Computation of Volterra and Fredholm Integro-Differential Nonlinear Boundary Value Problems by a Modified Regula Falsi-Bisection Shooting Approach

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Abstract: A modified Regula Falsi shooting method approach is deployed to solve a set of Volterra and Fredholm integro- nonlinear boundary value problems. Efforts to solve this class of problems with traditional shooting methods have generally failed and the outcomes from most domain based approaches are often plagued by ill conditioning. Our modification is based on an exponential series technique embedded in a shooting bracketing method. For the purposes of validation, we initially solved problems with known closed form solutions before considering those that do not come with this property but are singular, genuinely nonlinear, and are of practical interest. Although in most of these tests, convergence was found to be super linear, the errors decreased monotonically after few iterations. This suggests that the method is robust and can be trusted to yield faithful results and by far surpasses in simplicity various other techniques that have been applied to solve similar problems. In order to buttress the effectiveness and utility of this approach, we display both the graphical and error analysis outcomes. And for each test case, the method can be seen to demonstrate the closeness of numerically generated results to the analytical solutions.

Keywords: Integro differential equations; boundary value problems; Fredholm; Volterra; Regula Falsi; exponential series; shooting method.

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1. Introduction

An integro-differential equation (IDE) contains both the derivative as well as the integral of the unknown function. Equations of this kind represent situations where there is a strong link between the current behavior of the system together with its history at any starting point. They are applicable to several fields including travelling waves, spread of diseases, population spread, viscoelastic flows, particle dynamics, elasticity, mechanics of continuous media etc. Since only a few of them are amenable to closed form solutions, the majority is solved numerically.

Research in this field has been ongoing. Prominent among these are spectral techniques. Spectral methods adopt a basis function to construct an approximation to the solution. An automatic Chebyshev spectral approximation has been applied with great success by Driscoll *et al*. [1] to solve linear and nonlinear IDEs. Several of these applications can be found in [2, 3 4]. Adomian decomposition method yields a series expansion of the solution sought and has been extended by Hadizadeh and Maleknejad [5] to solve Volterra integral and higher order integral equations. Despite the fact that the method requires considerable symbolic manipulations of its components, sometimes only a few terms are needed to arrive at a satisfactory solution [6].

The sinc-derivative collocation numerical technique has been applied by Abdella and Ross [7] to arrive at a solution of a general second order IDE They deployed suitable transformations to reduce the non-homogeneous conditions to their homogeneous analogs. In addition they developed special .interpolation techniques to handle the derivatives of unknown variables. Their approach was found to be very competitive when the numerical results were compared with literature results. The method has been found to be very accurate and sometimes displays exponentially decaying errors[8] in addition to effectively control and damp out errors oftentimes associated with numerical differentiation [9,10].

Jaiswel [11] introduced a Newton-Steffensen method to effectively handle the derivative term in the Newton's method. He successively applied his derivative free technique to the solution of nonlinear algebraic equations. Overall it can be surmised that most of the research work done in this field has been geared towards the formulation of more elegant numerical approaches, to arrive at a higher order convergence , reduction in the level of complexity, differentiability enhancement and numerical reliability. Efficient ways of handling variations of derivative based Newton-Raphson approach are ubiquitously explored [12,13,14]. In addition, considerable has also been done on the shooting method bracketing-based algorithms. Noticeable among these are the work done by Brent [15] Dekker[16] and Wu[17].

So far, there is no single optimal shooting method for handling nonlinear problems. None is self sufficient ; each has its own strength and weaknesses. Any one method may outperform the other in a particular problem and may produce poor results in a different dataset. Hybrid techniques developed to counter some of these problems exist in literature. Sabharwal [18] developed a new hybrid Newton-Raphson algorithm and a blend of bisection, Regula Falsi and Newton-Raphson techniques. His method exploited the advantages of the three algorithms in each iteration to arrive at a better approximation of the sought result. Moreover he showed that the complexity of the method is far less than that of any of the three component algorithms. Sabharwal [18] considered the number of iterations and not the running time as a valid criterion for assessing. Overall performance. Badr et *al.* [19] adopted a different approach. They positioned their stand on the fact that there are some methods that take a relatively small number of iterations to solve a problem; despite that, the execution time may be quite large and vise versa. Hence both factors were considered as realistic assessments of a numerical algorithm. Their novel blended hybrid technique incorporated the advantages of the trisection and Regula Falsi methods and was claimed to outperform that of Sabharwal [18].

