, the expectation and variance of X are;

$$E(X) = \mu = \mu_1'(X) = \beta B\left(1 + \frac{1}{\theta}, \beta\right),$$

$$Var(X) = \mu_2 = \mu_2'(X) - \mu^2$$
$$= \beta B\left(1 + \frac{2}{\theta}, \beta\right) - \left[\beta B\left(1 + \frac{1}{\theta}, \beta\right)\right]^2.$$

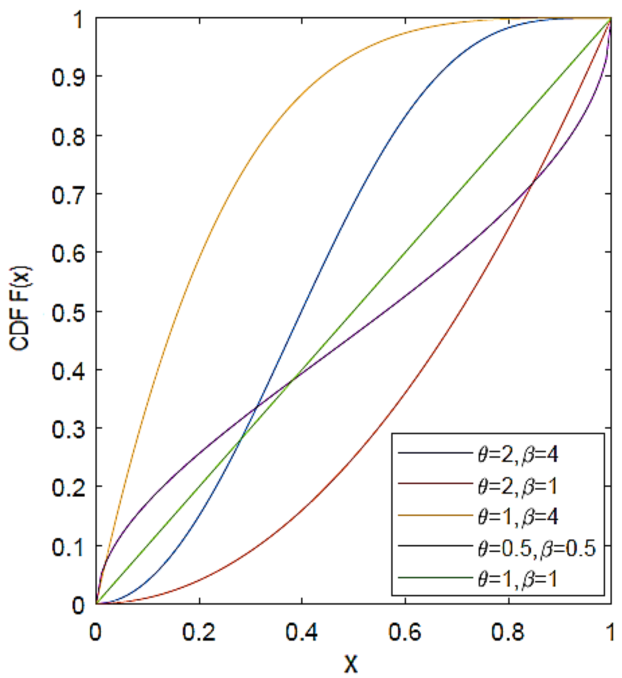

Fig. 2 The CDF plots of the Kumaraswamy distribution for different parameter values.

Many natural phenomena with lower and upper boundaries, such as individual heights, test scores, atmospheric temperatures, hydrological data, economic data (such as unemployment data), etc., are suitable to this distribution. Despite its adaptability, this distribution has received little statistical attention. However, recently, the genesis and the basic properties of the Kumaraswamy distribution were studied by Jones [18]. He noted that while this distribution has many of the same properties as the Beta distribution, it has some advantages in terms of tractability: its quantile function is simple and does not require any special functions, random variate generation is simple, L-moments and moments of order statistics have simple formulae and statistical meanings for the parameters, and so on.

Moreover, this distribution has a close relation with Beta and generalized Beta (first kind) listed below:

- If $X \sim Beta(1,\beta)$ then $X \sim K(1,\beta)$.
- If X $\sim Beta(\theta, 1)$ then $X \sim K(\theta, 1)$.
- If $X \sim K(\theta,\beta)$ then $X \sim GB\,1(\theta, 1, 1, \beta)$.

where $GB1$ stands for the generalized Beta distribution of the first kind.

Nadarajah [19] stated that the Kumaraswamy distribution can capture the shape of several well-known distributions such as the uniform distribution, triangular distribution, or practically any single modal distribution depending on the choice of the two shape parameters. The Kumaraswamy distribution is a specific instance of a three-parameter Beta distribution.

3. Fuzzy Systems

In the following, at first, we consider the fundamental notation and some basic definitions of fuzzy set theory which will be frequently used in this paper. Consider an experiment characterized by a probability space $X = (\Omega, \Psi, \Phi_{\alpha})$, where (Ω, Ψ) is a Borel measurable space and Φ_{λ} belongs to a specified family of probability measures $(\Phi_{\alpha}, \alpha \in \Theta)$ on (Ω, Ψ)Assume that the observer cannot distinguish or transmit with exactness the outcome in the performance of , but that rather the available observation may be described in terms of fuzzy information which is defined as follows, For details on this topic, see [20].

3.1 Basic Definitions

Definition 3.1: A fuzzy event x on X, characterized by a Borel measurable membership function $\mu_{\tilde{x}}(x)$ from X to [0,1], where $\mu_{\tilde{x}}(x)$ represents the "grade of membership" of x to $\tilde{x}$, is called fuzzy information associated with the experiment X. The set consisting of all observable events from the experiment X determines a fuzzy information system associated with it, which is defined as follows.

Definition 3.2: A fuzzy information system (henceforth, in short FIS) $\tilde{X}$ associated with the experiment X is a fuzzy partition with fuzzy events on X, that is a finite set of fuzzy events on X satisfying the orthogonality condition

$$\sum_{\tilde{x}\in\tilde{X}} (x) = 1,$$

for all $x \in X$. Alternatively, according to Zadeh (1968), given the experiment $X = (\Omega, \Psi, \Phi_\alpha), \alpha \in \Theta$ and a FIS $\tilde{X}$ associated with it, each probability measure Φ_α on (Ω, Ψ), induces a probability measure on $\tilde{X}$ defined as follows.

Definition 3.3: The probability distribution on $\tilde{X}$ induced by Φ_α is the mapping Φ from X to [0,1] such that;

$$\Phi(\tilde{x}) = \int_X \mu_{\tilde{x}}(x) d\Phi_\alpha(x),$$

for $\tilde{x} \in \tilde{X}$. In particular, the conditional density of a continuous random variable U with PDF $g(u)$ given the fuzzy event $\tilde{A}$ can be defined as;

$$g(u \mid \tilde{A}) = \frac{\mu_{\tilde{A}}(u) g(u)}{\int \mu_{\tilde{A}}(u) g(u) du}.$$

For more details about the membership functions and probability measures of fuzzy sets, one can refer to Pak et al. [21].

Definition 3.4: A fuzzy number is a subset, denoted by $\tilde{x}$, of the set of real numbers (denoted by $\mathbb{R}$) and is characterized by the so called membership function $\mu_{\tilde{x}}$, satisfying the following constraints:

(i) $\mu_{\tilde{x}}: \mathbb{R} \Longrightarrow [0,1]$ is Borel measurable.

(ii) For every $x_0 \in \mathbb{R}, \mu_{\tilde{x}}(x_0) = 1$.

(iii) The usual λ-cuts $(0 < \lambda \leq 1)$, defined as $B_\lambda(\tilde{x}) = \{x \in \mathbb{R}: \mu_{\tilde{x}}(x) \geq \lambda\}$, are all closed interval, i.e., $\lambda(\tilde{x}) = [\theta_\lambda, \beta_\lambda]$, $\forall \lambda \in (0,1]$.

Some widely known examples of membership functions to characterize fuzzy numbers are triangular and trapezoidal fuzzy numbers. For example, triangular fuzzy number is defined as $\tilde{x} = (\xi, \omega, \Upsilon)$ with the corresponding membership function;

$$\mu_{\tilde{x}} = \begin{cases} \frac{x-\xi}{\omega-\xi}, & \xi \leq x \leq \omega \\ \frac{\Upsilon-x}{\Upsilon-\omega}, & \omega \leq x \leq \Upsilon \\ 0, & otherwise \end{cases}$$

Similarly, a trapezoidal fuzzy number can be defined as $\tilde{x} = (\xi, \omega, \Upsilon, \alpha)$ with the corresponding membership function.

$$\mu_{\tilde{x}} = \begin{cases} \frac{x-\xi}{\omega-\xi}, & \xi \leq x \leq \omega \\ 1, & \omega \leq x \leq \Upsilon \\ \frac{\alpha-x}{\alpha-\Upsilon}, & \delta \leq x \leq \alpha \\ 0, & otherwise \end{cases}$$

Let us again revisit the example as mentioned earlier in the context of life length of an electric bulb.

3.2 Fuzzy Reliability Analysis

Reliability was defined as the probability that the unit or device will remain valid after a period of time (t) on use. If T is a continuous random variable,$T > 0$, the reliability function is:

$$R_T(t) = P(T \geq t) = \int_t^\infty f_T(x) dx$$

$$= 1 - \int_0^t f_T(x) dx = 1 - F_T(t)$$

Its properties are:

- $R(0) = p(T < 0) = 1.$
- $R(\infty) = 1.$
- $0 \leq R(t) \leq 1.$
- If $t_1 < t_2$ then, $R(t_1) \geq R(t_2)$

Now, we can say that the fuzzy reliability represents the probability of the unit performing the work required. It is with varying degrees of success for a specified period of time under normal conditions and symbolized by $\tilde{R}$, which is a function in the fuzzy set $\tilde{A}$:

$\tilde{R} = \mu_{\tilde{A}_i}(R). R$, While $R(t) = \int_t^\infty f(t)\, dt$ then; $\tilde{R} = \mu_{\tilde{A}_i}(R) \cdot \int_t^\infty f(t) dt, R(t) = \left(1 - x^\theta\right)^\beta$

We will assume that the values of the random variable $\tilde{T}$ are fuzzy number, $\tilde{t} = \{[0, \infty), \mu_{\tilde{t}_i}\}; \tilde{k}t\tilde{t} = t \in T$ So; the fuzziness is a real triangular fuzzy number, and:

$$\mu_{\tilde{k}}(k) = \frac{k - k_{max}}{1 - k_{max}} \begin{cases} \frac{k - k_{min}}{1 - k_{min}} & k \in (k_{min}, 1) \\ k \in (1, k_{max}) & \\ 0 & otherwise \end{cases}$$

where; $0 < k_{min} \leq 1 \leq k_{max}$.

