Basic Concept of the Beam Wave based Element for Mid and High Frequency Analysis

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Abstract: - This paper describes a Hermite beam wave based element of the steady-state dynamic response of a 1D structure system. This study focuses on the development of beam wave based elements. Compared with standard piecewise polynomial approximation, the wave basis is shown to give a considerable reduction in computational degrees of freedom.
In practical terms, it is concluded that the degrees of freedom for which accurate results can be obtained, using these new techniques, can be up to half of that of the conventional finite-element method.

Key-Words: - finite element, Trefftz method, beam wave based element, frequency depending.

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1 Introduction

The finite element method (FEM), [1], [2], [3], is a numerical technique that makes it possible to solve approximately the differential equations or with linear partial derivatives whatever the imposed boundary conditions, especially for composite structures, [4].

However, its implementation remains difficult and costly in some cases. Indeed, the mesh must obey certain rules, in particular, the elements must not be crushed, to avoid the degeneration of the associated *Jacobian*.

It is well known that the use of discrete numerical methods (finite element method FEM) for the solution of the dynamic structure equation is limited to problems in which the wavelength under consideration is not small in comparison with the domain size. The limitation arises because conventional elements, based on polynomial shape functions, can reliably capture only a limited portion of the sinusoidal waveform. In fact, an accurate description of the problem needs the use of about eight to ten degrees of freedom per full wavelength [5], [6]. To overcome these problems, we developed a beam wave based element, this method is based on the indirect Trefftz method, [1], [7], [8], [9], [10].

In this paper, we describe the basic concept of the beam wave based element. The idea is the enrichment of the conventional shape functions by the solution of the homogeneous equation. This technique makes the formation of matrices more complicated. To illustrate this technique two examples are presented. The numerical validation of this element is made by calculating the percentage of an error on the whole structure.

2 Problem Formulation

Consider an elastic thin beam Ω_s of length L, thickness t, density ρ_s, Poisson's coefficient υ and elasticity modulus E.
The beam makes an angle α from the horizontal, the Fig.1 below shows the problem geometry.
The problem to study is governed by the dynamic equation of the structure and the boundary conditions given by the following equation:

- Dynamic equation

$$\frac{d^4 w\left(x^{'}\right)}{d^4 x^{'}} - k_b^4 w\left(x^{'}\right) = \frac{q\left(x^{'}\right)}{D}, \quad \text{in } \Omega_s \qquad (1)$$

With

$$k_b = \sqrt[4]{\frac{\rho_s t \omega^2}{D}} : \text{Structural bending} \qquad (1)$$

$$D = \frac{Et^3}{12\left(1-\upsilon^2\right)} : \text{Bending stiffness.} \qquad (2)$$

- Boundary Conditions
  - Clamped - clamped beam

$$\begin{cases} w(0)=0 \\ w(L)=0 \\ \dfrac{dw}{dx'}(0)=0 \\ \dfrac{dw}{dx'}(L)=0 \end{cases} \tag{3}$$

- simply supported beam

$$\begin{cases} w(0)=0 \\ w(L)=0 \\ \dfrac{d^2w}{d^2x'}(0)=0 \\ \dfrac{d^2w}{d^2x'}(L)=0 \end{cases} \tag{4}$$

Either the virtual displacement $v(M)$, arbitrary and regular in the domain Ω_s, the weighting of the structure dynamic equation by $v(M)$ leads after integrations to:

$$\int_{\Omega_s} \left(\frac{d^4 w(x')}{d^4 x'} - k_b^4 w(x') - \frac{q(x')}{D} \right) v(x')\, d\Omega = 0 \quad (6)$$

$\forall v$ kinematically admissible

The mathematical transformation by two integrations by parts we arrive at the following weak form:

$$\int_{\Omega_s} \frac{d^2w(x')}{d^2x'} \frac{d^2v(x')}{d^2x'} d\Omega - \int_{\Omega_s} k_b^4 w(x') v(x') d\Omega - \int_{\Omega_s} \frac{q(x')}{D} v(x') d\Omega \quad (5)$$
$$+v(L)\frac{d^3w}{d^3x'}(L) - v(0)\frac{d^3w}{d^3x'}(0) - \frac{dv}{dx'}(L)\frac{d^2w}{d^2x'}(L) + \frac{dv}{dx'}(0)\frac{d^2w}{d^2x'}(0) = 0$$

In the case of a beam simply supported on both sides the displacement and the bending moment are zero where the second derivatives of w are zero and we can take $v(M)$ equal to zero at $x'=0$ and $x'=L$.