We hasten to comment that quite a good number of these methods are aimed at nonlinear algebraic equations and contain attempts to attenuate handle numerical challenges by extensive algebraic manipulations. However considerable computational efforts are needed to translate and adapt some of these advantages to the solution of nonlinear integral boundary value problems (BVPs). Having noted the above, the foremost objective of the work reported herein is to adopt a blended bracketing shooting method approach to provide accurate results for a nonlinear integro-differential equation. In line with this, two types of shooting bracketing methods are considered, namely: the Regula Falsi and the bisection techniques. Each of them has its own peculiar characteristic that provides valuable insights into their strengths and weaknesses. The bisection method, depends solely on the continuity of the function being investigated, as a consequence, its value at the midpoint should lie between those at the endpoints.However, if the function is differentiable, the Regula Falsi takes advantage of this by using linear interpolation (or extrapolation) to provide a better estimate of the root within the range. Both methods known as closed or bracketing techniques require two guesses that bracket or contain the sought root. With each iteration, the range is further refined to facilitate the location of the root.

The manner of approaching this quest varies with the method. Whereas with the bisection method, the endpoints of the function should contain the root (right in the middle of the latest refinement); with the Regular Falsi, this guaranteed method of locating the root is compromised in exchange for a better guess for the location of the root. It does this by finding the secant line between two pints and then locating the root at the point where the line cuts the x-axis. Since the location of the root is no longer situated at half the interval space ϕ , it will now lie anywhere between 0 and 1 $\left(0<\phi<1\right)$. This is a very useful information in a hybrid algorithm formulation that includes a switching mechanism. Keeping a tab on the value of ϕ will yield a clear directive as to which of the two methods to use. Several modifications of the Regula Falsi have been reported in literature consisting of several variants of the classical approach [20-23]. We must comment that relevant literature in this field contains numerous applications to nonlinear algebraic problems; and similar solutions of nonlinear boundary value problems is often lacking, it is this gap we aim to lessen. Traditionally, problems with varying degrees of nonlinearity and even singularity have been handled with domain based techniques such as finite difference, finite element, boundary element and several variants of collocation techniques. Their limits and advantages have also been widely discussed. We do not attempt to resort to such methods. In this work, we introduce an algorithm , that is a blend of two shooting method bracketing techniques. It a does not rely on the differentiability of a function or a predictor-corrector approach. Instead the iterative procedure is more heuristic and proceeds in a manner that is in consonance with the optimal ratio at which the iteration interval is monotonically increasing or decreasing.

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2.1 Numerical Formulation

The hybrid technique reported here is a considerable modification of Thota [24] where a root-finding method was used to detect non-zero real roots of transcendental nonlinear algebraic equations. We deal with a family of techniques which constrain the root within an interval. Let us assume that [a,b] is such an interval, that contains such a root, then an approximation of such a root with the Regula Falsi method is given as :

$$\xi=\frac{a.f\left(b\right)-bf\left(a\right)}{f\left(b\right)-f\left(a\right)} \qquad (1)$$

Linear interpolation yields :

$$y=f\left(a\right)-\frac{\left(f\left(a\right)+f\left(b\right)\right)}{\left(b-a\right)}\left(x-a\right) \qquad (2)$$

In fact if $f\left(\xi\right).f\left(a\right)<0$, then the interval $\left[a,\xi\right]$ must contain a root. After each iteration, more information is needed to refine the range of the root identification. This can be said of both the secant and the Regula Falsi methods. The major difference between the two is that like the bisection method , the Regula Falsi ensures that the root is bracketed within range of the function; whereas the secant method obtains a better approximation by modifying the previous estimate of the root.

