Isochronous oscillatory motions and the quantum spectrum

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Abstract: Necessary and sufficient conditions for isochrony of oscillatory motions introduced in the paper "Physica Scripta vol 94, N 12" are discussed. Thanks to the WKB perturbation method expressions are derived for the corrections to the equally spaced valid for analytic isochronous potentials.
In this paper, we bring some improvements and we suggest another quantization of the quantum spectrum. These results will be illustrated by several examples

Key–Words: oscillatory motions, isochronicity, WKB method, quantum spectrum

Received: April 17, 2022. Revised: November 11, 2022. Accepted: December 12, 2022. Published: December 31, 2022.

1 Introduction

Consider the Schrodinger equation

$$\mathcal{H}\Psi = E\Psi$$

Ψ being the wave function associated with the state of the particle and

$$\mathcal{H} = \frac{p^2}{m} + G$$

the Hamiltonian operator who describes this evolution.

In the sequel we are interested in the case of isochronous potentials. This means the frequency of the classical motion in such potentials is energy-independent, it is natural to expect their quantum spectra to be equally spaced. However, as it has already been shown in some specific examples, this property is not always true.

This second order partial differential equation is linear and homogeneous. This is not trivial to solve in the case of complex potentials, apart from numerical resolution. There is however an approximate method of resolution, the WKB approximation, named after the physicists Wentzel, Kramers and Brillouin. This approximation is based on the fact that the solutions of the Schrodinger equation can be approximated by a function comprising usually conventional quantities, provided that the potential does not vary strongly over distances of the order of the length of wave.

These is a connection between classical and quantum transformations. This fact has been established by Eleonskii and al. [2]. They show that the classical limit of the isospectral transformation for the Schrodinger equation is precisely the isochronicity preserving the energy dependence of the oscillation frequency. In quantum mechanics, the energy levels of a parabolic well are regularly spaced by a certain quantity. Moreover, it is possible to construct potentials, essentially different from the parabolic well, whose spectrum is exactly harmonic.

The semiclassical WKB method is one of powerful approximations for computing the energy eigenvalues of the Schrodinger equation. The field of its applicability is larger than standard perturbation theory which is restricted to perturbing potentials with small coupling constants. In particular, it permits to write the quantization condition as a power series in $\hbar$ (such series are generally non convergent). The solvable potentials are those whose series can be explicitly summed. This problem has motivated a lot of authors who highlight some exactly solvable, in a sense that the exact eigenenergies and eigenfunctions can be obtained explicitly, see [3] for example. Our method described below permits also another approach of two-dimensional superintegrability, see [4].

Consider the scalar equation with a center at the origin 0

$$\ddot{x} + g(x) = 0 \qquad (1)$$

or its planar equivalent system

$$\dot{x} = y, \qquad \dot{y} = -g(x) \qquad (2)$$

where $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}$ and $g(x) = \frac{dG(x)}{dx}$ is analytic on R where $G(x)$ is the potential of (1).

Suppose system (2) admits a periodic orbit in the phase plane with energy E and $g(x)$ has bounded period for real energies E. Given $G(x)$, Let $T(E)$ denotes the minimal period of this periodic orbit. Its expression is

$$T(E) = 2\int_a^b \frac{dx}{\sqrt{2E - 2G(x)}}. \qquad (3)$$

$T(E)$ is well defined and there is a neighborhood of the real axis for which $T(E)$ is analytic.

We suppose that the potential $G(x)$ has one minimum value which, for convenience locate at the origin 0 and $\frac{d^2G(x)}{dx^2}(0) = 1$. The turning points a, b of this orbit are solutions of $G(x) = E$. Then the origin 0 is a center of (2). This center is isochronous when the period of all orbits near $0 \in R^2$ are constant ($T = \frac{2\pi}{\sqrt{g'(0)}} = 2\pi$). The corresponding potential $G(x)$ is also called isochronous.

Since the potential $G(x)$ has a local minimum at 0, then we may consider an involution A by

$$G(A(x)) = G(x) \text{ and } A(x)x < 0$$

for all $x \in [a, b]$. So, any closed orbit is A-invariant and A exchanges the turning points: $b = A(a)$.