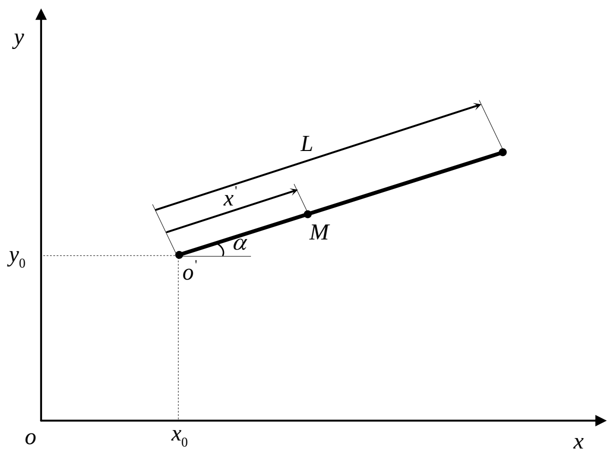

Fig.1: Elastic thin beam.

3 Finite Element Approximations

The FEM, [11], [12], is a well-known simulation technique to model the steady-state dynamic behaviour of complex structures. The technique determines an approximate solution to the problem described by the beam dynamic equation (1) and the imposed structural boundary conditions (4) and (5).
The finite element used in this study is the beam linear finite element with two degrees of freedom per node. Fig.2 shows the geometry of the beam element and these degrees of freedom.

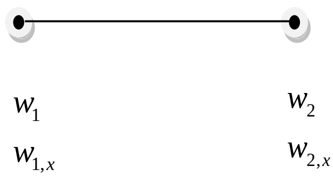

Fig. 2: Finite beam element.

The FEM approximates the exact solution for each of the structural deformation fields by a weighted sum of simple (polynomial) shape functions.
The displacement of the structure is approximated on a finite element by:

$$w = \left\langle N_s^1, N_s^2, N_s^3, N_s^4 \right\rangle \begin{Bmatrix} w_1 \\ w_{1,x} \\ w_2 \\ w_{2,x} \end{Bmatrix} = \left\langle N_s \right\rangle \{u_n\} \quad (6)$$

With $\{N_s^m; m=1,2,3,4\}$ are the shape functions of high precision of Hermite type given by: (9)

$$N^s(\xi) = \frac{1}{4}\left\langle (1-\xi)^2(2+\xi) \quad \frac{l}{2}(1-\xi^2)(1-\xi) \quad (1+\xi)^2(2-\xi) \quad \frac{l}{2}(-1+\xi^2)(1+\xi) \right\rangle$$

Applying a Galerkin weighted residual formulation, [10], these functions are expressed as a linear

combination of the same basis functions as used in the deformation approximations (8)
The equation (5) is written in the following matrix form: (10)

$$\langle v\rangle \int_0^L \left[\{N_{s,xx}\}\langle N_{s,xx}\rangle - k_b^4\{N_s\}\langle N_s\rangle\right]dx\{u_n\} - \langle v\rangle\int_0^L \{N_s\}\, f(x)\,dx =$$

$$-\langle v(0) \quad -v_{,x}(0) \quad v(L) \quad -v_{,x}(L)\rangle \begin{Bmatrix} \frac{d^3w}{d^3x^{'}}(0) \\ \frac{d^2w}{d^2x^{'}}(0) \\ \frac{d^3w}{d^3x^{'}}(L) \\ \frac{d^2w}{d^2x^{'}}(L) \end{Bmatrix}$$

4 Enriched Finite Element

The idea is to enrich the basis of the standard finite elements with a base derived from the homogeneous solution of the dynamic equation, [13], of the structure.