If the random variable T has a traditional fractional distribution with $K(\theta, \beta)$, the corresponding $\tilde{T}$ of random variable will have $K(\tilde{\theta}, \tilde{\beta})$ variable. For each $t \in [0, \infty)$, the cumulative fuzzy distribution function is;

$$\tilde{F}(\tilde{t}) = 1 - \left(1 - \tilde{k}t^{\hat{\theta}}\right)^{\hat{\beta}} \; ; \; \tilde{t} > 0 \quad (9)$$

Then, the fuzzy reliability function is:

$$\tilde{R}(t) = \left(1 - \tilde{k}t^{\hat{\theta}}\right)^{\hat{\beta}} \quad (10)$$

4. Classical Point Estimation Methods

In this section, we describe some methods for estimating the parameters θ and β of the Kumaraswamy distribution. We assume throughout that $x = (x_1, \ldots, x_n)$ is a random sample of size n from the Kumaraswamy distribution with both parameters θ and β unknown. We let $x_{1:n} < \cdots < x_{n:n}$ denote the associated order statistics. The parameters estimations of the Kumaraswamy distribution is investigated using seven estimations methods, namely, the maximum likelihood (ML), maximum product spacing (MPS), method of moments (MM), probability weighted moment (PWM), ordinary least squares (OLS), weighted least squares (WLS) and Cramér–von Mises (CVM) will be discussed in details.

4. 1 Maximum Likelihood Estimation

The method of maximum likelihood (ML) estimation is the most frequently used method of parameter estimation. Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. For the random sample $x = (x_1, \ldots, x_n)$, the log-likelihood function can be written as:

$$l(\theta, \beta \mid x) = n\log(\theta\beta) + (\theta - 1)\sum_{i=1}^{n} \log(x_i) + (\beta - 1)\sum_{i=1}^{n} \log\left(1 - x_i^{\theta}\right). \quad (11)$$

The ML estimations, $\hat{\theta}_{ML}$ and $\hat{\beta}_{ML}$, can be obtained by maximizing Eq. (11) with respect to θ and β. The partial derivatives, $U_\theta = \frac{\partial}{\partial\theta} l(\theta, \beta \mid x)$ and $U_\beta = \frac{\partial}{\partial\beta} l(\theta, \beta \mid x)$, are;

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i) - (\beta - 1)\sum_{i=1}^{n} \frac{x_i^{\theta}\log(x_i)}{1 - x_i^{\theta}}.$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left(1 - x_i^{\theta}\right)$$

Setting these to zero, we obtain the ML estimation $\hat{\theta}_{\mathrm{ML}}$ is the solution of:

$$\frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i) - [\beta - 1]\sum_{i=1}^{n} \frac{x_i^{\theta}\log(x_i)}{1 - x_i^{\theta}} = 0 \quad (12)$$

The ML estimation $\hat{\beta}_{ML}$, that is:

$$\hat{\beta}_{ML} = -\frac{n}{\sum_{i=1}^{n} \log\left(1 - x_i^{\theta}\right)}. \quad (13)$$

In determining the estimation in equations (12) and (13) of Kumaraswamy distribution by the ML estimation method which cannot be solved analytically, this can be solved by numerical iteration method that is Newton Raphson's method.

➢ **Newton Raphson Algorithm**

The steps in Newton Raphson's (NR) algorithm are:

1. Determining the starting value $\hat{\theta}$ and $\hat{\beta}$.
2. Determining the first derivative and the second derivative of $z = f(\theta, \beta)$ i.e. :

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) + (\beta - 1)\sum_{n=1}^{n} \ln\left(1 - x_i^{\theta}\right)$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1 - x_i^{\theta}\right)$$

$$\frac{\partial^2 \ln L}{\partial^2\theta} = -\frac{n}{\theta^2} + (\beta - 1)\sum_{i=1}^{n} \frac{-xi^{\theta}\ln^2(x_i)}{\left(1 - x_i^{\theta}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial^2 \beta} = -\frac{n}{\beta^2}$$

3. Defining g_η as the first gradient vector and derivative vector of its parameters:

$$g_\eta = \begin{bmatrix} \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) + (\beta - 1)\sum_{i=1}^{n} \ln\left(1 - x_i^{\theta}\right) \\ \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1 - x_i^{\theta}\right) \end{bmatrix}$$

4. Next defining the Hessian matrix H_η where the Hessian Matrix or second derivative matrix to its parameters, denoted by H_η are:

$$H_\eta = \begin{bmatrix} -\frac{n}{\theta^2} + (\beta - 1)\sum_{i=1}^{n} \frac{-x_i^{\theta}\ln^2(x_i)}{\left(1 - x_i^{\theta}\right)^2} & \frac{n}{\beta} + \sum_{i=1}^{n} \frac{-\theta x_i^{\theta - 1}}{1 - xi^{\theta}} \\ (\beta - 1)\sum_{i=1}^{n} \frac{-x_i^{\theta}\ln^2(x_i)}{\left(1 - x_i^{\theta}\right)^2} & -\frac{n}{\beta^2} \end{bmatrix}$$

5. Iteration will stop when $\left\| \begin{matrix} \theta_{\eta+1} - \theta_\eta \\ \beta_{\eta+1} - \beta_\eta \end{matrix} \right\| < \varepsilon$, where ε is the specified error limit.

4.2 Maximum Product of Spacing Estimation

In statistics, maximum product spacing (MPS) estimator is a method for estimating the parameters of univariate statistical models. The method requires maximization of the geometric mean of spacings in the data, which are the differences between the values of the cumulative distribution function at neighbouring data points. One of the most common methods for estimating the parameters of a distribution from data, the method of ML estimates, can break down in various cases, such as involving certain mixtures of continuous distributions. In these cases the method of MPS method may be successful. The MPS method chooses the parameter values that make the observed data as uniform as possible, according to a specific quantitative measure of uniformity, see [22].

The uniform spacing of the random sample $x = (x_1, \ldots, x_n)$ can be defined as:

$D_i(\theta,\beta) = F(x_{i:n} \mid \theta,\beta) - F(x_{i-1:n} \mid \theta,\beta)$
$F(x_{0:n} \mid \theta,\beta) = 0,\ F(x_{n+1:n} \mid \theta,\beta) = 1$
$;i = 1, \dots, n,$

Clearly,
$D_0(\alpha,\beta) + D_1(\alpha,\beta) + \cdots + D_{n+1}(\alpha,\beta) = 1.$

Following Cheng and Amin [22], the MPS estimation, $\hat{\theta}_{\text{MPS}}$ and $\hat{\beta}_{\text{MPS}}$, are the values of θ and β maximizing the geometric mean of the spacing:

$$G(\theta,\beta) = \left[\prod_{i=1}^{n+1} D_i(\theta,\beta)\right]^{\frac{1}{n+1}}$$

or, equivalently, maximizing the function:

$$H(\theta,\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\theta,\beta)$$

The estimators, $\hat{\theta}_{\text{MPS}}$ and $\hat{\beta}_{\text{MPS}}$, can also be obtained by solving:

$$\frac{\partial}{\partial\theta}H(\theta,\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{D_i(\theta,\beta)} \left[\Delta_1(x_{i:n} \mid \theta,\beta) - \Delta_1(x_{i-1:n} \mid \theta,\beta)\right] = 0$$

$$\frac{\partial}{\partial\beta}H(\theta,\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{D_i(\theta,\beta)} \left[\Delta_2(x_{i:n} \mid \theta,\beta) - \Delta_2(x_{i-1:n} \mid \theta,\beta)\right] = 0$$

where;

$$\Delta_1(x_{i:n} \mid \theta,\beta) = -\left(1 - x_{i:n}^{\theta}\right)^{\beta}\log\left(1 - x_{i:n}^{\theta}\right) \quad (14)$$

and;

$$\Delta_2(x_{i:n} \mid \theta,\beta) = \beta\left(1 - x_{i:n}^{\theta}\right)^{\beta-1}x_{i:n}^{\theta}\log x_{i:n} \quad (15)$$

Cheng and Amin [22] showed that maximizing H as a method of parameter estimation is as efficient as ML estimation and that the MPS estimators are consistent under more general conditions than ML estimators.

4.3 Method of Moments Estimation

The method of moments (MM) estimation can be obtained by equating the mean and variance of Eq. (3) to their sample counterparts, that is;

$$E(X \mid \theta,\beta) = \theta\beta\left(1 + \frac{1}{\theta}\beta\right) = \bar{x},$$

$$\text{Var}(X \mid \theta,\beta) = \theta\beta\left(1 + \frac{2}{\theta}\beta\right) - \left[\theta\beta\left(1 + \frac{1}{\theta}\beta\right)\right]^2 = s^2,$$

where $\bar{x}$ and s^2 are the sample mean and sample variance, respectively.

