Researching of various realizations of the semi-Markov random process with positive tendency and negative jump

U. KERIMOVA[1], V. POLADOVA[2]
[1] Azerbaijan Technical University,
AZERBAIJAN

[2] Institute of Radiation Problems,
AZERBAIJAN

Abstract: - In recent years, semi-Markov random walks have been employed to address numerous intriguing problems across fields such as inventory management, risk management, insurance, and reliability theories. Particularly, many significant problems in stock control, queue management, and reliability theory can be represented using random walks constrained by two barriers. These barriers, depending on the specific problem, may be reflecting, delaying, absorbing, elastic, or of other types. A key problem in stochastic process theory is determining the Laplace transforms for the distribution of semi-Markov random processes. In this context, this article explores semi-Markov random processes characterized by a positive tendency and negative jumps.

Key-Words: - Laplace transforms, semi-Markov random process, random variable, process with positive tendency and negative jumps

Received: April 15, 2026. Revised: May 9, 2026. Accepted: June 11, 2026. Published: July 22, 2026.

1 Introduction

A Markov process is a stochastic process that models the transition of an individual between a finite number of defined states, with the condition that the individual must occupy exactly one state at any given time. For the process to terminate, at least one state must be absorbing, meaning that once an individual enters this state, the probability of leaving it becomes zero. It is well established that most problems in stock control theory are frequently formulated using random walks or random walks with delaying barriers. Numerious studies of Levy [1], W.L.Smith [2] and L. Takacs [3] have been done about step processes of semi-Markov random walk with one or two barriers of their practical and theoretical importance. The essential developments of semi-Markov processes theorywere proposed by R. Pyke [4]-[6], E.Çinlar [7], A.Turbin [8], N. Limniosand G. Oprisan [9], D. C. Silvestrov [11], F. Grabski [12], A.A.Borovkov [14]. Random walks wandering in a stripe studied by Lotov, V. I. [10]. The asymptotic of distributions in two-sided boundary problems for random walks defined on a Markov chain is given by Harlamov B.P [13]. But in the most of these studies the distribution of the process has free distribution. Busarov V. A [15] is considered the asymptotic behavior of random wanderings in random medium with delaying screen. The various semi-Markov random walk processes with or without barriers was studied by Nasirova T.I. [16], [17], [18], [19], [20], [21]

This article focuses on the study of semi-Markov random processes with a positive tendency and negative jumps, while also addressing the lower and upper bounded semi-Markov models in resource control theory. To define the probability characteristics of the proposed model, random delays, jumps delayed at zero, and screened semi-Markov random processes are examined.

2 Problem Formulation

Let a sequence of independent and identically distributed pairs of random variables $\{\xi_k, \varsigma_k\}_{k\geq 1}, k=\overline{1,\infty}$ defined on a probability space (Ω, F, P) such that ξ_k and ς_k are independent random variables and $\xi_k > 0$, $\varsigma_k > 0$. Using these random variables we will derive the following step processes of semi-Markov random walk:

$X_z(t) = z + t - \sum_{i=1}^{k-1} \zeta_i$,

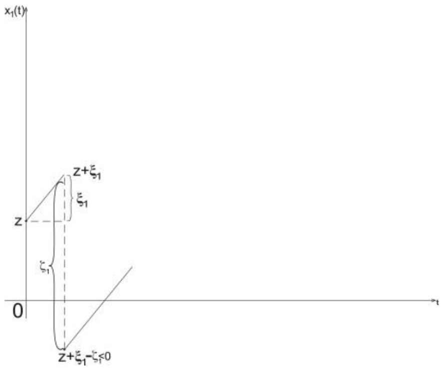
if $\sum_{i=1}^{k-1} \xi_i \leq t < \sum_{i=1}^{k} \xi_i,\ k=\overrightarrow{1,\infty}$, $z \geq 0$

$X_z(t)$process is the (asymptotic) semi-Markov random processes with positive tendency and negative jump. One of the realizations of the process $X_z(t)$ will be in the following form:

a) $X_z(t) = z + t$ if $t < \xi_1$ (see, Figure 1),

b) $X_z(t) = z + t - \zeta_1$, if $\xi_1 \le t < \xi_1 + \xi_2$ (see, Figure 2)