Without any loss in generality, the linear interpolation for both methods is written as :

$$z_{n+1} = z_n - \frac{z_n - z_{n-1}}{f(z_n) - f(z_{n-1})} f(z_n) \qquad (3)$$

Equation (3) is actually the first two terms of the expansion:

$$z_{n+1} = z_n - \frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})} + \frac{1}{2z_n}\left(\frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})}\right)^2 - \frac{1}{6z_n^2}\left(\frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})}\right)^3 + \ldots (4)$$

Adopting the standard form of exponential expansion, equation (4) can be recast to read:

$$z_{n+1} = z_n \left[\begin{array}{l} 1 - \frac{1}{z_n}\left(\frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})}\right) + \\ \frac{1}{2}\left\{\frac{1}{z_n}\right\}^2 \left(\frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})}\right)^2 - \\ \frac{1}{6}\left\{\frac{1}{z_n}\right\}^3 \left(\frac{(z_n - z_{n-1}) f(z_n)}{f(z_n) - f(z_{n-1})}\right)^3 + \ldots\ldots \end{array} \right] \qquad (5)$$

hence:

$$z_{n+1} = z_n \exp\left(\frac{(z_{n-1} - z_n) f(z_n)}{z_n\left(f(z_n) - f(z_{n-1})\right)} \right) \qquad (6)$$

where the order of approximation order of equation (6) is :

$$\mathrm{O}\left(\frac{1}{24 z_n^{\,3}} \left\{ \frac{(z_n - z_{n-1}) f(z_n)}{\left(f(z_n) - (z_{n-1})\right)} \right\}^4 \right)$$

Essentially we are trying to achieve three things; namely (i) develop a very simple numerical algorithm to locate a root (ii) bracket the root and (iii) speed up the whole process. The hybrid method discussed herein should be able to identify the root, but in reality, due to its heavy bias towards the Regula Falsi method, it will exchange the more reliable interval halving bisection approach for a better guess of the root location. To optimize performance, the interval ratio ϕ is monitored as iteration proceeds., and whenever $\phi > 0.5$, a switch is made to the bisection method [25].

2.2 The Algorithm

Step 1: Supply the two guess values, error tolerance and other parameters for computation.

Step 2: Set up an iteration loop and apply equation 6

Step 3 : Compute the value of λ for each iteration

Step 4: Use an 'if' statement to determine the appropriate shooting technique to deploy; based on the value of λ

Step 5 : Terminate or continue with the computation based on the magnitude of the error criterion