We proved the following results in [1]

Theorem 1-1 *Let $g(x)$ be an analytic function and $G(x) = \int_0^x g(s)ds$ and A be the analytic involution defined by $G(A(x)) = G(x)$. Suppose that for $x \neq 0, xg(x) > 0$. Then the equation*

$$\ddot{x} + g(x) = 0 \qquad (1)$$

has an isochronous center at 0 if and only if the function

$$\frac{d}{dx}[G(x)/g^2(x)]$$

is A-invariant i.e. $\frac{d}{dx}[G/g^2](x) = \frac{d}{dx}[G/g^2](A(x))$ in some neighborhood of 0. .

Theorem 1-2 *Let $G(x) = \int_0^x g(s)ds$ be an analytic potential. Suppose that for $x \neq 0, xg(x) > 0$. Then the equation*

$$\ddot{x} + g(x) = 0 \qquad (1)$$

has an isochronous center at 0 if and only if

$$x - \frac{2G}{g} = F(G) \qquad (4)$$

where F is an analytic function defined in some neighborhood of 0.

2 An alternative result

As consequences we prove the following

Theorem 2-1 *Let $G(x) = \int_0^x g(s)ds$ be an analytic potential defined in a neighborhood of 0. Suppose equation*

$$\ddot{x} + g(x) = 0 \qquad (1)$$

has an isochronous center at 0. Let $g^{(n)}(x)$ be the n-th derivative of the potential (with respect to x): $g^{(n)}(x) = \frac{d^n}{dx^n}G(x)$, $n \geq 1$ then these derivatives may be expressed under the form

$$g^{(n)}(x) = a_n(G)x + b_n(G), \quad n \geq 0 \qquad (5)$$

where a_n and b_n are analytic functions with respect to G.

In fact, as we had see in [1], the functions a_n and b_n are only dependent on G_1 the odd part of $G = G(x)$.

Proof By Proposition 3-4 of [1], condition $x(G) = \sqrt{2G} + P(G)$ with $P = P(G)$ is a non-zero analytic function implies that equation (1) has an isochronous center at 0. Deriving with respect to G one obtains

$$\frac{dx}{dG} = \frac{1}{x} + P'(G) = \frac{1}{g}$$

or equivalently

$$\frac{g}{x} = \frac{1}{1 + xP'(G)} = a_1(G) + \frac{b_1(G)}{x}$$

with

$$a_1(G) = \frac{-1}{2GP'^2 - 1} \quad and \quad b_1(G) = \frac{2GP'}{2GP'^2 - 1}.$$

Notice that by hypothesis G is defined defined in a neighborhood of 0 then $2GP'^2 - 1$ is necessary non zero.
The functions $a_1(G)$ and $b_1(G)$ are analytic since P and P' they are too.
Derive now $g'(x)$ it yields

$$g'(x) = \frac{dg}{dx} = \frac{d}{dx}a_1(G)x + a_1(G) + \frac{d}{dx}b_1(G) =$$

$$\frac{d}{dG}a_1(G)gx + \frac{d}{dG}b_1(G)g + a_1(G).$$

$$g'(x) = \left(\left(\frac{d}{dG}a_1\right)x + \frac{d}{dG}b_1\right)(a_1x + b_1) + a_1(G)$$

where the symbol prime $'$ means $\frac{d}{dG}$ and a_1 or b_1 stands for $a_1(G)$ or $b_1(G)$.

After replacing $g(x) = a_1(G)x + b_1(G)$ one obtains

$$2\,Ga_1'a_1 + a_1'xb_1 + {a_1}^2 + 1/2\,\frac{a_1\sqrt{2}b_1}{\sqrt{G}} + b_1'a_1x + b_1'b_1$$

By simplifying one find the expression of $g'(x) = a_2(G)x + b_2(G)$ with

$$a_2(G) = a_1'b_1 + \frac{a_1b_1}{2G} + b_1'a_1$$

$$b_2(G) = 2Ga_1a_1' + a_1^2 + b_1b_1'$$

Here

$$\frac{a_1b_1}{2G} = \frac{P'}{(2GP'^2 - 1)^2}$$

which is analytic. Then the functions $a_2(G)$ and $b_2(G)$ are analytically dependent on the functions $a_1(G), b_1(G)$ and their derivatives.