The solutions of the homogeneous equation are given by:

$$w_n\psi_n = w_n e^{-j^n k_b x^{'}} \qquad (7)$$

With $n \in [\![1,4]\!]$

We enriched the shape functions **Σφάλμα! Το αρχείο προέλευσης της αναφοράς δεν βρέθηκε.** on the basis of the structure mode, using some of the propagating modes $\left(\psi_1 = e^{-ik_b x},\ \psi_2 = e^{ik_b x}\right)$.

Therefore the new shape functions are given by:

$$\aleph = \langle N_1\mathbf{\Psi}\ \ N_2\mathbf{\Psi}\ \ N_3\mathbf{\Psi}\ \ N_4\mathbf{\Psi}\rangle = \frac{1}{4}\langle(1-\xi)^2(2+\xi)\mathbf{\Psi}\quad \frac{l}{2}(1-\xi^2)(1-\xi)\mathbf{\Psi}\quad (1+\xi)^2(2-\xi)\mathbf{\Psi}\quad \frac{l}{2}(-1+\xi^2)(1+\xi)\mathbf{\Psi}\rangle \quad (12)$$

With $\mathbf{\Psi} = \langle \psi_1 \quad \psi_2\rangle$ and l is the element length.
The displacement of the structure will be approximated on an element by:

$$w^e = \aleph\{w_i\} \qquad (8)$$

With $\{w_i : i = 1,2,\dots,8\}$ are the structure waves amplitudes.
The test function v is chosen equal to the conjugated shape function.
Fig. 3 shows the geometry of the beam-enriched element and these degrees of freedom.

Fig. 3: Enriched finite beam element.

5 Numerical Results

A comparison between the numerical results obtained by the enriched finite element and the standard finite element is made. The example of a simply supported beam is presented, and two cases are studied. The first case is the case of a load distributed over the beam and the second is the case of a concentrated force applied in the middle of the beam.
The percentage of error between numerical values and analytical ones in the middle of the beam is calculated.
The error according to the number of degrees of freedom is presented.

The percentage of relative error, [14], is given by:

$$Error(\%) = 100 \times \frac{\|w_{Analytic} - w_{Numerical}\|}{\|w_{Analytic}\|} \qquad (14)$$

With
$w_{Analytic}$: The analytic displacement of the beam,
$w_{Numerical}$: The numerical displacement of the beam.

In this study we use an aluminum beam whose characteristics are the following:
$\rho_s = 2790\ m^3/Kg$: Density,
$I = 70 \times 10^9\ MPa$: elasticity modulus,
$\upsilon = 0.3$: Poisson's coefficient,
$I = 0.25 \times 10^{-4}\ Kg.m^2$: Moment of inertia.

The displacement of the beam can be decomposed on its modal base as follows:

$$w = \sum_{n=1}^{\infty} \varphi_n(x')\delta_n \qquad (9)$$

With
φ_n : Mode n,
δ_n : Modal component of displacement corresponds to the mode n.

In the case of a simply supported beam the modes of the beam can be written as [15], [16]:

$$\varphi_n = \sqrt{\frac{2}{\rho_s L}} \sin\left(\frac{n\pi}{L} x'\right) \qquad (10)$$

And the modal component of the displacement is given by:

$$\delta_n = \frac{\int_0^L q(x')\varphi_n(x')dx'}{(\omega_n^2 - \omega^2)} \qquad (11)$$

With

$$\omega_n = (n\pi)^2 \sqrt{\frac{D}{mL^4}} \qquad (12)$$

5.1 Loading Distributed on the Beam

Fig.4 shows a simply supported beam excited by a distributed load $q = 1000N$.

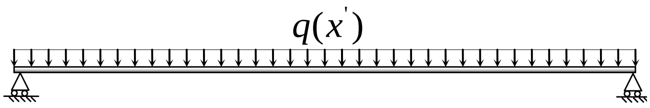

Fig. 4: Simply supported beam with a distributed loading.