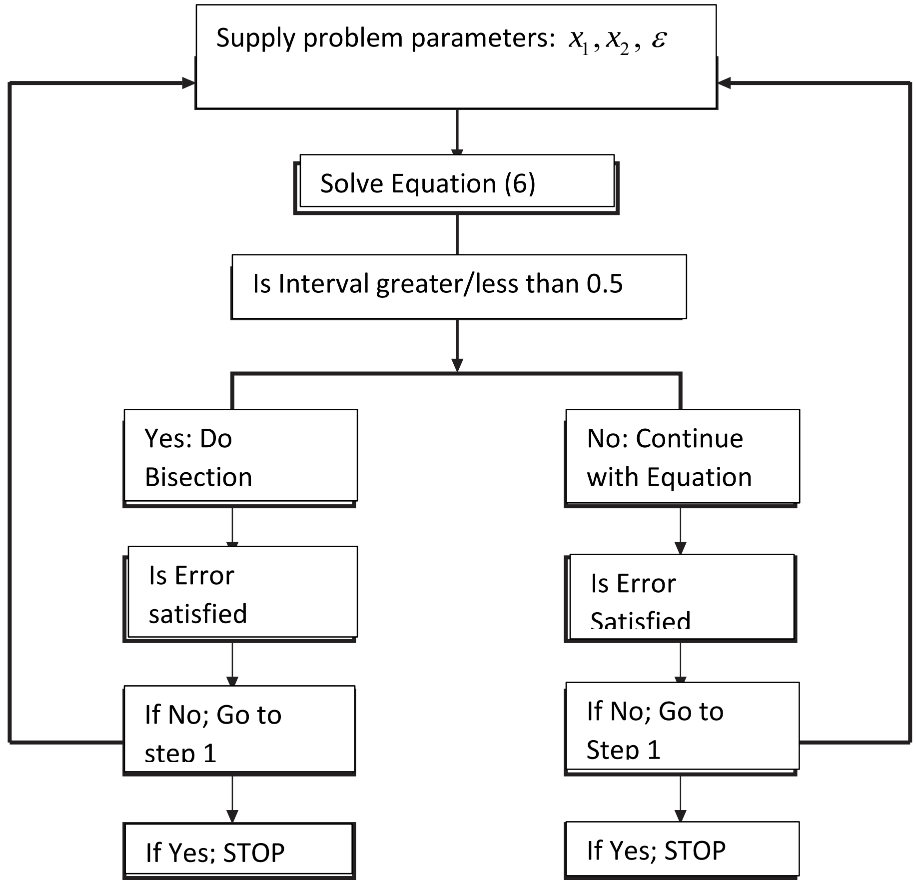

Fig. 1a : Algorithm Fowchart

3.Numerical Experiments :

For simulation purposes, we use a termination error tolerance of $\varepsilon = 10^{-6}$ together with an upper bound iteration of 50 . Examples of Integro-differential equations with analytic solutions where available are chosen from literature. .

3.1.Example 1

The first example from Filipov et al.[26] is of a Volterra integral type.

$$u''(t) = 3u(t)'(G(t) + 1/3)/u^2 \;, t \in (1,2)$$

$$G(t) = \int_1^t u(s)ds, \; u(1) = 1, \; u(2) = 4 \qquad (7a)$$

The analytic solution is given as : $u(t) = t^2 \qquad (7b)$

Fig. 1b shows excellent comparison between numerical and analytic results. A monotonically decreasing error profile is displayed in Fig. 1c. The convergence of the guessed slopes at the left hand side (LHS) boundary as the shooting method algorithm proceeds is displayed in Fig. 1d