By recurrence we easily prove that

$$g^{(p)}(x) = a_p(G)x + b_p(G)$$

where the function $a_p(G)$ and $b_p(G)$ are analytic with respect to G. Thank to *Maple* we are able to carry out the calculations.

Theorem 2-2 *Let $G(x) = \int_0^x g(s)ds$ be an analytic potential and $\phi(x)$ a function defined in a neighborhood of 0. A be the analytic involution defined by $G(A(x)) = G(x)$. Then for $a < 0 < b = A(a)$ and $G(a) = G(b) = E$ the following integrals equality holds*

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = \int_0^b \frac{\phi(x) - \phi(A(x))}{\sqrt{E - G(x)}}g(x)dx$$

In particular, if we may expressed $\phi(x) = u(G)x + v(G)$ then

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = \int_0^b \frac{2u(G)x}{\sqrt{E - G(x)}}g(x)dx$$

Proof It suffices to split the integral

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = \int_a^0 \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx +$$

$$\int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx$$

Recall that $a < 0 < b$. By definition when $x \in [a, 0]$ then $A(x) \in [0, b]$ and conversely. By a change of variable $x = A(y)$ the integral becomes

$$\int_a^0 \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = -\int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}}g((A(y))A'(y)dy =$$

$$-\int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}}g(y)dy$$

since $g((A(y))A'(y) = g(y)$. Therefore

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = \int_0^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx -$$

$$\int_0^b \frac{\phi(A(y))}{\sqrt{E - G(y)}}g(y)dy = .$$

On the other hand, suppose $\phi(x) = u(G)x + v(G)$. Then the following integral may be written

$$\int_a^b \frac{\phi(x)}{\sqrt{E - G(x)}}g(x)dx = \int_a^b \frac{u(G)x + v(G)}{\sqrt{E - G(x)}}g(x)dx =$$

$$\int_a^b \frac{u(G)x}{\sqrt{E - G(x)}}g(x)dx + \int_a^b \frac{v(G)}{\sqrt{E - G(x)}}g(x)dx$$

The last integral can be written

$$\int_a^b \frac{v(G)}{\sqrt{E - G(x)}}g(x)dx = \int_0^E \frac{v(G)}{\sqrt{E - G}}dG = 0$$

since $v(G)$ is analytic. The other integral can be written

$$\int_a^b \frac{u(G)x}{\sqrt{E - G(x)}}g(x)dx = \int_a^0 \frac{u(G)x}{\sqrt{E - G(x)}}g(x)dx +$$

$$\int_0^b \frac{2u(G)x}{\sqrt{E-G(x)}} g(x)dx$$

$$= \int_0^b \frac{u(G)y}{\sqrt{E-G(y)}} g(A(y))A'(y)dy =$$

$$\int_0^b \frac{-u(G)y}{\sqrt{E-G(y)}} g(y)dy = \int_0^b \frac{u(G)y}{\sqrt{E-G(y)}} g(y)dy$$

since $y = A(x)$. Finally,

$$\int_a^b \frac{\phi(x)}{\sqrt{E-G(x)}} g(x)dx = \int_0^b \frac{u(G)x}{\sqrt{E-G(x)}} g(x)dx +$$

$$\int_0^b \frac{u(G)x}{\sqrt{E-G(x)}} g(x)dx$$

We may also derive

Corollary 2-3 *Under hypotheses of Theorem 2-1, consider the derivatives of g :* $g^{(j)}(x) = \frac{d^j g}{dx^j}$ *Then the analytic function*

$$V_{m,\nu}(x) = \prod_{j=1}^{m} \left(\frac{d^j g}{dx^j}\right)_j^{\nu}$$

may be expressed under the form :

$$V_{m,\nu}(x) = u_{m,\nu}(G)x + v_{m,\nu}(G) \qquad (6)$$

where $\nu = (\nu_1, \nu_2, ..., \nu_m)$ *and* $u_{m,\nu}$ *and* $v_{m,\nu}$ *are analytic functions with respect to G.*