The analytic displacement of the beam writes:

$$w(x') = \sum_n \frac{-2q}{n\pi m(\omega_n^2 - \omega^2)}((-1)^n - 1)\sin\left(\frac{n\pi}{L}x'\right) \qquad (19)$$

Subsequently, we present in Fig.5, Fig.6, Fig.7, Fig.8, Fig.9, Fig.10 the error according to the number of degrees of freedom for different frequencies of excitations.

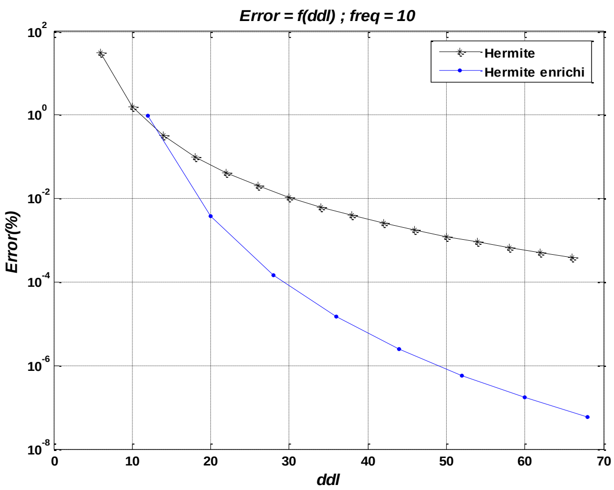

Fig. 5: Error according to degrees of freedom for $f = 10Hz$.

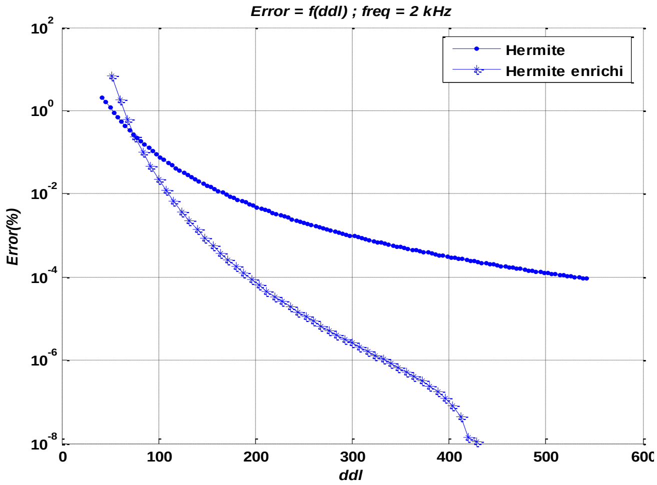

Fig. 6: Error according to degrees of freedom for $f = 2\,KHz$.

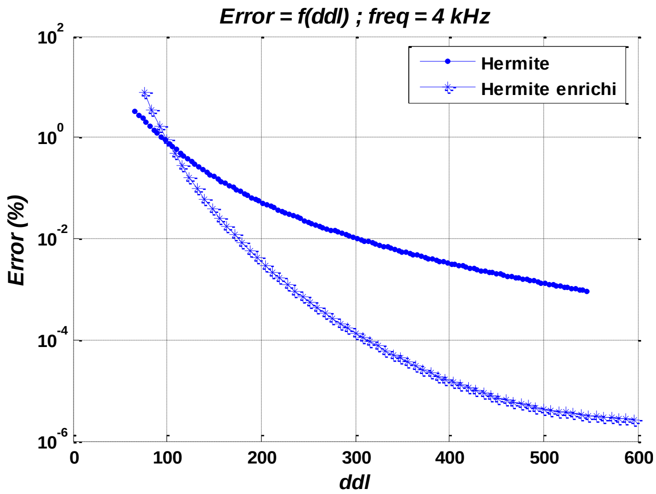

Fig. 7: Error according to degrees of freedom for $f = 4\,KHz$.