4.4 Probability Weighted Moment Estimation

To estimate the parameters of Kumaraswamy distribution (θ,β) we have to find the inverse function of the cumulative distribution function, while the inverse function as is obtained as follows:

$$x = \left[1 - (1-F)^{\frac{1}{\beta}}\right]^{\frac{1}{\theta}}$$

Next to search for probability weighted moment (PWM) estimation of Kumaraswamy distribution by searching for the-r moment is as follows:

$$M_{1,\,s,0} = \int_0^1 x(F)[F(x)]^s df$$
$$= \int_0^1 \left[1 - (1-F)^{\frac{1}{\beta}}\right]^{\frac{1}{\theta}} [\,F(x)]^s df$$

So the obtained PWM form to estimate parameter θ and β from Kumaraswamy distribution is:

$$M_{1,\,s,0} = \beta\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{\Gamma\left(\frac{1}{\theta}+1\right)\Gamma(\beta k+\beta)}{\Gamma\left(\frac{1}{2}+1+\beta k+\beta\right)}$$

After obtaining the-r moment (M_s) then the next step is to determine the estimators for parameters $\hat{\theta}_{\text{PWM}}$ and $\hat{\beta}_{\text{PWM}}$ respectively;

$$\hat{\theta}_{\text{PWM}} = \frac{\left(\frac{\hat{M}_0}{\Gamma\left(\frac{1}{\theta}+1\right)\Gamma(\beta)}\right)\Gamma\left(\frac{1}{\theta}\right)\left(\frac{\Gamma(\beta)\Gamma\left(\frac{1}{\theta}+1+2\,\beta\right)-\Gamma(2\beta)\Gamma\left(\frac{1}{\theta+1+\beta}\right)}{\Gamma\left(\frac{1}{\theta}+1+2\beta\right)}\right)}{\hat{M}_1}$$

$$\hat{\beta}_{\text{PWM}} = \frac{\hat{M}_0\Gamma\left(\frac{1}{\theta}+1+\beta\right)}{\Gamma\left(\frac{1}{\theta}+1\right)\Gamma(\beta)}$$

Moreover, after obtaining the respective estimators from the Kumaraswamy distribution, the next step is to examine the unbiased characteristic for estimator a and b with the definition of unbiased are $E(\hat{\theta}) = \theta$, and $E(\hat{\beta}) = \beta$, respectively.

$$E(\hat{\theta}) = E\left[\frac{\left(\frac{\hat{M}_0}{\Gamma\left(\frac{1}{\theta}+1\right)\Gamma(\beta)}\right)\Gamma\left(\frac{1}{\theta}\right)\left(\frac{\Gamma(\beta)\Gamma\left(\frac{1}{\theta}+1+2\beta\right)-\Gamma(2\beta)\Gamma\left(\frac{1}{\theta}+1+\beta\right)}{\Gamma\left(\frac{1}{\theta}+1+2\beta\right)}\right)}{\hat{M}_1}\right]$$

$$\text{E}(\hat{\theta}) = \theta$$

$$E(\hat{\beta}) = E\left[\frac{\hat{M}_0\Gamma\left(\frac{1}{\theta}+1+\beta\right)}{\Gamma\left(\frac{1}{\theta}+1\right)\Gamma(\beta)}\right]$$

$$E(\hat{\beta}) = \beta$$

Thus, $\hat{\theta}$ and $\hat{\beta}$ are unbiased estimator for θ and β.

4.5 Ordinary Least Squares Estimation

Swain et al. [23] introduced the ordinary least squares estimator (OLS) method is used to estimate the parameters of Beta distribution.

Let $x_{1:n} < \cdots < x_{n:n}$ be the order statistics of a sample from the Kumaraswamy distribution and then the OLS of $\hat{\theta}_{\text{OLS}}$ and $\hat{\beta}_{\text{OLS}}$ can be obtained by minimizing the following function with respect to θ and β:

$$S(\theta,\beta) = \sum_{i=1}^{n}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right]^2,$$

with respect to θ and β, where $F(\cdot)$ is given by Eq. (2). Equivalently, they can be obtained by solving:

$$\sum_{i=1}^{n}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right]\Delta_1(x_{i:n} \mid \theta,\beta) = 0,$$
$$\sum_{i=1}^{n}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right]\Delta_2(x_{i:n} \mid \theta,\beta) = 0,$$

where $\Delta_1(\cdot| \theta,\beta)$ and $\Delta_2(\cdot| \theta,\beta)$ are given by equations (14) and (15), respectively.

4.6 Weighted Least Squares Estimation

Swain et al. [23] introduced the weighted least square (WLS) estimators. We use the WLS procedure for estimating the parameters θ and β of the Kumaraswamy distribution. The WLS, $\hat{\theta}_{\mathrm{WLS}}$ and $\hat{\beta}_{\mathrm{WLS}}$, can be obtained by minimizing the following function;

$$W(\theta,\beta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right]^2$$

These estimators can also be obtained by solving:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right] \Delta_1(x_{i:n} \mid \theta,\beta) = 0$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n} \mid \theta,\beta) - \frac{i}{n+1}\right] \Delta_2(x_{i:n} \mid \theta,\beta) = 0$$

where $\Delta_1(\cdot| \theta,\beta)$ and $\Delta_2(\cdot| \theta,\beta)$ are given by equations (14) and (15), respectively.

4.7 Cramér–von-Mises Estimation

Cramér–von Mises type minimum distance estimators are based on minimizing the distance between the theoretical and empirical cumulative distribution functions. Choi and Bulgren [24] provided empirical evidence that the bias of these estimators is smaller than the bias of other minimum distance estimators. The Cramér–von Mises (CVM) estimator, $\hat{\theta}_{\mathrm{CVM}}$ and $\hat{\beta}_{\mathrm{CVM}}$, are the values of θ and β minimizing the following function;

$$C(\theta,\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n} \mid \theta,\beta) - \frac{2i-1}{2n}\right]^2$$

The estimators can also be obtained by solving:

$$\sum_{i=1}^{n} \left(F(x_{i:n} \mid \theta,\beta) - \frac{2i-1}{2n}\right) \Delta_1(x_{i:n} \mid \theta,\beta) = 0,$$
$$\sum_{i=1}^{n} \left(F(x_{i:n} \mid \theta,\beta) - \frac{2i-1}{2n}\right) \Delta_2(x_{i:n} \mid \theta,\beta) = 0,$$

where $\Delta_1(\cdot| \theta,\beta)$ and $\Delta_2(\cdot| \theta,\beta)$ are given by equations (14) and (15), respectively.

5. Fuzzy Bayesian Estimation

Fuzzy Bayesian (FB) approach has been adopted to enhance the probability updating process with fuzzy evidences by utilizing the conditional probability densities and the membership functions of the evidence's values. This approach has been widely applied in structural reliability to access the safety of the constructed projects, see [14].

This technique can be used to calculate the likelihood probability and posterior probability for a given fuzzy value using the likelihood density function. However, determining the likelihood density function is difficult. The density functions are approximated as a certain sort of distribution, such as Guassian or Weibull, in many of its applications, and the parameters of the approximated distributions are calculated using laboratory tests and statistical methodology. This estimation is complicated and time consuming. Furthermore, once we know the likelihood density function for the continuous valued data, determining the likelihood probability for the fuzzy valued evidence is a waste of time. The likelihood density function for continuous valued evidence should be calculated using the likelihood probability for fuzzy valued data.

The Bayes theorem can be expressed as the conditional distribution of β given x is given as;

$$\Pi(\beta|x) = \frac{f(x|\beta)\, h(\beta)}{\int f(x|\beta)\, h(\beta)\, \partial\beta},$$

where $f(x|\beta)$ is the likelihood function of the distribution, $h(\beta)$ is the prior probability distribution for the parameter β and $\Pi(\beta|x)$ is the posterior probability distribution. The initial stage needs to be done is to determine the prior distribution. Prior Gamma distribution is used as a conjugate prior distribution for β, see [25]. Suppose β random variable with following prior density function;

$$\Pi(\beta) = \frac{\beta^{\delta-1}}{\Gamma(\delta)\vartheta^{\delta}} e^{-\frac{\beta}{\vartheta}},\ \vartheta > 0, \delta > 0$$

or it can be written $h(\beta) \propto \beta^{\delta-1} e^{-\frac{\beta}{\vartheta}}$.

Note that β is Gamma distribution with parameter δ and ϑ. By using the method of moments will be obtained parameter values as follows;

$$\delta = \frac{\bar{X}^2}{\sum_{i=1}^{n} \frac{x_i^2}{n} - \bar{X}^2},$$

and;

$$\vartheta = \frac{\sum_{i=1}^{n} \frac{x_i^2}{n} - \bar{X}^2}{\bar{X}}.$$

In the Bayesian estimation unknown parameter is assumed to behave as random variable with distribution commonly known as prior probability distribution. Here, we consider the following independent gamma priors for all the parameters θ and β given as follows:

- Prior for θ: $\Pi(\theta) \sim \Gamma(\delta,\vartheta)$.
- Prior for β: $\Pi(\beta) \sim \Gamma(\delta,\vartheta)$.

We make no claims that these hyperparameter selections are optimum or uniformly best in all cases. However, we found this to be a fair option in all of the simulations/examples we explored. Of course, there could be more. By combining Eq. (11) with the above set of independent priors, the joint density functions of the data and the parameters θ and β becomes;

$$\Pi(x,\theta,\beta) \propto \left(\theta^{\delta}\beta^{\vartheta} exp(-[\delta\theta+\vartheta\beta])\right)$$
$$\times\, n(\log\theta+\log\beta)+(\theta-1)\sum_{i=1}^{n}\log\int x_i\mu_{\tilde{x}_i}(x)dx$$
$$+(\beta-1)\sum_{i=1}^{n}\log\int\left(1-x_i^{\theta}\right)\mu_{\tilde{x}_i}(x)dx \quad (16)$$

Therefore, the marginal posterior density functions of θ and β respectively given the data (x) can be obtained as;

- $\Pi(\theta \mid x) \propto \int_0^{\infty}\Pi(x,\theta,\beta)\,d\beta.$
- $\Pi(\beta \mid x) \propto \int_0^{\infty}\Pi(x,\theta,\beta)\,d\theta.$

Note that the FB estimate of any function of θ, say $h(\theta)$, under squared error loss function is the posterior mean which is given by;

$$\int_0^{\infty}\Pi(\theta \mid x)h(\theta)\,d\theta \quad (17)$$

and similarly for the other parameter β as well.