Proof By Theorem 2-1 any derivative of g may be written when G is isochronous

$$g^{(n)}(x) = a_n(G)x + b_n(G), n \geq 0$$

where a_n and b_n being analytic functions. It is easy to realize that it is the same for any power of any derivative $(g^{(n)}(x))_n^{\nu}$. We may prove that by recurrence

$$(g^{(n)}(x))_n^{\nu} = a_{n,\nu}(G)x + b_{n,\nu}(G).$$

More generally, we may also prove by recurrence that a product of power of derivatives have the similar expression

$$(g^{(n)}(x))_1^{\nu}(g^{(p)}(x))_2^{\nu} = a_{n,p,\nu}(G)x + b_{n,p,\nu}(G).$$

Thus we may write for any product

$$V_{m,\nu}(x) = \prod_{j=1}^{m} \left(\frac{d^j g}{dx^j}\right)_j^{\nu} = u_{m,\nu}(G)x + v_{m,\nu}(G)$$

3 Applications to the WKB quantization

3.1 The quantum spectrum

Consider the Schrodinger equation

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + G(x)\right]\psi(x) = E\psi(x). \qquad (7)$$

The Hamiltonian of the system is given by

$$\mathcal{H} = \frac{p^2}{m} + G(x)$$

where the mass $m = 1$.
This Hamiltonian is a constant of motion, whose value is equal to the total energy E.
The wave function can always be written as

$$\psi(x) = exp(\frac{i}{\hbar}\sigma(x))$$

The WKB expansion for the phase is a power series in $\hbar$:

$$\sigma(x) = \sum_0^{\infty} (\frac{\hbar}{i})^k \sigma_k(x).$$

Following [1], [5] rewrite the quantisation condition as

$$\sum_0^{\infty} I_{2k}(E) = (n + \frac{1}{2})\hbar, \ n \in N$$

1where

$$I_{2k}(E) = \frac{1}{2\pi}(\frac{\hbar}{i})^{2k} \int_{\gamma} d\sigma_{2k}, \ k \in N. \qquad (8)$$

When $G(x)$ is analytic and $xg(x) > 0$, it has been proved that the contour integrals can be replaced by equivalent Rieman integrals between the two turning points. More precisely,

$$I_2(E) = -\frac{\hbar^2}{24\sqrt{2}\pi}\frac{\partial^2}{\partial E^2}\int_a^b \frac{g^2(x)}{\sqrt{E-G(x)}}dx$$

and

$$I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi}[\frac{1}{120}\frac{\partial^3}{\partial E^3}\int_a^b \frac{g'^2(x)}{\sqrt{E-G(x)}}dx$$

$$-\frac{1}{288}\frac{\partial^4}{\partial E^4}\int_a^b \frac{g^2(x)g'(x)}{\sqrt{E-G(x)}}dx]$$

One establishes the following

$$\oint_\gamma d\sigma_m = 2\sum_{L(\nu)=m} \frac{2^{\frac{m}{2}-1+|\nu|}i}{(m-3+2\mid\nu\mid)!!}\frac{\partial^{\frac{m}{2}-1+|\nu|}}{\partial E^{\frac{m}{2}-1+|\nu|}}\times$$

$$\int_a^b \frac{U_\nu G^{(\nu)}}{\sqrt{E-G}}dx$$

where

$$G^{(\nu)}(x) = \prod_{j=1}^m \left(\frac{d^jG}{dx^j}\right)_j^\nu$$

Coefficients U_ν are defined by a recurrence equation.
Notice that higher order corrections quickly increase in complexity and we know only few cases where a WKB expansion can be worked to all orders. Which resulting in a convergent series whose sum is identical to the exact spectrum.

3.2 Applications

Here are some easy examples in order to verify Theorem 2-1.

3.2.1 The isotonic potential

It is known, the spectrum of a potential is generally not strictly regularly spaced, except for the harmonic $G(x) = \frac{1}{2}x^2$ and the isotonic ones, which are very particular cases :

$$G(x) = \frac{1}{8\alpha^2}[\alpha x + 1 - \frac{1}{\alpha x+1}]^2$$

There are isochronous potentials with a strictly equally spaced (harmonic) spectrum. As we have seen in [1] its inverse is

$$x = \sqrt{2G}+P(G) = \frac{\sqrt{2}\sqrt{G\alpha^2} - 1 - \sqrt{2\,G\alpha^2+1}}{\alpha} =$$

$$\sqrt{2G} - \frac{1+\sqrt{2\,G\alpha^2+1}}{\alpha}.$$

3.2.2 A generalization

Let us consider now a three-parameters family of potentials more general than the isotonic case

$$x = \frac{2G}{g} - 2\,\frac{\alpha\left(-1+\sqrt{1+\beta G}\right)}{b\sqrt{1+\beta G}},$$

where α and β are real parameters such that $2\alpha^2 \le \beta$.
Here

$$x = \sqrt{2G}+P(G) = \frac{-2\,\alpha + \sqrt{2G}\beta + \sqrt{4\,\alpha^2+4\,G\beta\alpha^2}}{\beta}$$