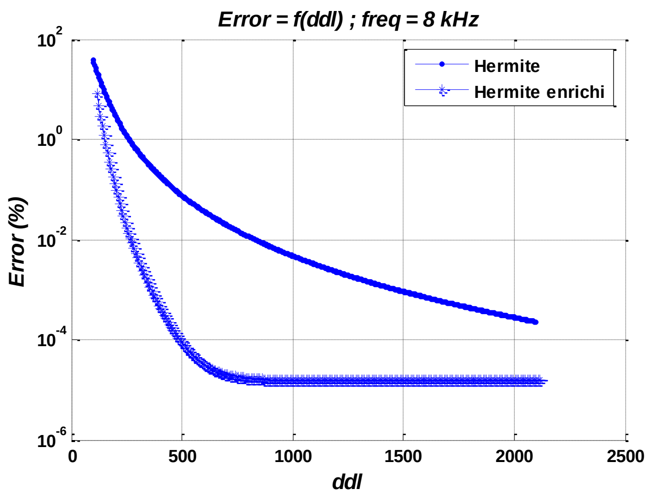

Fig. 8: Error according to degrees of freedom for $f = 8\,KHz$.

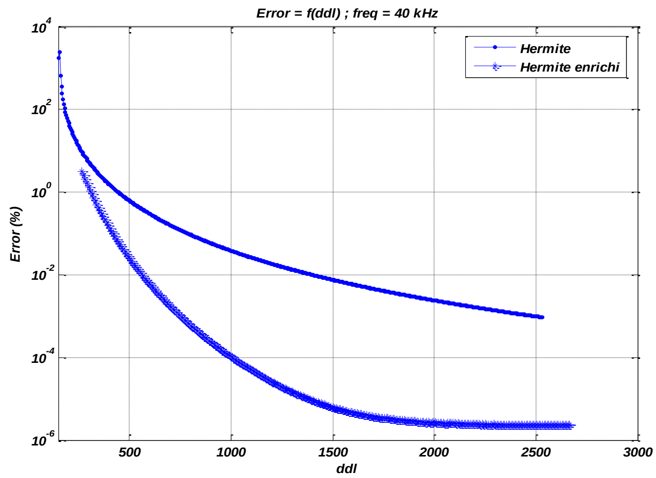

Fig. 9: Error according to degrees of freedom for $f = 40\ KHz$.

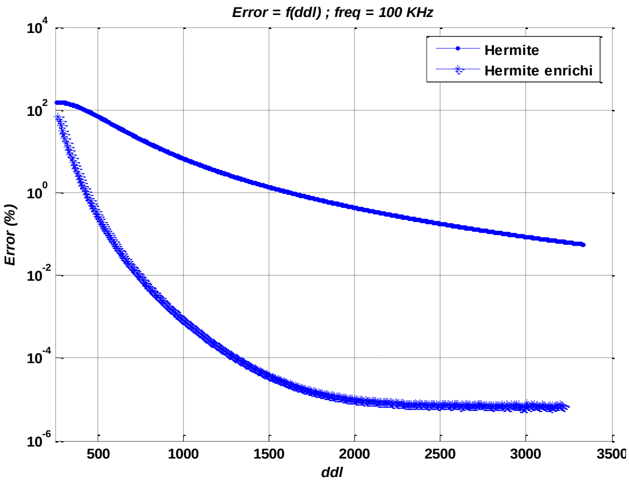

Fig. 10: Error according to degrees of freedom for $f = 100\ KHz$.

According to these results, it is noted that to have the same error for the two elements, it is necessary to use more than the double degrees of freedom for the not enriched Hermite element. And we note that the enriched element converges faster than the non-enriched Hermite element.

5.2 Concentrated Force Applied in the Middle of the Beam

Fig. 11 shows a simply supported beam submitted to a concentrated force in the middle.

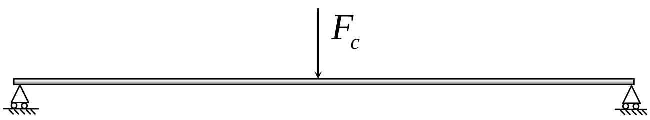

Fig. 11: Concentrated force.

The analytical displacement of the beam writes:

$$w(x') = \sum_n \frac{F_c}{(\omega_n^2 - \omega^2)} \frac{2}{mL} \sin\left(\frac{n\pi}{L} x'_M\right) \sin\left(\frac{n\pi}{L} x'\right) \quad (20)$$

Subsequently, we present in Fig.12, Fig.13, Fig.14, Fig.15, Fig.16, and Fig.17 the error according to the number of degrees of freedom for different frequencies of excitations.