However, the Equations (16) and (17) are not available in analytically tractable and closed nice form due to the complex form of the likelihood function. Therefore, we use Tierney and Kadane approximation as well as Markov Chain Monte Carlo (MCMC) technique for computing the FB estimate of θ and β, see [26].

First, we rewrite the expression in Eq. (16) as (for both the parameters θ and β respectively);

$$\int_0^{\infty}\Pi(\theta \mid x)h(\theta)d\theta=\frac{\int_0^{\infty}exp(nF^*(\theta))d\theta}{\int_0^{\infty}exp(nF(\theta))d\theta}, \quad (18)$$

and;

$$\int_0^{\infty}\Pi(\beta \mid x)h(\beta)\,d\beta=\frac{\int_0^{\infty}exp(nF^*(\beta))d\beta}{\int_0^{\infty}exp(nF(\beta))d\beta}, \quad (19)$$

where;

$$F(\theta)=\frac{1}{n}\log\Pi(x,\theta),$$

and;

$$F^*(\theta)=F(\theta)+\frac{1}{n}\log h(\theta).$$

Tierney and Kadane [26] applied Laplaces method to produce an approximation of Eq. (22) as follows:

$$\hat{h}_{BT}(\theta)=\left[\frac{\lambda^*}{\lambda}\right]^{1/2} exp(n[F^*(\bar{\theta}^*)-F(\bar{\theta})]), \quad (20)$$

where $\bar{\theta}^*$ and $\bar{\theta}$ maximize $F^*(\bar{\theta}^*)$ and $F(\bar{\theta})$, respectively, and λ^* and λ are minus of the inverse of the second derivatives of $F^*(\theta)$ and $F(\theta)$ at $\bar{\theta}^*$ and $\bar{\theta}$ respectively.

Similar operation will be assumed for the other parameter β as well. Next, we apply this approximation to obtain the FB estimate of the parameter θ. Setting $h(\theta)=\theta$, we have;

$$F(\theta)=\frac{1}{n}\int_0^{\infty}\{\delta\log\theta-(\delta\theta+\vartheta\beta)+\vartheta\log\beta$$
$$+n(\log\theta+\log\beta)+(\theta-1)\sum_{i=1}^{n}\log\int x_i\mu_{\tilde{x}_i}(x)\,dx$$
$$+(\beta-1)\sum_{i=1}^{n}\log\int\left(1-x_i^{\theta}\right)\mu_{\tilde{x}_i}(x)dx\}\,d\beta \quad (21)$$

and;

$$F^*(\theta)=\frac{1}{n}\int_0^{\infty}\{\delta\log\theta-(\delta\theta+\vartheta\beta)+\vartheta\log\beta$$
$$+n(\log\theta+\log\beta)+(\theta-1)\sum_{i=1}^{n}\log\int x_i\mu_{\tilde{x}_i}(x)dx$$
$$+(\beta-1)\sum_{i=1}^{n}\log\int\left(1-x_i^{\theta}\right)\mu_{\tilde{x}_i}(x)dx\}\,d\beta \quad (22)$$

On substitution of equations (21) and (22) in Eq. (20), one can obtain the FB estimate of θ under squared error loss function (SELF). Similar approach can also be made to obtain the FB estimate of β under SELF.

6 Numerical Studies

6.1 Simulation Study

In this section, simulation study are conducted to compare the performances of the different estimators of unknown parameters (θ,β) and reliability function for Kumaraswamy distribution and to illustrate the effect of the estimation method of the reliability function. Our main objective is to compare the performances of the seven classical estimation methods and the proposed method FB estimates of the unknown parameters and reliability function for various sample sizes and parameter values.

All the computations are performed in an R-programming environment "version 4.1.2". App For simulation purposes, we have considered $n=(50,75,100,150,200)$, $\theta=(0.5,1.5,2,3)$ and $\beta=(0.5,1.5,2.5,3.5)$. For each combination of n,θ and β, we simulated $L=10{,}000$ samples each of size n for each simulation experiment to obtain homogenous in estimating the reliability function of the Kumaraswamy distribution. The inversion method was used. For each n, we have generated random sample from the Kumaraswamy distribution with different parameter values. Then, using the method as proposed by Pak et al. [27], each realization of the generated samples was fuzzified by employing FIS. The estimates of the parameters θ and β for the fuzzy sample were computed using the maximum likelihood method and under the Bayesian paradigm (with independent priors set up).

For initial choices of the parameters (θ, β) required for the ML method, we have taken values that are wide apart from the actual values of the parameters. For computing the FB estimates, we have assumed that both θ and β have independent gamma priors with specific choices of the hyperparameters (described earlier). The generation of a variable follows a uniform distribution $u \sim U\,(0,1)$ using the rand term. Generate fuzzy data following the Kumaraswamy distribution by inverse transformation method using the following formula:

$$t_i = \beta\left(ln\left(\frac{1}{u}\right)\right)^{-\frac{1}{\theta}};\ i = 1,2,\dots\dots,n \qquad (23)$$

The sample is represented by vector t of Kumaraswamy distribution. The t-sample vector is converted to fuzzy using the fuzzy hypothetical information system as in Figure 3, corresponding to the following membership functions:

$$\mu_{\tilde{t}_1}(t) = \begin{cases} 1 & t \le 0.05 \\ \frac{0.25-t}{0.2} & 0.05 \le t \le 0.25 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_2}(t) = \begin{cases} \frac{t-0.05}{0.2} & 0.05 \le t \le 0.25 \\ \frac{0.5-t}{0.25} & 0.25 \le t \le 0.5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_3}(t) = \begin{cases} \frac{t-0.25}{0.25} & 0.25 \le t \le 0.5 \\ \frac{0.75-t}{0.25} & 0.5 \le t \le 0.75 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_4}(t) = \begin{cases} \frac{t-0.5}{0.25} & 0.5 \le t \le 0.75 \\ \frac{1-t}{0.25} & 0.75 \le t \le 1 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_5}(t) = \begin{cases} \frac{t-0.75}{0.25} & 0.75 \le t \le 1 \\ \frac{1.5-t}{0.5} & 1 \le t \le 1.5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_6}(t) = \begin{cases} \frac{t-1}{0.5} & 1 \le t \le 1.5 \\ \frac{2-t}{0.5} & 1.5 \le t \le 2 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_7}(t) = \begin{cases} \frac{t-1.5}{0.5} & 1 \le t \le 1.5 \\ 3-t & 2 \le t \le 3 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{t}_8}(t) = \begin{cases} t-2 & 2 \le t \le 3 \\ 1 & t \ge 3 \\ 0 & otherwise \end{cases}$$

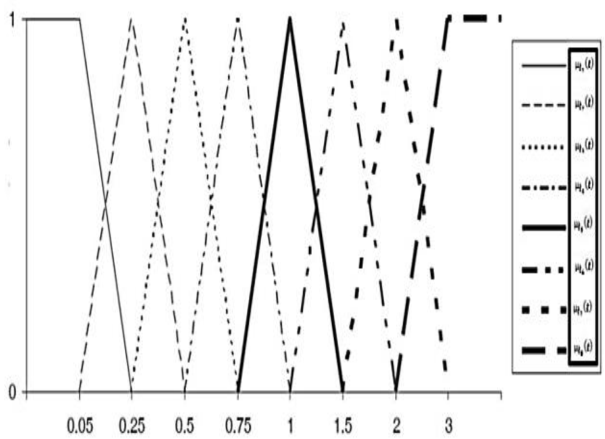

Fig. 3 The FIS hypothetical used in simulating simulation data.

The estimates of θ, β and reliability function for after the creation of the randomized fuzzy values $\tilde{t}_i$ of the CDF function according to the size of the given samples and the default values of initial parameters according to the formula $\tilde{R}(t_i)$, the values of t_i and the initial parameters were computed according to the functions of the $\mu_{\tilde{t}_i}$(t) for each fuzzy unit $\tilde{t}_i$. Then, extract for each $\tilde{R}(t_i)$ and find expectation of $\tilde{R}(t_i)$ as follows:

$$\tilde{R}(t) = \hat{E}\left(\tilde{R}(t_i)/\tilde{x}_i\right) = \frac{1}{L}\sum_{h=1}^{L} R^{(h)}(t) \qquad (24)$$

The parameters θ, β and fuzzy reliability function were estimated by each of the seven methods and FB estimates for each of the simulated samples. We compare the performances of the ML, the MPS, the MM, the PWM, the OLS, the WLS, the CVM and the FB estimates in terms of Bias and root mean squared error (RMSE) criteria defined by:

$\text{Bias}(\hat{\theta}) = \frac{1}{L}\sum_{i=1}^{L}\ (\hat{\theta}_i - \theta)$,
$\text{Bias}(\hat{\beta}) = \frac{1}{L}\sum_{i=1}^{L}\ (\hat{\beta}_i - \beta)$,
$\text{Bias}(\tilde{R}) = \frac{1}{L}\sum_{i=1}^{L}\ (\tilde{R}_i - R)$,
and;
$\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}\ (\hat{\theta}_i - \theta)^2}$,
$\text{RMSE}(\hat{\beta}) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}\ (\hat{\beta}_i - \beta)^2}$,
$\text{RMSE}(\tilde{R}) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}\ (\tilde{R}_i - R)^2}$.