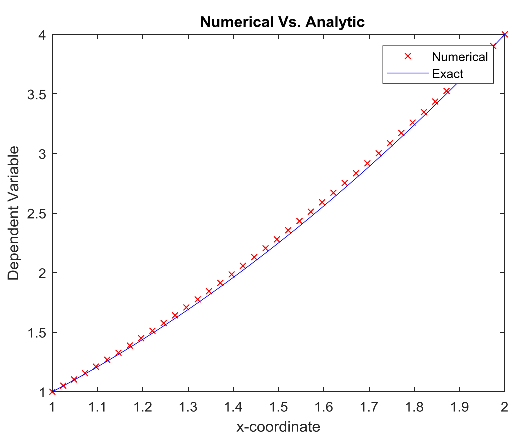

Fig.1b Comparison of numerical and analytic

solutions

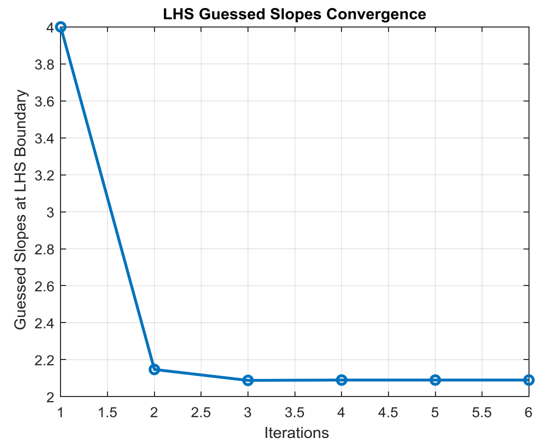

Fig. 1c : Guessed slopes per iteration at the LHS boundary

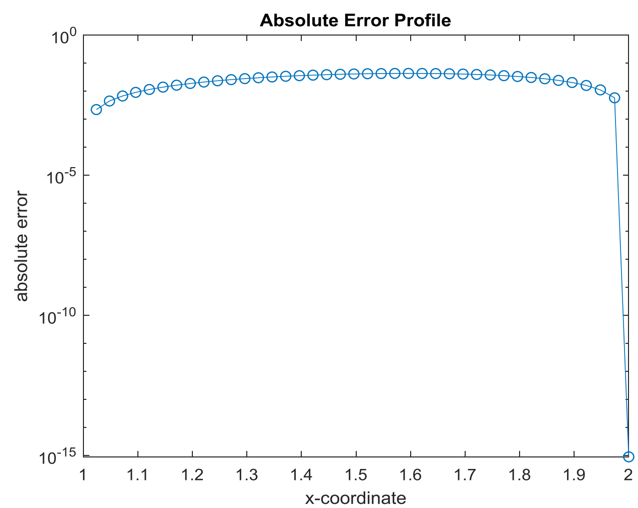

Fig. 1d Absolute error profile

3.2.Example 2

Problem 2 is taken from Abdella and Ross [7] and is of the Fredholm type

$$u'' - 2u - \int_{-1}^{1} te^{-t}\cos(t)udt = x \quad t \in (-1,1)$$

$$u(-1) = -1, \; u(1) = 1 \qquad (8a)$$

The solution is given as

$$u(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3 \cos(t) - 0.5t \qquad (8b)$$

where

$$c_1 = 0.4206057265;\ c_2 = -0.3545613032;$$
$$c_3 = -0.2662525683$$

Figures 1a and 1b show the profiles of both the numerical solution and the absolute error between the numerical and the analytic solution. The magnitude of the errors displayed confirms that the numerical results are accurate .

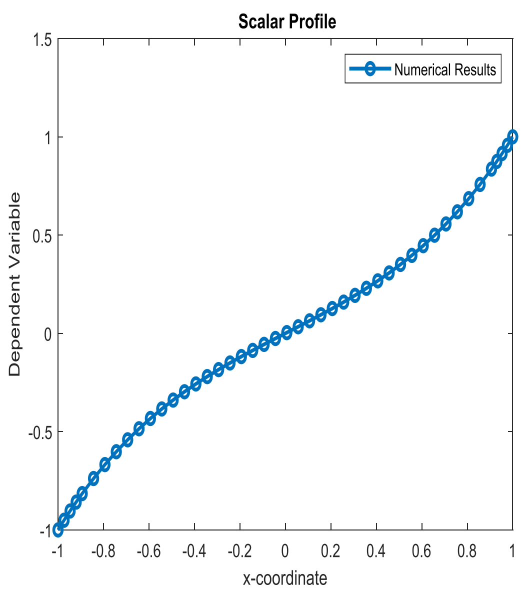

Fig. 2a : Numerical solution profile

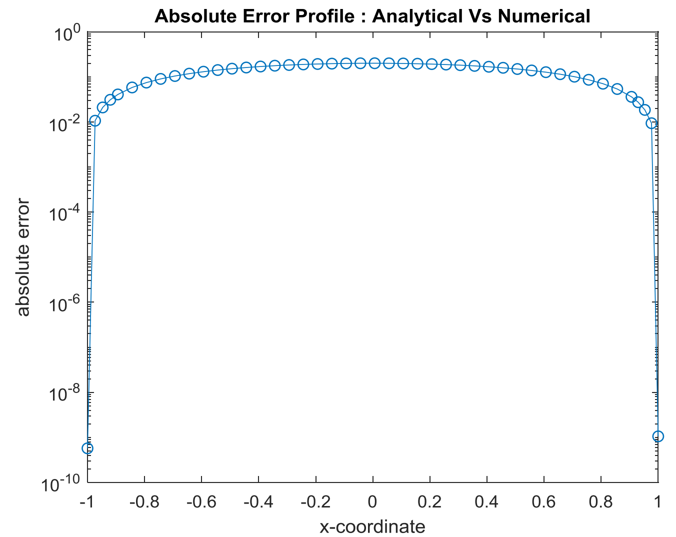

Fig. 2b : Absolute error profile

3.3.Example 3

This is a Fredholm type Integro-differential equation given as:

$$u'' = \frac{2u^2}{(t+1)} - t + \int_0^1 t(t+1)u\,dt \qquad u(0) = 1,\ u(1) = 0.5 \quad (9a)$$