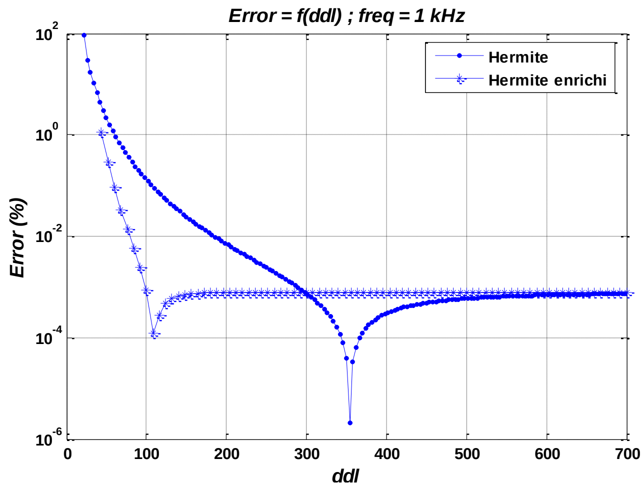

Fig. 12: Error according to degrees of freedom for $f = 1\ KHz$, concentrated force case.

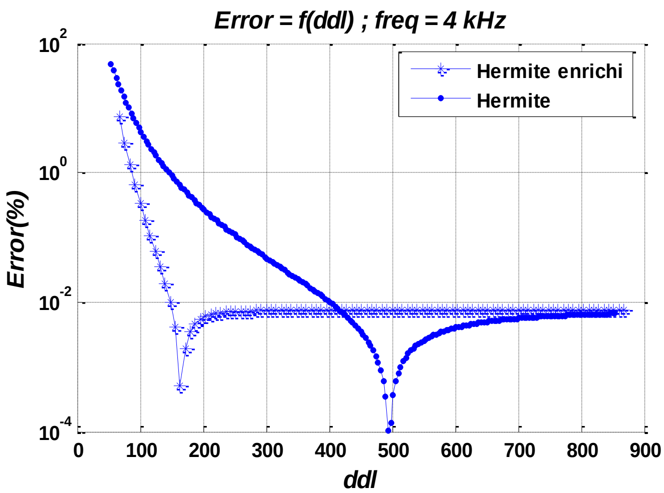

Fig. 13: Error according to degrees of freedom for $f = 4\ KHz$, concentrated force case.

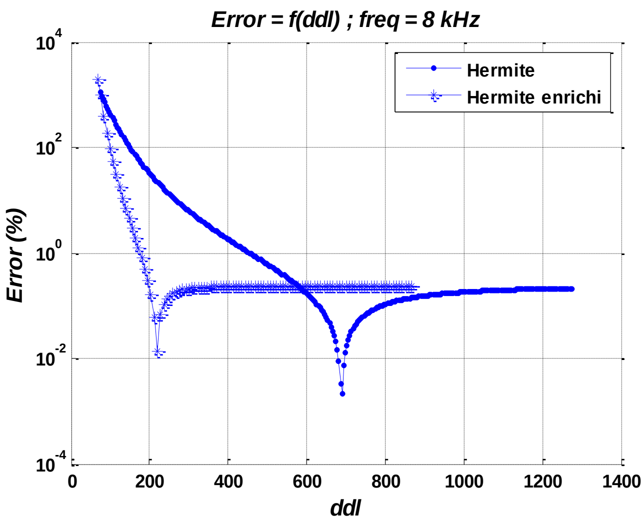

Fig. 14: Error according to degrees of freedom for $f = 8\ KHz$, concentrated force case.