Where $\hat{\theta}_i$, $\hat{\beta}_i$ and $\tilde{R}_i$ is the estimated of θ , β and $R(t)$ respectively, at i^{th} experiment of $L = 10{,}000$ Monte Carlo experiments.

The results of the simulation study have been provided in Appendix A, Tables (A.1-A.5). The following concluding remakes are noticed based on these results as follows;

1. All the estimates reveal the property of consistency, i.e., the Biases and the RMSE of $\hat{\alpha}$, $\hat{\beta}$ and $\widetilde{R}$ always decreased as n increased
2. The MPS method has more relative efficiency than ML, PWM and OLS for most parameters of Kumaraswamy distribution in all tables.
3. The FB estimation method exceeds the estimate of the seven classical methods are used. The fuzzy reliability was estimated using FB method with the lowest Bias and the least RMSE.
4. In the FB method, when sample size is larger, the Bias and the RMSE are reduced to as little as the sample size is 200.
5. The Biases and the RMSE of $\hat{\theta}$ and $\hat{\beta}$ always appeared smallest for the MPS, WLS and CVM methods.
6. The parameter estimated by FB method and MPS and CVM methods with the default values as well as the fuzzy reliability converges from the default reliability as the size of the sample increases.
7. FB estimation method has achieved the lowest value of the seven classical estimation methods. This indicates that the duration data of linear accelerator is more consistent with the fuzzy Kumaraswamy distribution when estimating the parameters of this distribution in Bayes.

6.2 An Application to Cancer Data Set

Cancer is a deadly disease that spreads quickly. Millions of people are affected with this disease each year. This lethal disease is a major focus of scientific research and new breakthroughs in the field of treatment. As a result of these efforts, new technology gadgets have been developed that provide new sources of cancer detection and therapy. As a result, the significance of gadgets that show disease has to be addressed. One of the most important is the linear accelerator device, which is a modern and advanced device in the detection of cancer and radiation treatment. These data are taken from a cancer study described by [28].

6.2.1 Linear Accelerator

The linear accelerator device is one of the most advanced and cutting-edge devices for detecting and killing cancer cells using radiation. In the following situations, this gadget is used:

1. To treat cancer by destroying cancer cells.
2. Control cancer by preventing cancer cells from growing and spreading.
3. Relieving cancer symptoms such as pain.

The device is one of the most modern medical devices used to treat tumors in the Babylonian center for tumor therapy. The center only operates one piece of equipment for a linear accelerator that provides continuous service to citizens. However, if the first device fails or stops working for technical reasons, the second device is used to offer ongoing service. However, it should be emphasized that there is no reliable tracking of the device's operating and halting times. In the event that the gadget malfunctions, the operators must have a precise understanding of the operation periods and holidays. For example, the device operator must inform the center's management and the hospital's management orally. In turn, management must contact the device's manufacturer to arrange for the devices imprecision in recording operating times and holidays to be repaired. As a result, data on the linear accelerator's operation time are fuzzy integers that belong to fuzzy times with varying degrees of membership, as shown in [29].

Approximate information was obtained on the length of operation of the equipment until the work of the specialists in charge of the device, the supervising engineers and the administration of the center. These times were arranged in Table 1, measured in months for the period from the beginning of installation of the equipment at the center.

Table 1
Cancer data set extend the speed of the linear accelerator system until (i) it stops working in months

1.3, 1.3, 1.4, 1.6, 1.7, 1.7, 1.8, 1.8, 1.9, 2, 2, 2, 2, 2.1, 2.2, 2.2, 2.4, 2.4, 2.4, 2.5, 2.6, 2.8, 3, 3, 3.1, 3.3, 3.3, 3.3, 3.4, 3.5, 3.5, 3.6, 3.7, 3.8, 3.8, 3.9, 4.1, 4.2, 4.5, 4.5, 4.6, 5.1, 5.2, 5.3, 5.5, 5.8, 5.8, 5.9, 6.1, 6.3, 6.5, 6.6, 6.8, 7.7, 8.5, 10.4, 10.6, 10.6, 10.9, 11.4, 12.6, 14, 17.7

6.2.2 Data Fuzzification

The real sample vector x was converted to mist using the FIS as in Figure 4, corresponding to the following functions;

$$\mu_{\tilde{t}_1}(x) = \begin{cases} 1 & x \le 1.3 \\ \frac{1.7-x}{0.4} & 1.3 \le x \le 1.7 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-1.3}{0.4} & 1.3 \le x \le 1.7 \\ \frac{1.8-x}{0.1} & 1.7 \le x \le 1.8 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-1.7}{0.1} & 1.7 \le x \le 1.8 \\ \frac{2.1-x}{0.3} & 1.8 \le x \le 2.1 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-1.8}{0.3} & 1.8 \le x \le 2.1 \\ \frac{2.6-x}{0.5} & 2.1 \le x \le 2.6 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-2.1}{0.3} & 2.1 \le x \le 2.6 \\ \frac{3.3-x}{0.7} & 2.6 \le x \le 3.3 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-2.6}{0.7} & 2.6 \le x \le 3.3 \\ \frac{3.8-x}{0.5} & 3.3 \le x \le 3.8 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-3.3}{0.5} & 3.3 \le x \le 3.8 \\ \frac{3.9-x}{0.1} & 3.8 \le x \le 3.9 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} \frac{x-3.8}{0.1} & 3.8 \le x \le 3.9 \\ \frac{5.1-x}{1.1} & 3.9 \le x \le 5.1 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_9}(x) = \begin{cases} \frac{x-3.9}{1.1} & 3.9 \le x \le 5.1 \\ \frac{6.1-x}{1} & 5.1 \le x \le 6.1 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_{10}}(x) = \begin{cases} \frac{x-5.1}{1} & 5.1 \le x \le 6.1 \\ \frac{6.1-x}{4.5} & 6.1 \le x \le 10.6 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_{11}}(x) = \begin{cases} \frac{x-6.1}{7.1} & 10.6 \le x \le 17.7 \\ 1 & x \ge 17.7 \\ 0 & otherwise \end{cases}$$

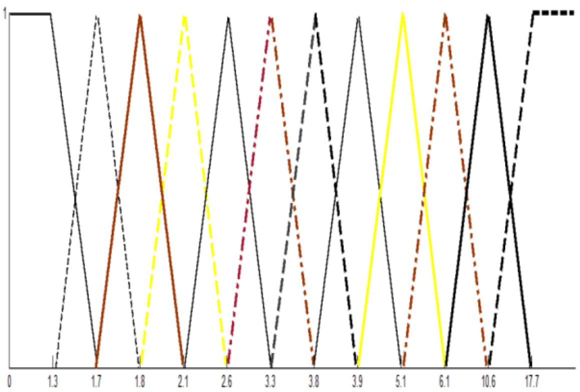

Fig. 4 FIS used to process cancer data set.

We have used the relationship between the arithmetic mean and the variance for the distribution of the parameter matrix. This is to obtain the initial value of the algorithms used in the estimation of the parameters by solving the non-linear equations and using the NR algorithm. In Table 2, we computed the Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted Kumaraswamy functions.

Table 2
Goodness of-fit for cancer data set

D	0.0648
P-value	0.7513

The distance (D) between the fitted and the empirical distribution functions for the data is 0.0648 and the corresponding p-value is 0.7513. Therefore, it indicates that Kumaraswamy distribution can be fitted to the data set, by use empirical cumulative distribution function (ecdf) to obtain a data graph to confirm the accuracy of the K-S test as Figure 5.

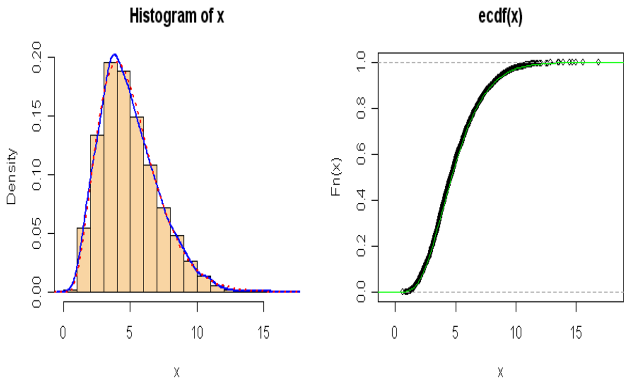

Fig. 5 Kolmogorov-Smirnov test and the plot of max distance between two ecdf Curves.

The selection of models for specific data is one of the basic tasks of the scientific study in choosing a predictive model from a group of candidate models. Several statistical methods are available to determine the best method of estimation, where the most widely used are the Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). However, the better estimation method corresponds to the smaller values of AIC, CAIC, BIC and HQIC. These methods are determined according to the following formulas respectively.
The AIC is evaluated as follows:

$$\text{AIC} = 2k - 2\ell, \qquad (25)$$

The CAIC is:

$$\text{CAIC} = \frac{2nk}{n-k-1} - 2\ell, \qquad (26)$$

The BIC is given by:

$$\text{BIC} = k\log(n) - 2\ell, \quad (27)$$

The HQIC is:

$$\text{HQIC} = 2k\log(\log(n)) - 2\ell. \quad (28)$$

where ℓ is the ML estimation log-likelihood function value, k is parameters count in the distribution in the proposed distribution , and n is considered as the size of the sample used in calculations.