The analytic solution is

$$u(t) = 1/(t+1) \qquad (9b)$$

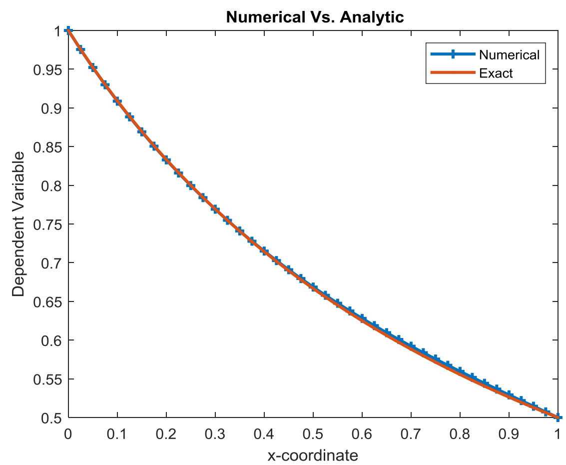

Fig. 3a Comparison of numerical and analytic solution

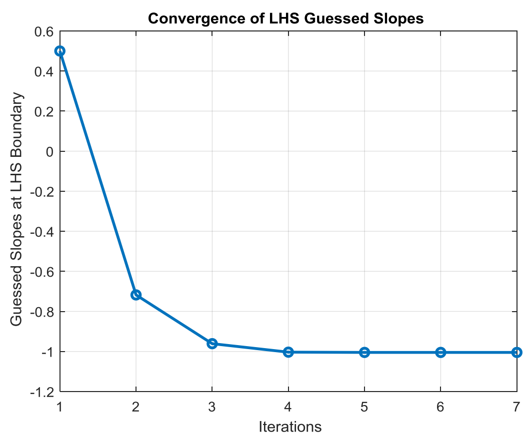

Fig. 3b : Convergence of guessed slopes per iteration

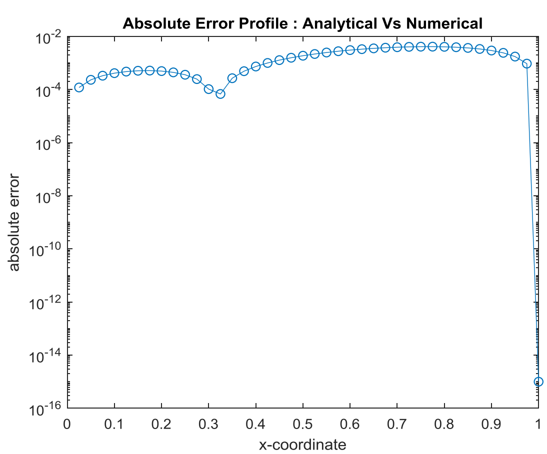

Fig. 3c. Absolute error profile

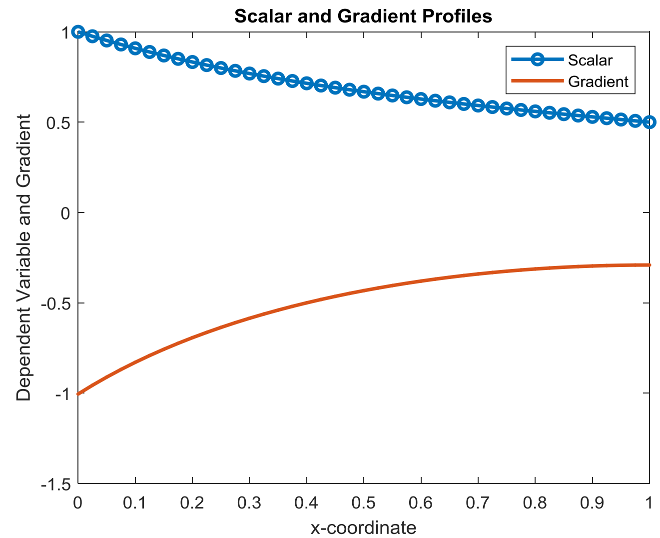

Fig. 3d : Scalar profile and gradient

3.4.Example 4

This example is a Volterra type Integro-differential equation taken from Abdella and Ross [7].