A resolution of these equations yields

$$G(x) = \frac{8\alpha^2 + (\beta+2\alpha^2)(4\alpha x + \beta x^2)}{2(\beta-2\alpha^2)^2} -$$

$$\frac{(4\alpha^2 + 2\alpha\beta x)\sqrt{2(2+\beta x^2+4\alpha x)}}{2(\beta-2\alpha^2)^2}$$

Then, the above potential is isochronous according to Theorem 2-1. Applying scaling property of isochronous potentials. The potentials $G(x)$ and $\frac{1}{\gamma^2}G(\gamma x)$ have the same period. That means the following three-parameters potentials family is isochronous

$$G(x) = \frac{1}{2\gamma^2}X^2(\gamma x) =$$

$$\frac{[2\alpha + \beta\gamma x - \alpha\sqrt{2(2+\beta\gamma^2x^2+4\alpha\gamma x)}]^2}{2\gamma^2(\beta-2\alpha^2)^2}$$

So the case $2\alpha^2 = \beta$ and $\gamma = 1$ yields the isotonic potential.

3.2.3 WKB corrections

By Theorem 2-1, we may write $g = \frac{dG}{dx} = a(G)x + b(G)$. Writing

$$I_2(E) = \frac{-\hbar^2}{24\sqrt{2\pi}}\frac{\partial^2}{\partial E^2}\int_a^b \frac{g^2(x)}{\sqrt{E-G(x)}}dx =$$

$$\frac{-\hbar^2}{24\sqrt{2\pi}}\frac{\partial^2}{\partial E^2}\int_a^b \frac{g(x)}{\sqrt{E-G(x)}}g(x)dx$$

and by Theorem 1-2 we may express

$$-\frac{\hbar^2}{24\sqrt{2\pi}}\frac{\partial^2}{\partial E^2}\int_0^b \frac{2a(G)x}{\sqrt{E-G(x)}}g(x)dx =$$

$$-\frac{\hbar^2}{24\sqrt{2\pi}}\frac{\partial^2}{\partial E^2}\int_0^E \frac{2a(v)\sqrt{x(v)}}{\sqrt{E-v}}dv.$$

Then making the change of variables $u = \frac{v}{E}$ (we suppose here $\omega = 1$)

$$I_2(E) = -\frac{\hbar^2}{24\pi}\frac{\partial^2}{\partial E^2}[E\int_0^1 \frac{2a(uE)x(u)}{\sqrt{1-u}}\ du].$$

A similar calculation gives the fourth order correction

$$I_4(E) = -\frac{\hbar^4}{4\sqrt{2}\pi}[\frac{1}{120}\frac{\partial^3}{\partial E^3}\int_a^b \frac{g'^2(x)}{g(x)\sqrt{E-G(x)}}g(x)dx$$

$$-\frac{1}{288}\frac{\partial^4}{\partial E^4}\int_a^b \frac{g(x)g'(x)}{\sqrt{E-G(x)}}g(x)dx].$$

By Theorem 1-1 one gets $g = \frac{dG}{dx} = a(G)x + b(G)$ and $g' = \frac{dg}{dx} = a_1(G)\sqrt{2G} + b_1(G)$. However, we may easily prove

$$\frac{g'^2}{g} = a_{1,2}(G)x + b_{1,2}(G)$$

where $a_{1,2}(G)$ and $b_{1,2}(G)$ are analytic functions. As well as

$$g(x)g'(x) = c_{1,2}(G)x + d_{1,2}(G)$$

where $c_{1,2}(G)$ and $c_{1,2}(G)$ are analytic functions. Similar to I_2, I_4 may be expressed through Abel integrals :

$$I_4(E) = -\frac{\hbar^4}{4\pi}[\frac{E^{-3}}{120}\int_0^E \frac{(x(v))^5}{\sqrt{E-v}}\frac{\partial^3}{\partial v^3}a_{1,2}(v)\ dv$$

$$-\frac{E^{-4}}{288}\int_0^E \frac{(x(v))^7}{\sqrt{E-v}}\frac{\partial^4}{\partial v^4}c_{1,2}(v)\ dv].$$

Thus one can choose $I_2(E)$ and deduce the corresponding analytic isochronous potential such that, its asymptotic decay is faster than the asymptotic decay of $I_4(E)$. Therefore, $I_2(E)$ and $I_4(E)$ grow exponentially fast as E grows to ∞. We will get the similar for higher order corrections.