Table 3
Parameter estimates and goodness-of-fit measures for cancer data set

Methods	Estimate			Goodness-of-Fit Measures			
	$\hat{\theta}$	$\hat{\beta}$	$\tilde{R}$	AIC	BIC	CAIC	HQIC
ML	31.3795	1.9606	0.7165	243.6362	284.2593	257.3357	228.175
MPS	24.1267	1.8244	0.4678	158.7529	178.1246	191.8282	156.247
MM	34.0458	2.0133	0.8305	173.0514	194.3498	184.0239	168.398
PWM	34.3898	1.9509	0.5224	172.6098	139.2743	182.3851	177.436
OLS	27.8196	1.9587	0.9187	241.3962	209.0632	253.1743	219.513
WLS	29.2665	1.9628	0.6153	229.7521	238.6071	233.0747	252.8726
CVM	32.4227	2.0424	0.4321	151.6935	131.3964	172.7658	167.409
FB	22.4979	1.8102	0.2452	132.0517	117.67383	160.9147	145.5824

Table 3, summarizes the estimates of the methods of Kumaraswamy distribution parameters, reliability function and the rate of uncertainty function and the values of goodness-of-fit measures for cancer data set. We note that from the results in Table 3, the four varieties of goodness-of-fit measures for FB estimation method have achieved the lowest value of the seven classical estimation methods. This indicates that the duration data of the linear accelerator is more consistent with the fuzzy Kumaraswamy distribution when estimating the parameters of this distribution in FB proposed estimation method.

7. Conclusions

In this paper, we have discussed several estimation procedures for the Kumaraswamy distribution. In particular, we have discussed seven classical estimation methods and propose a fuzzy Bayesian procedure to estimate the unknown parameters and fuzzy reliability function. The seven classical methods are; maximum likelihood estimation, maximum product spacing estimation, method of moments, probability-weighted moment, ordinary least squares, weighted least squares and Cramér–von Mises estimation. It is not feasible to compare these methods theoretically. We have performed an extensive simulation study to compare these methods. R software is used to perform this study, see Appendix B. We have also compared estimators by cancer data set application; the results show that the four varieties of goodness-of-fit measures for the FB estimation method have achieved the lowest value of the seven classical estimation methods. In terms of overall comparison (with respect to Bias and RMSE) the performance of the proposed FB estimates is generally the best.

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Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Appendices

Appendix A: Simulation Results

Table A.1

Bias and RMSE values of the parameters and reliability for $\theta = 0.5$ and $\beta = 0.5$

n	Estimate	ML		MPS		MM		PWM		OLS		WLS		CVM		FB	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
50	$\hat{\theta}$	0.2089	0.2382	0.1589	0.1688	0.1559	0.198	0.2038	0.1520	0.1524	0.1042	0.1936	0.1159	0.1252	0.1071	0.0963	0.0816
	$\hat{\beta}$	0.1438	0.1699	0.0702	0.0609	0.1293	0.1243	0.0766	0.0801	0.0907	0.0959	0.1038	0.1092	0.0858	0.0906	0.0742	0.0659
	$\tilde{R}$	0.1398	0.1650	0.1292	0.1365	0.1522	0.1784	0.1783	0.2029	0.1227	0.1304	0.1728	0.2085	0.1123	0.1210	0.1070	0.0903
75	$\hat{\theta}$	0.1652	0.1884	0.0762	0.0646	0.1257	0.1335	0.1233	0.1566	0.1531	0.0917	0.1612	0.1202	0.0990	0.0847	0.1205	0.0824
	$\hat{\beta}$	0.1138	0.1344	0.0678	0.0716	0.0555	0.0474	0.0821	0.0864	0.0870	0.0983	0.0718	0.0758	0.0587	0.0521	0.0606	0.0633
	$\tilde{R}$	0.1411	0.1605	0.1367	0.1586	0.1204	0.1411	0.1106	0.1305	0.0888	0.0957	0.1022	0.1080	0.0970	0.1032	0.0846	0.0714
100	$\hat{\theta}$	0.0994	0.1056	0.0953	0.0652	0.1307	0.1491	0.1275	0.0951	0.0975	0.1239	0.0783	0.0670	0.1211	0.0725	0.0602	0.0511
	$\hat{\beta}$	0.0900	0.1063	0.0439	0.0375	0.0688	0.0778	0.0649	0.0683	0.0537	0.0567	0.0568	0.0600	0.0479	0.0501	0.0464	0.0412
	$\tilde{R}$	0.1116	0.1269	0.0703	0.0757	0.1081	0.1254	0.0875	0.1033	0.0952	0.1116	0.0768	0.0816	0.0808	0.0854	0.0669	0.0565
150	$\hat{\theta}$	0.1034	0.1179	0.0787	0.0835	0.0771	0.0980	0.0620	0.0530	0.0958	0.0573	0.1009	0.0752	0.0754	0.0516	0.0477	0.0404
	$\hat{\beta}$	0.0712	0.0841	0.0424	0.0448	0.0544	0.0615	0.0347	0.0297	0.0449	0.0474	0.0513	0.0540	0.0379	0.0396	0.0367	0.0326
	$\tilde{R}$	0.0817	0.0855	0.0556	0.0599	0.0753	0.0883	0.0692	0.0883	0.1004	0.0992	0.0607	0.0645	0.0639	0.0675	0.0530	0.0447
200	$\hat{\theta}$	0.0818	0.0933	0.0597	0.0408	0.0622	0.0661	0.0610	0.0775	0.0798	0.0595	0.0758	0.0454	0.0490	0.0419	0.0377	0.0320
	$\hat{\beta}$	0.0563	0.0665	0.0300	0.0313	0.0430	0.0487	0.0406	0.0427	0.0355	0.0375	0.0336	0.0355	0.0275	0.0235	0.0290	0.0258
	$\tilde{R}$	0.0596	0.0699	0.0440	0.0474	0.0698	0.0794	0.0547	0.0646	0.0676	0.0785	0.0506	0.0534	0.0480	0.0511	0.0419	0.0353

Table A.2

Bias and RMSE values of the parameters and reliability for $\theta = 1.5$ and $\beta = 0.5$

n	Estimate	ML		MPS		MM		PWM		OLS		WLS		CVM		FB	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
50	$\hat{\theta}$	0.4958	0.5654	0.2971	0.2543	0.3772	0.4006	0.4837	0.3607	0.4594	0.2750	0.3700	0.470	0.3617	0.2473	0.2285	0.1899
	$\hat{\beta}$	0.3414	0.4032	0.1649	0.1423	0.2610	0.2951	0.2463	0.2592	0.2035	0.2150	0.2153	0.2275	0.1760	0.1446	0.1817	0.1900
	$\tilde{R}$	0.4101	0.476	0.3612	0.4235	0.4233	0.4816	0.3319	0.3917	0.2912	0.3096	0.3066	0.3239	0.2665	0.2873	0.2540	0.1668
75	$\hat{\theta}$	0.3681	0.4197	0.2685	0.1836	0.2800	0.2973	0.3590	0.2678	0.3410	0.2042	0.2746	0.3489	0.2206	0.1888	0.1697	0.1410
	$\hat{\beta}$	0.2534	0.2993	0.1236	0.1057	0.1938	0.2190	0.1828	0.1924	0.1511	0.1596	0.1598	0.1689	0.1349	0.1411	0.1306	0.1073
	$\tilde{R}$	0.3142	0.3575	0.2276	0.2405	0.2681	0.3144	0.2161	0.2298	0.2464	0.2908	0.3045	0.3532	0.1978	0.2133	0.1885	0.1238
100	$\hat{\theta}$	0.2217	0.2528	0.1617	0.1106	0.1687	0.1791	0.1654	0.2101	0.1329	0.1137	0.2054	0.1230	0.2163	0.1613	0.1022	0.0849
	$\hat{\beta}$	0.1101	0.1159	0.0963	0.1017	0.1167	0.1319	0.1527	0.1803	0.0745	0.0637	0.0910	0.0961	0.0813	0.0850	0.0787	0.0646
	$\tilde{R}$	0.1893	0.2153	0.1371	0.1448	0.1615	0.1894	0.1484	0.1751	0.1834	0.2127	0.1302	0.1384	0.1192	0.1285	0.1136	0.0746
150	$\hat{\theta}$	0.1506	0.1913	0.1472	0.1007	0.2018	0.2301	0.1968	0.1468	0.1870	0.1119	0.1535	0.1630	0.1209	0.1035	0.0930	0.0773
	$\hat{\beta}$	0.1389	0.1641	0.0828	0.0875	0.1062	0.1201	0.0678	0.0579	0.1002	0.1055	0.0876	0.0926	0.0716	0.0588	0.0740	0.0773
	$\tilde{R}$	0.1723	0.1960	0.1669	0.1936	0.1470	0.1724	0.1351	0.1594	0.1248	0.1318	0.1185	0.1260	0.1085	0.1169	0.1034	0.0679
200	$\hat{\theta}$	0.1634	0.1864	0.0979	0.0838	0.1243	0.1320	0.1219	0.1549	0.1514	0.0907	0.1594	0.1189	0.1192	0.0815	0.0753	0.0626
	$\hat{\beta}$	0.0860	0.0973	0.0599	0.0626	0.1125	0.1329	0.0812	0.0854	0.0549	0.0469	0.0671	0.0709	0.0710	0.0750	0.0580	0.0477
	$\tilde{R}$	0.1191	0.1396	0.0878	0.0947	0.1395	0.1587	0.1352	0.1568	0.1094	0.1291	0.1011	0.1068	0.0960	0.1020	0.0837	0.0550