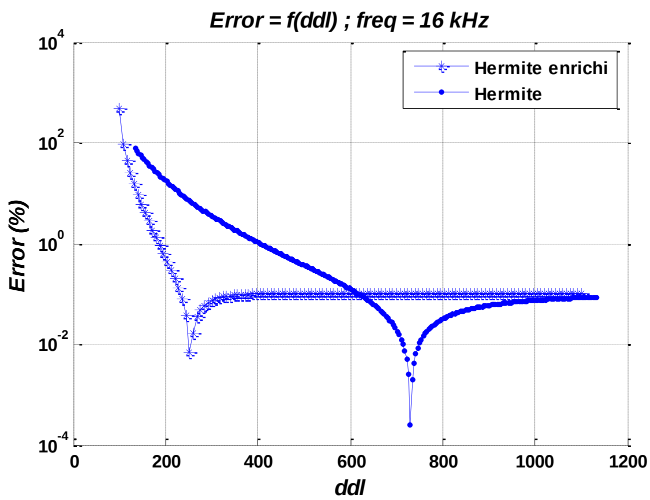

Fig. 15: Error according to degrees of freedom for $f = 16\ KHz$, concentrated force case.

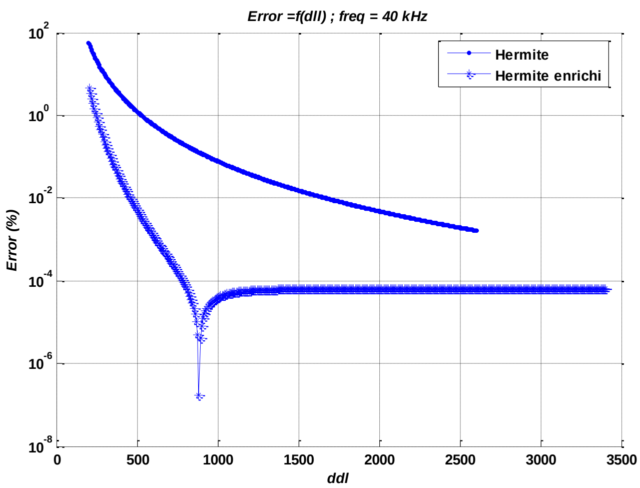

Fig. 16: Error according to degrees of freedom for $f = 40\ KHz$, concentrated force case.

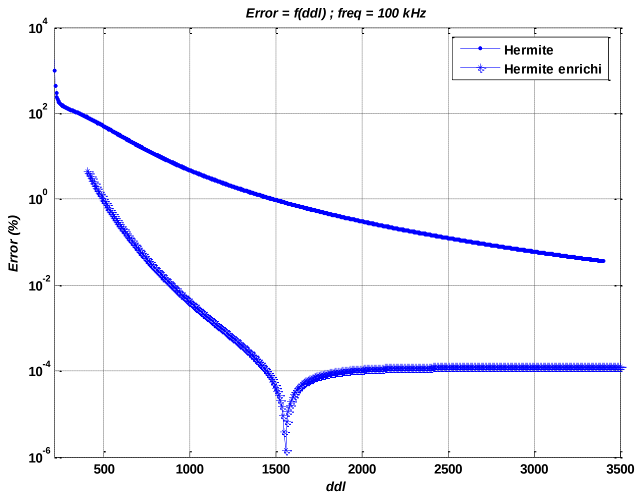

Fig. 17: Error according to degrees of freedom for $f = 100\ KHz$, concentrated force case.

These results show the efficiency of the Hermite-enriched element developed in low, medium, and high frequencies.
According to these results, we note that the enriched element converges faster than the non-enriched Hermite element, in addition, the use of this element allows us to reduce the number of degrees of freedoms necessary to half.

6 Conclusion

This article describes the beam plane wave element. This paper aimed to study this enriched element according to the frequency so the comparison with the standard finite element.

This element is of Hermite type enriched by a base deduced from the homogeneous solution of the dynamic equation of the structure. The validation was done by treating two examples of a simply supported beam. The first was the case of a distributed constant loading and the second was the case of a concentrated force. The results found showed the effectiveness of the developed element at low, medium, and high frequencies. Thus, these results showed that the developed enriched Hermite elements converged faster than those of the non-enriched Hermite type.

The obtained results show that, while increasing the frequency of excitation, the necessary number of degrees of freedom for the solution of problems with a given level of error decreases. So the results prove that the enriched element converges more quickly than the Hermite non-enriched element.
From the perspective of the continuity of this work and to broaden its field of application, it would be interesting to develop the extension of the method to

cases of problems associated with 2D, 3D, and composite material.