$$u'' + \frac{1}{\left(1+u^2\right)} + te^u + \int_0^t (tsu)\,ds = f(t) \quad (10a)$$

$$u(0) = 1, u(1) = 2 \quad (10b)$$

$$f(t) = 2 + \frac{1}{1+\left(1+t^2\right)^2} + te^{\left(1+t^2\right)} + \frac{t^3}{4}\left(2+t^2\right) \quad (10c)$$

The results displayed are quite reliable. The boundary conditions are not only satisfied in figure 4a, the reliability of the algorithm is also confirmed by the closeness of analytic and numerical results. Fig. 4b is a candid picture of what obtains at the left end boundary as the shooting iteration proceeds. Essentially, convergence starts at the fourth iteration as the difference between the ‘hits and target (specified Dirichlet right end boundary condition)’ gets smaller per iteration. Fig. 4c is the absolute error profile.. A salient observation shows that the magnitude of the errors increases monotonically as we move away from the left boundary, but starts decreasing as we move closer to the right boundary as the boundary conditions are satisfied at both ends. This must have been caused by the restrictive nonlinear forcing function that comes with this problem. Figure 4d shows the ascending solution profile from the left to the right boundaries. as well as the accompanying gradient.. This serves to validate the reliability of the results as well as the physics of the problem.