Table A.3

Bias and RMSE values of the parameters and reliability for $\theta = 2$ and $\beta = 1.5$

n	Estimate	ML		MPS		MM		PWM		OLS		WLS		CVM		FB	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
50	$\hat{\theta}$	0.9418	1.0740	0.7027	0.8927	0.7165	0.7608	0.9187	0.6852	0.8726	0.5224	0.5644	0.4830	0.6870	0.4698	0.4341	0.3607
	$\hat{\beta}$	0.6485	0.7659	0.3164	0.2704	0.4958	0.5605	0.4678	0.4923	0.4090	0.4321	0.3866	0.4083	0.3452	0.3610	0.3343	0.2746
	$\tilde{R}$	0.8040	0.9147	0.7790	0.9037	0.6861	0.8045	0.6305	0.7440	0.5824	0.6153	0.5531	0.5880	0.5062	0.5457	0.4824	0.3169
75	$\hat{\theta}$	0.5675	0.6026	0.5565	0.7070	0.7459	0.8506	0.7276	0.5427	0.4470	0.3825	0.6911	0.4137	0.3041	0.2721	0.3438	0.2857
	$\hat{\beta}$	0.5136	0.6066	0.2506	0.2142	0.3926	0.4439	0.3705	0.3899	0.3239	0.3423	0.3062	0.3234	0.2734	0.2859	0.2648	0.2175
	$\tilde{R}$	0.6368	0.7244	0.6170	0.7157	0.5434	0.6371	0.4380	0.4657	0.4613	0.4873	0.4993	0.5893	0.4009	0.4322	0.3821	0.2510
100	$\hat{\theta}$	0.5213	0.5945	0.3089	0.2481	0.3966	0.4211	0.5085	0.3793	0.4830	0.2892	0.3124	0.2674	0.3803	0.2600	0.2403	0.1997
	$\hat{\beta}$	0.3590	0.4239	0.1751	0.1497	0.2744	0.3103	0.2589	0.2725	0.2264	0.2392	0.2140	0.2260	0.1850	0.1520	0.1911	0.1998
	$\tilde{R}$	0.4450	0.5063	0.4312	0.5002	0.3798	0.4453	0.3490	0.4118	0.3224	0.3406	0.3061	0.3255	0.2802	0.3021	0.2670	0.1754
150	$\hat{\theta}$	0.3456	0.3941	0.2579	0.3276	0.2629	0.2792	0.3202	0.1917	0.3372	0.2515	0.2071	0.1773	0.2521	0.1724	0.1593	0.1324
	$\hat{\beta}$	0.2380	0.2811	0.1161	0.0992	0.1819	0.2057	0.1717	0.1807	0.1419	0.1499	0.1501	0.1586	0.1267	0.1325	0.1227	0.1008
	$\tilde{R}$	0.2951	0.3357	0.2859	0.3316	0.2518	0.2952	0.2314	0.2730	0.2137	0.2258	0.2030	0.2158	0.1858	0.2003	0.1770	0.1163
200	$\hat{\theta}$	0.2076	0.2367	0.1549	0.1968	0.2025	0.1510	0.1579	0.1677	0.1244	0.1065	0.1923	0.1151	0.0957	0.0795	0.1514	0.1035
	$\hat{\beta}$	0.1429	0.1688	0.0697	0.0596	0.1093	0.1235	0.1031	0.1085	0.0901	0.0953	0.0852	0.0900	0.0761	0.0796	0.0737	0.0605
	$\tilde{R}$	0.1772	0.2016	0.1717	0.1992	0.1390	0.1640	0.1512	0.1773	0.1219	0.1296	0.1284	0.1356	0.1116	0.1203	0.1063	0.0698

Table A.4

Bias and RMSE values of the parameters and reliability for $\theta = 2$ and $\beta = 2.5$

n	Estimate	ML		MPS		MM		PWM		OLS		WLS		CVM		FB	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
50	$\hat{\theta}$	0.6267	0.7146	0.4571	0.3126	0.4768	0.5063	0.4676	0.5940	0.5806	0.3476	0.6113	0.4559	0.3755	0.3214	0.2888	0.2449
	$\hat{\beta}$	0.4315	0.5096	0.2572	0.2717	0.3629	0.4298	0.3112	0.3276	0.3299	0.3730	0.2721	0.2875	0.2224	0.1977	0.2105	0.1799
	$\tilde{R}$	0.5350	0.6086	0.4195	0.4951	0.4565	0.5353	0.5184	0.6013	0.3368	0.3631	0.3875	0.4094	0.3680	0.3913	0.3210	0.2709
75	$\hat{\theta}$	0.4320	0.4926	0.1991	0.1688	0.3286	0.3490	0.4214	0.3143	0.3223	0.4094	0.2589	0.2215	0.4002	0.2396	0.3151	0.2155
	$\hat{\beta}$	0.2274	0.2571	0.1773	0.1873	0.2974	0.3513	0.2501	0.2963	0.1876	0.1982	0.2145	0.2258	0.1533	0.1363	0.1451	0.1240
	$\tilde{R}$	0.3688	0.4195	0.2322	0.2503	0.3573	0.4145	0.2892	0.3413	0.3147	0.3690	0.2537	0.2697	0.2671	0.2822	0.2213	0.1867
100	$\hat{\theta}$	0.2978	0.3396	0.1784	0.1527	0.2692	0.2822	0.2265	0.2406	0.2759	0.1652	0.2905	0.2166	0.2172	0.1485	0.1372	0.1164
	$\hat{\beta}$	0.2050	0.2421	0.1222	0.1291	0.1724	0.2042	0.1567	0.1772	0.1479	0.1557	0.1293	0.1366	0.1057	0.0939	0.1062	0.0855
	$\tilde{R}$	0.2542	0.2892	0.1600	0.1725	0.2169	0.2543	0.1993	0.2352	0.2463	0.2857	0.1749	0.1859	0.1841	0.1945	0.1525	0.1287
150	$\hat{\theta}$	0.2053	0.2341	0.1497	0.1024	0.1562	0.1658	0.1531	0.1945	0.1902	0.1139	0.2002	0.1493	0.1230	0.1053	0.0946	0.0802
	$\hat{\beta}$	0.1413	0.1669	0.0690	0.0589	0.1189	0.1408	0.1019	0.1073	0.0891	0.0942	0.1080	0.1222	0.0843	0.0890	0.0729	0.0648
	$\tilde{R}$	0.1495	0.1753	0.1205	0.1282	0.1752	0.1994	0.1698	0.1969	0.1269	0.1341	0.1374	0.1622	0.1103	0.1189	0.1051	0.0887
200	$\hat{\theta}$	0.1415	0.1613	0.1032	0.0706	0.1076	0.1143	0.1380	0.1029	0.1056	0.1341	0.0848	0.0726	0.1311	0.0785	0.0652	0.0553
	$\hat{\beta}$	0.0819	0.0970	0.0581	0.0613	0.0974	0.1151	0.0745	0.0842	0.0703	0.0740	0.0614	0.0649	0.0502	0.0446	0.0475	0.0406
	$\tilde{R}$	0.1208	0.1374	0.0760	0.0820	0.1031	0.1209	0.1170	0.1358	0.0875	0.0924	0.0947	0.1118	0.0831	0.0883	0.0725	0.0612

Table A.5
Bias and RMSE values of the parameters and reliability for $\theta = 3$ and $\beta = 3.5$