References:

[1] P. Bettess, Short-wave scattering: problems and techniques, *Phil.Trans. R. Soc. Lond.*, 362:421–443,2004

[2] *(text in French)* Zienkiewicz R. L. Taylor, La méthode des éléments finis : formulation de base et problèmes linéaires, *trad. par Jacques-Hervé Saïac, Jérôme Jaffré, Michel Kern... et al. .- [4e éd.] .- Paris-La Défense : AFNOR (Paris), Impr. Jouve .- XVIII-620 p : ill., couv. ill : 24 cm .- AFNOR technique - ISBN 2-12-301111-8 (rel.)* , 1991

[3] PINCA, Camelia Bretotean, TIRIAN, Gelu Ovidiu, et JOSAN, Ana., Application of finite element method to an overhead crane bridge. *Wseas transactions on applied and theoretical mechanics*,vol.4, no2,p.64-73, 2009.

[4] Eva Kormaníková, Kamila Kotrasová, Dynamic Behavior of Composite Sandwich Panel with CFRP Outer Layers, *WSEAS Transactions on Applied and Theoretical Mechanics*, vol. 17, pp. 263-269, 2022

[5] E. Perrey-Debain, J. Trevelyan, P. Bettess, Wave boundary elements: a theoretical overview presenting applications in scattering of short waves, *Engineering Analysis with Boundary Elements*, 28 (2004) 131–141

[6] F. Ihlenburg, Finite Element Analysis of Acoustic Scattering, *Springer- Verlag, New York*, 1998.

[7] M. Gyimesi, I. Tsukerman, D. Lavers, T. Pawlak and D. Ostergaard, Hybrid finite element-Trefftz method for open boundary analysis, *in IEEE Transactions on Magnetics*, vol. 32, no. 3, pp. 671-674, May 1996

[8] Adam Wroblewski, Andrzej P. Zielinski, sarvey and applications of special purpose T-complete systems, *Computational Fluid and Solid Mechanics*, 2003

[9] J. Jirousek, M. N'diaye, hybrid –Trefftz p-method elements for analysis of flat slabs with drops, *Computers & Structures*, Vol. 43. No. 1. pp. 163-179. 1992

[10] J. Jirousek, A. Wriiblewski, T-elements: a finite element approach with advantages of boundary solution methods, *Advances in Engineering Software*, 24 (1995) 71-88

[11] T.W. Preston, A.B.J. Reece, and P.S. Sangha, Induction motor analysis by time-stepping techniques, *IEEE Transactions on Magnetics*, Vol.24, No.1, 1988, pp. 471-474.

[12] Meknani Bassem, Messaoudi Rima, Talaat Abdelhamid, Nasserdine Kechkar, Ehab S.Selima, Numerical solution of quadratic general Korteweg-de Vries equation by Galerkin quadratic finite element method, *WSEAS Transactions on Mathematics*, Volume 17, 2018, pp. 220-228

[13] Soufien Essahbi, Emmanuel Perrey-Debain, Mabrouk Ben Tahar, Lotfi Hammami, Mohamed Haddar, Plane wave based method: Analytic integration and frequency behaviour, *WSEAS Transactions on Applied and Theoretical Mechanics*, Issue 1, Volume 7, January 2012.

[14] Ioannis Doltsinis, Spring Cell Equivalence of Simplex Finite Elements – Exploration of an Iterative Approach, *WSEAS Transactions on Applied and Theoretical Mechanics*, vol. 15, pp. 222-235, 2020

[15] B. Drouin , J.- M. Senicourt, F. Lavaste, G. Frezans, De la mécanique vibratoire classique à la méthode des éléments finis, *Volume 1 AFNOR*, 1993 ISBN : 2-12-309111-1

[16] Carl Q. Howard, Modal mass of clamped beams and clamped plates, *Journal of Sound and Vibration*, 301(2007), 410 – 414.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Soufien ESSAHBI contributed to this research at all stages, from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflict of interest to declare that is relevant to the content of this article.