Turn now to upper order WKB correction. Following [1], [3] the explicit expression for $I_{2n}(E)$ is given by

$$I_{2n}(E) = -\frac{\sqrt{2}}{\pi}\hbar^{2n}\sum_{L(\nu)=2n}\frac{2^{|\nu|}}{(2n-3+2\mid\nu\mid)!!}J_\nu(E) \tag{9}$$

where

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}\int_a^b \frac{U_\nu G^{(\nu)}}{\sqrt{E-G}}dx$$

where

$$G^{(\nu)}(x) = \prod_{j=1}^{2n}(\frac{d^jG}{dx^j})_j^\nu$$

and where $\nu = (\nu_1, \nu_2, ..., \nu_{2n}), \nu_j \in N, L(\nu) = \sum_{j=1}^{2n} j\nu_j$ and $\mid\nu\mid = \sum_{j=1}^{2n}\nu_j$. The coefficients U_ν satisfy a certain recurrence relation.

By Corollary 2-3 $G^{(\nu)}$ may be expressed under the form

$$G^{(\nu)}(x) = \prod_{j=1}^{n}\left(\frac{d^jG}{dx^j}\right)_j^\nu = u_{n,\nu}(G)x + v_{n,\nu}(G)$$

where $u_{n,\nu}$ and $v_{n,\nu}$ are analytic functions with respect to G. Therefore,

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}\int_a^b \frac{U_\nu u_{n,\nu}(G)x}{\sqrt{E-G}}g(x)dx$$

By Corollary 1-1, we can write

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}\int_0^b \frac{2U_\nu u_{n,\nu}(G)x}{\sqrt{E-G}}g(x)dx$$

$$J_\nu(E) = \frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}\int_0^E \frac{2U_\nu u_{n,\nu}(v)x(v)}{\sqrt{E-v}}dv$$

Another equivalent formulation via Abel integrals

$$J_\nu(E) = A_{(n,\nu)}\int_0^E \frac{(x(v))^{n-2+|\nu|}}{\sqrt{E-v}}\frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}u_{n,\nu}(v),$$

where $A_{(n,\nu)} = 2U_\nu E^{-n+1-|\nu|}$.

Similar to I_2 and I_4 the nth correction I_{2n} will be expressed through Abel integrals :

$$I_{2n}(E) = \frac{-\hbar^{2n}}{\pi}\sum_{L(\nu)=2n}\frac{2^{|\nu|+1}U_\nu E^{-n+1-|\nu|}}{(2n-3+2\mid\nu\mid)!!}\times$$

$$\int_0^E \frac{(x(v))^{n-2+|\nu|}}{\sqrt{E-v}}\frac{\partial^{n-1+|\nu|}}{\partial E^{n-1+|\nu|}}u_{n,\nu}(v)dv$$

where $u_{n,\nu}(G)$ is such that

$$G^{(\nu)}(x) = \prod_{j=1}^{m}\left(\frac{d^jG}{dx^j}\right)_j^\nu = u_{n,\nu}(G)x + v_{n,\nu}(G).$$

4 Conclusion

It is well known that the quantum Schrodinger solvable potentials is rather small because quantum exactly solvability is a very strong condition. We have therefore highlighted another no less complicated but interesting expression of $I_{2n}(E)$ in the general case. The natural question is which of the two is more appropriate to use. In fact, it will depend on the type of isochronous potentials that we consider. In some situations, it may be easier to use either of these formulas. As we remarked higher order corrections quickly increase in complexity. Here too the WKB corrections $I_{2n}(E)$ grow exponentially fast as E grows to ∞. The WKB series should be summed for any isochronous potential and would be finite as E grows to ∞.

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