n	Estimate	ML		MPS		MM		PWM		OLS		WLS		CVM		FB	
		Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
50	$\hat{\theta}$	0.8140	0.9283	0.4878	0.4175	0.6193	0.6576	0.6073	0.7715	0.7940	0.5922	0.7542	0.4515	0.5938	0.4060	0.3752	0.3181
	$\hat{\beta}$	0.4285	0.4844	0.3341	0.3529	0.5605	0.6619	0.4714	0.5583	0.4043	0.4255	0.3535	0.3735	0.2889	0.2568	0.2734	0.2337
	$\tilde{R}$	0.6949	0.7906	0.4375	0.4717	0.6733	0.7810	0.5449	0.6431	0.5930	0.6953	0.5034	0.5318	0.4780	0.5082	0.4169	0.3518
75	$\hat{\theta}$	0.6020	0.6865	0.3608	0.3088	0.4492	0.5706	0.4580	0.4864	0.5578	0.3339	0.5873	0.4380	0.2775	0.2353	0.4392	0.3003
	$\hat{\beta}$	0.3169	0.3583	0.2137	0.1899	0.4145	0.4896	0.3486	0.4129	0.2614	0.2762	0.2990	0.3147	0.2471	0.2610	0.2022	0.1729
	$\tilde{R}$	0.4030	0.4756	0.3535	0.3759	0.4980	0.5777	0.5139	0.5847	0.4386	0.5142	0.3723	0.3933	0.3236	0.3488	0.3084	0.2602
100	$\hat{\theta}$	0.3851	0.4391	0.2308	0.1975	0.2929	0.3111	0.2873	0.3650	0.3756	0.2801	0.3568	0.2136	0.2809	0.1921	0.1775	0.1505
	$\hat{\beta}$	0.2651	0.3131	0.1912	0.2013	0.2027	0.2292	0.1672	0.1767	0.2230	0.2641	0.1581	0.1670	0.1294	0.1106	0.1367	0.1215
	$\tilde{R}$	0.2805	0.3289	0.2070	0.2231	0.3287	0.3740	0.3185	0.3695	0.2381	0.2516	0.2578	0.3042	0.2261	0.2404	0.1972	0.1664
150	$\hat{\theta}$	0.1941	0.2213	0.1163	0.0995	0.1448	0.1839	0.1476	0.1568	0.1893	0.1412	0.1798	0.1076	0.1416	0.0968	0.0894	0.0758
	$\hat{\beta}$	0.0964	0.1014	0.0843	0.0890	0.1022	0.1155	0.1336	0.1578	0.1124	0.1331	0.0797	0.0841	0.0689	0.0612	0.0652	0.0557
	$\tilde{R}$	0.1657	0.1885	0.1043	0.1124	0.1605	0.1862	0.1299	0.1533	0.1414	0.1658	0.1206	0.1268	0.1140	0.1212	0.0994	0.0839
200	$\hat{\theta}$	0.1062	0.1211	0.0775	0.0530	0.0808	0.0858	0.1036	0.0773	0.0793	0.1007	0.0637	0.0545	0.0984	0.0589	0.0490	0.0415
	$\hat{\beta}$	0.0731	0.0864	0.0436	0.0461	0.0615	0.0729	0.0528	0.0555	0.0461	0.0487	0.0559	0.0632	0.0357	0.0305	0.0377	0.0335
	$\tilde{R}$	0.0774	0.0907	0.0571	0.0615	0.0907	0.1032	0.0711	0.0839	0.0879	0.1019	0.0624	0.0663	0.0657	0.0694	0.0544	0.0459

Appendix B: R-Codes

```
###################################
Parameters Kumaraswamy Distribution
###################################
# theta > 0
# beta > 0
# 0 < x < 1
###################################
## clean up everything
remove(list=objects())
options(warn = -1)
          ### Packages ###
library("stats4")
library("MASS")
library("bbmle")
library("maxLik")
            ### PDF ###
dkuma <- Vectorize(function(x,
theta , beta,log = FALSE){
logden <-  log(theta ) + log(beta)
+(theta -1)*log(x) + (beta-
1)*log(1-x^ theta )
val<- ifelse(log, logden,
exp(logden))
return(val)
})
            ### CDF ###
pkuma <-
Vectorize(function(q,theta ,
beta,log.p = FALSE){
  cdf <-  1 - (1-q^theta)^beta
  val <- ifelse(log.p, log(cdf),
cdf)
  return(val)
})
     ### Quantile function ###
qkuma <- Vectorize(function(u,theta
,Beta){
 val <-  ( 1 - (1-u)^(1/beta))^(1/
theta )
  return(val)
})
   ### Moments of order n of the
      Kumaraswamy(theta,beta)
          Distribution ###
mn <- function(theta,beta,n){
  log.num <- log(beta) + lgamma(1 +
n/ theta) + lgamma(beta)
  log.den <- lgamma(1 + beta + n/
theta)
  return(exp(log.num-log.den))
}
       ### NR-Algorithm ###
rm(list=ls(all=TRUE))
library("rootSolve")
n=100; theta<-1.5; beta<-3;y<-c()
a<-c();b<-c();w<-c();u<-c();H<-c()
y<- rkuma(n,theta,beta)
betahat<-c();thetahat<-c()
for(it in 1:10000){
for(i in 1:n){
a[i]=runif(1,y[i]-1,y[i])
b[i]=runif(1,y[i],y[i]+1)
w[i]<-runif(1)
u[i]<-runif(1,0,1-w[i])
H[i]<-(1+w[i]-u[i])/2}
f<-function(t,theta,beta){
f<-pkuma(t,shape=theta,beta)}
mu<-function(t,i){
if(t>=a[i] & t<=y[i])
return(H[i]*(t-a[i])/(y[i]-a[i]))
if(t>y[i] & t<=b[i])
return(H[i]*(b[i]-t)/(b[i]-y[i]))
else
return(0)}
h1<-function(t,i,theta,beta){
h1<-mu(t,i)*f(t,theta,beta)}
I<-function(z){
theta<-z[1]
beta<-z[2]
i<-z[3]
I<-log(integrate(h1,max(0,a[i]),
b[i],i,theta,beta)$value)}
logl<-function(z){
theta <-z[1]
beta <-z[2]
ss<-0
for(j in 1:n){
c<-c(theta,beta,j)
ss<-ss+I(c)}
logl<--ss}
c<-c(theta,beta)
out<-
suppressWarnings(nlminb(c,logl))
thetahat[it]<-out$par[1]
betahat[it]<-out$par[2]}
mean(thetahat);mean(betahat)
mean(thetahat)-theta;
mean(betahat)- beta
mean((thetahat-theta)ˆ2)
mean((betahat-beta)ˆ2)
x<-5
R<-exp(-((x/theta)ˆbeta))
print(R)
RR<-exp(-((x/thetahat)ˆbetahat))
print(mean(RR))
print(mean(RR-R))
print(mean((RR-R)ˆ2))
       ### FB-Estimation ####
rm(list=ls(all=TRUE))
library("numDeriv")
```

```
betahat<-c();thetahat<-c()
n=50;theta<-5;beta<-10;y<-c()
a<-c();b<-c();w<-c();u<-c();H<-c()
thet.<-c();bet.<-c()
for(it in 1:10000){
d1<- rkuma(50,theta,beta)
l.prim<-function(z){
theta<-z[1]
beta<-z[2]
l.prim<--sum(log(pkuma(d1,theta,
beta)))}
c<-c(theta,beta)
out.<-
suppressWarnings(nlm(l.prim,c))
thet.[it]<-out.$estimate[1];
bet.[it]<-out.$estimate[2]}
V1<-mean((thet.-theta)^2)
V2<-mean((bet.-beta)^2)
b1<-mean(thet.)/(V1)
a1<-mean(thet.)*b1/2
a2<-(mean(bet.))^2/(12*V2)
b2<-(mean(bet.))*(a2-1)/12
for(it in 1:10000){
y<- rkumar(n,theta,beta)
for(i in 1:n){
a[i]=runif(1,y[i]-1,y[i])
b[i]=runif(1,y[i],y[i]+1)
w[i]<-runif(1)
u[i]<-runif(1,0,1-w[i])
H[i]<-(1+w[i]-u[i])/2}
f<-function(t,theta,beta){
f<-pkuma(t,shape=theta,beta)}
mu<-function(t,i){
if(t>=a[i] & t<=y[i])
return(H[i]*(t-a[i])/(y[i]-a[i]))
if(t>y[i] & t<=b[i])
return(H[i]*(b[i]-t)/(b[i]-y[i]))
function(t,i){
if(t>=a[i] & t<=y[i])
return(H[i]*(t-a[i])/(y[i]-a[i]))
if(t>y[i] & t<=b[i])
return(H[i]*(b[i]-t)/(b[i]-y[i]))
else
return(0)}
h1<-function(t,i,theta,beta){
mu(t,i)*f(t,theta,beta)}
I1<-function(theta,beta,i){
I1<-integrate(h1,max(0,a[i]),
b[i],i,theta,beta)$value}
I.1<-function(theta,beta){
ss<-0
for(j in 1:n){
ss<-ss+log(I1(theta,beta,j))}
I.1<-ss}
HH<-function(z){
theta<-z[1]
beta<-z[2]
HH<--((1/n)*((n+a1-1)*log(theta)-
b1*theta-(n+a2+1)*log(beta)-
b1/beta+I.1(theta,beta)))}
c<-c(theta,beta)
out<-suppressWarnings(nlminb(c,HH))
c1<-c(out$par[1],out$par[2])
hess1<-suppressWarnings(hessian(
func=HH, x=c1))
sigma1<--solve(hess1)
Hstar1<-function(z){
theta<-z[1]
beta<-z[2]
cstar<-c(theta,beta)
Hstar1<-(-
(1/n)*log(theta)+(HH(cstar)))}
c<-c(theta,beta)
out.star1<-
suppressWarnings(nlminb(c,Hstar1))
c2<-
c(out.star1$par[1],out.star1$par[2)
hess2<-suppressWarnings(hessian(
func=Hstar1, x=c2))
sigma2<--solve(hess2)
Hstar2<-function(z){
theta <-z[1]
beta <-z[2]
cstar<-c(theta,beta)
Hstar2<-(-
(1/n)*log(beta)+(HH(cstar)))}
c<-c(theta,beta)
out.star2<-suppressWarnings(nlminb(
c,Hstar2))
c3<-
c(out.star2$par[1],out.star2$par[2)
hess3<-suppressWarnings(hessian(
func=Hstar2, x=c3))
sigma3<--solve(hess3)
thetahat[it]<-
suppressWarnings(((det(sigma2)
)/(det(sigma1)))^(1/2)*exp(-n*(
Hstar1(c2)-HH(c1))))
mean(thetahat);mean(betahat)
mean(thetahat)-theta;
mean(betahat)-beta
mean((thetahat-theta)^2)
mean((betahat-beta)^2)
x<-5
R<-exp(-((x/theta)^beta))
print(R)
RR<-exp(-((x/betahat)^thetahat))
print(mean(RR))
print(mean(RR-R))
print(mean((RR-R)^2)
```

END Codes