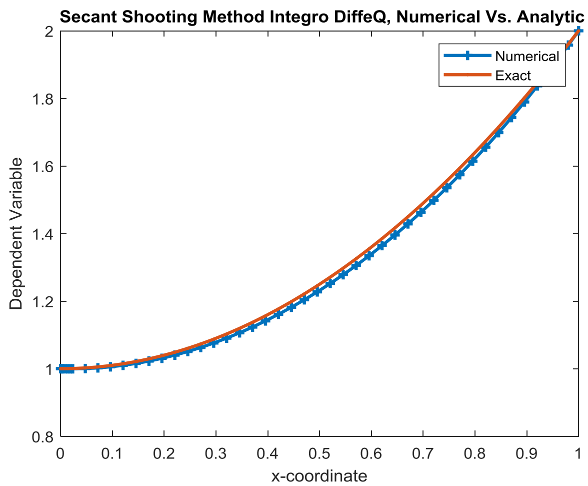

Fig. 4a : Profiles of numerical and analytic results

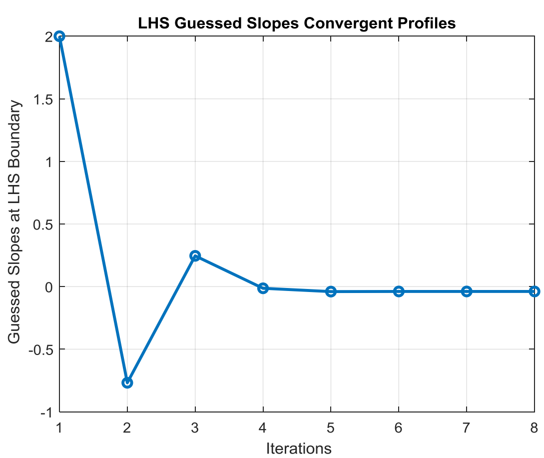

Fig. 4b : Convergence of guessed slopes at left end boundary

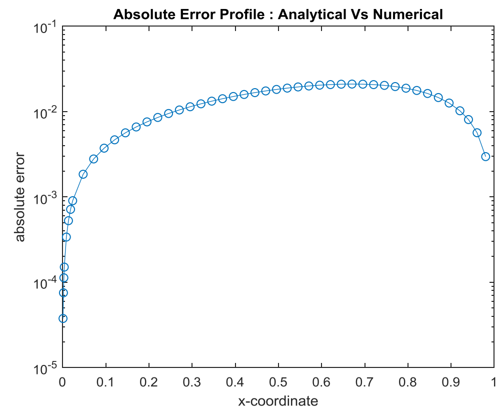

Fig. 4c: Absolute error profile

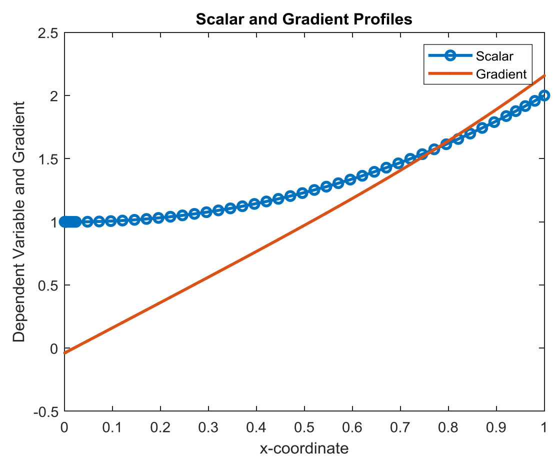

Fig. 4d: Scalar and gradient profiles

3.5.Example 5

Here we consider a Fredholm integro-differential equation with a Neumann boundary condition at the left-end of the problem interval definition as well as a singularity. We have purposely included this problem to show that the hybrid formulation is capable of dealing with different boundary conditions as well as singularity.

$$u''+\frac{2u'}{t}+\int_0^1 2s\,ds=\frac{u}{(1+u)} \qquad (11a)$$

$$u'(0)=0,\quad u(1)=1 \qquad (11b)$$

The analytic solution to this problem is unknown. Fig. 5a shows the scalar solution profile. The boundary conditions at both ends are satisfied. In addition Fig. 5b illustrates the convergence between 'hits' and 'target' as computation proceeds. Based on these results, the method has been able to cope with additional challenges contributed by this type of problem.

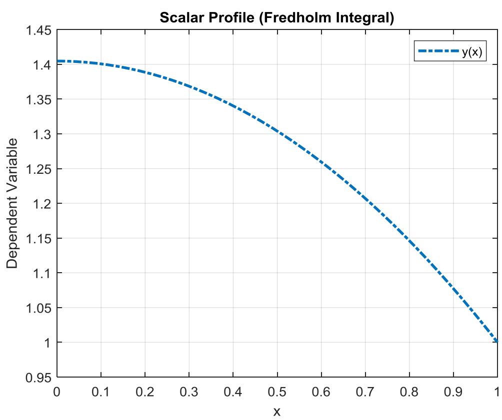

Fig. 5a : Scalar profile

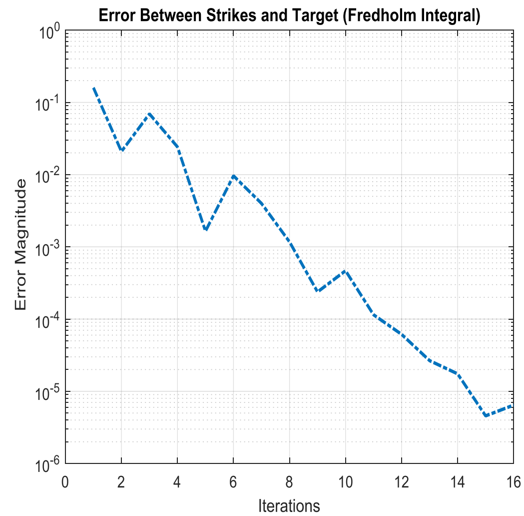

Fig. 5b: Error between strikes and target at the right end boundary

4 Conclusion

This article describes a blended numerical shooting algorithm which handles integro-differential equations straightforwardly. The effort to arrive at a simple and robust algorithm is deliberate and a key motivation. Results obtained herein confirm that the formulation is robust and the overall convergence is satisfactory as the iteration proceeds. The maximum iteration steps for all the problems handled herein is less than 10. Beyond being easy to implement, the method requires only continuity and neither derivatives nor a predictor-corrector component. More importantly because of its bracketing approach, it is better guaranteed to converge to a root. We believe that further numerical experimentation included theoretical analysis are needed to arrive at firmer conclusions.

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