Fig. 1: Fig. 1



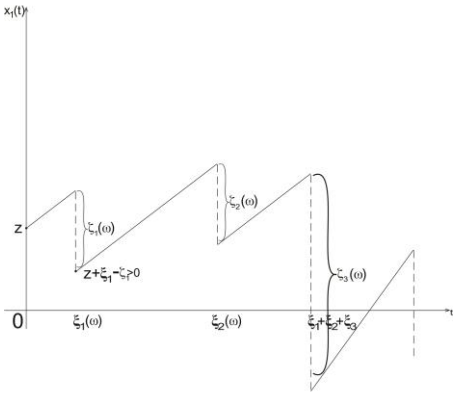


Fig. 2

3 Problem Solution

Let's include the τ_z^0 random variable defined as below:

$$\tau_z^0 = min\ \{t: X_z(t) \le 0\}$$

where τ_z^0 is the time of the first passage of X(t) process. We need to find Laplace transform for distribution of τ_z^0 random variable. Let us set Laplace transform for the distribution of τ_z^0 random variable as

$$L(\theta) = Ee^{-\theta \tau_1^0} \ , \ \theta > 0$$

$$L(\theta \mid z) = E(e^{-\theta \tau_z^0} \mid X(0) = z) \ , \ z \ge 0$$

The main aim of this study is to express the Laplace transform of the distribution function for the duration of the process $X(t)$ by means of some probability characteristics of random variables ξ_k and ς_k.

According to the formula of total probability, we can find the following integral equation of Laplace transforms for the distribution of the first passage of the zero level of the process.

$$L(\theta \mid z) = \int_{s=0}^{t} e^{-\theta s}\, \mathrm{P}\{\varsigma_{k1} > z + s\}\, \mathrm{P}\{\xi_k \in ds\}$$
$$- \int_{s=0}^{\infty} e^{-\theta s} \int_{\alpha=0}^{z+s} L(\theta \mid \alpha)\, d_\alpha\, \mathrm{P}\{\varsigma_k < z + s - \alpha\}\, dP\{\xi_k < s\}$$
(1)

This integral equation can be solved by the method of successive approximation. However, the obtained expressions are not suitable for applications. Therefore, we restrict the class of walks in order to obtain the explicit form.

Let's assume that ξ_k random variable has the Erlang distribution of m construction, while ς_k random variable has the single construction Erlang distribution:

$$P\{\xi_k(\omega) < t\} = \left\{1 - \left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{m-1}}{(m-1)!}\right] e^{-\mu t}\right\} \varepsilon(t)\ , \qquad \lambda > 0,$$

$$\mathrm{P}\{\varsigma_k(\omega) < t\} = \left[1 - e^{-\mu t}\right] \varepsilon(t), \qquad \mu > 0$$

(2)

In this case equation (1) will be as follows:

$$L(\theta \mid z) = \frac{\lambda^m}{(\lambda + \mu + \theta)^m} e^{-\mu z} -$$

$$\frac{\lambda^m\ \mu}{(m\ -1)!} e^{-\mu z} \int_{s=0}^{\infty} e^{-(\lambda+\mu+\theta)\, s} s^{m-1} \int_{\alpha=0}^{z+s} e^{\mu \alpha} L(\theta \mid \alpha)\, d\alpha\, ds$$

(3)

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (3) by $e^{\mu z}$ and derive on z .

In this case we will receive the following differential equation with (m+1) construction:

$$\sum_{k=0}^{m} C_m^k \left[\mu\, L^{(k)}(\theta \mid z) + L^{(k+1)}(\theta \mid z)\right] e^{(\lambda+\theta)z}\ (-1)^{m-k}\ (\lambda + \theta)^{m-k} =$$
$$(-1)^m\ \lambda^m\ \mu\, e^{-(\lambda+\theta)z}\ L(\theta \mid z)$$

This differencial equation has been solved within the boundary conditions and we obtained**:**

$$L(\theta \mid z) = \frac{\lambda^m}{[\lambda + \theta - k_1(\theta)]^m} e^{k_1(\theta) z}$$

This expression is the Laplace transform for relative distribution of τ_z^0 random variable. Then we will need to find $L(\theta)$ -Laplace transform for non-relative distribution of τ_z^0:

3.1. Investigation of Laplace transform with respect to time, Laplace - Stieltjes transform for the ergodic distribution of the semi-Markov random process with positive tendency and negative jump

At the next stage we will find Laplace transform with respect to time, Laplace - Stieltjes transform for the ergodic distribution of the semi-Markov random process with positive tendency and negative jump.
In accordance with formula of total probability for $x \geq 0$ we have:

$$P\{X(t) < x|X(0) = z\} =$$
$$= P\{X(t) < x; \xi_k > t|X(0) = z\} +$$
$$+P\{X(t) < x; \xi_k < t|X(0) = z\} +$$
$$+\int_{s=0}^{\infty}\int_{y=0}^{\infty} P\{\xi_k \in ds; X(s) \in dy|X(0) = z\} \times$$
$$\times P\{X(t-s) < x|X(0) = y\} \quad (4)$$

We denote

$\tilde{R}(t,x|z) = P\{X(t) < x|X(0) = z\}$, $x > 0$, $z \geq 0$- Laplace transform with respect to time

$\tilde{\tilde{R}}(\theta,\alpha|z) = \int_0^{\infty} e^{-\alpha x} d_x\tilde{R}(t,x|z)$, $\alpha > 0$, $z \geq 0$- Laplace - Stieltjes transform for the ergodic distribution of the semi-Markov random process.

In this case equation (4) will be as follows :

$$\tilde{\tilde{R}}(\theta,\alpha|z) = e^{\theta z}\int_z^{\infty} e^{-(\alpha+\theta)x} P\{\xi_1 > x - z\}dx +$$
$$\tilde{\tilde{R}}(\theta,\alpha|0)\int_0^{\infty} e^{-\theta t} P\{\zeta_1 > z + t\}\, \mathrm{dP}\{\xi_1 < t\} -$$
$$-\int_{y=0}^{z} \tilde{\tilde{R}}(\theta,\alpha|y)\int_{t=0}^{\infty} e^{-\theta t} d_yP\{\zeta_1 < z + t - y\}\, \mathrm{dP}\{\xi_1 < t\} -$$
$$-\int_{y=z}^{\infty} \tilde{\tilde{R}}(\theta,\alpha|y)\int_{t=y-z}^{\infty} e^{-\theta t} d_yP\{\zeta_1 < z + t - y\}\, \mathrm{dP}\{\xi_1 < t\} \quad (5)$$

We will solve this integral equation in special case. Let's assume that ξ_k random variable has the Erlangian distribution of m construction, while ς_k random variable has the single construction Erlangian distribution(2).In this case equation (5) will be as follows:

$$\tilde{\tilde{R}}(\theta,\alpha|z) = \frac{(\alpha+\mu+\theta)^m - \mu^m}{(\alpha+\mu+\theta)^m\,(\alpha+\theta)}\, \mathrm{e}^{-\alpha z} +$$
$$+\left[\frac{\mu}{\lambda+\mu+\theta}\right]^m e^{-\lambda z}\, \tilde{\tilde{R}}(\theta,\alpha|0) -$$
$$-\lambda\left[\frac{\mu}{\lambda+\mu+\theta}\right]^m \mathrm{e}^{-\lambda z}\int_0^z e^{\lambda y}\,\tilde{\tilde{R}}(\theta,\alpha|y)\mathrm{dy} -$$
$$-\frac{\lambda\mu^m}{(n-1)!}\,\mathrm{e}^{-\lambda z}\int_{y=z}^{\infty} e^{\lambda y}\,\tilde{\tilde{R}}(\theta,\alpha|y) \times$$
$$\times \int_{t=y-z}^{\infty} e^{-(\lambda+\mu+\theta)t}\, t^{m-1}\,\mathrm{dtdy} \quad (6)$$

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (6) by $e^{\lambda z}$ and derive on z . Then we will multiply both sides of last equation by $e^{(\lambda+\mu+\theta)z}$ and derive on z . If repeat this process (m-1) time we will have differential equation. The general solution of this differential equation will be

$\tilde{\tilde{R}}(\theta,\alpha|z) = \sum_{i=1}^{m} C_i(\theta,\alpha)\, \mathrm{e}^{k_i(\theta)z} + \tilde{\tilde{R}}_{\mathrm{sp}}(\theta,\alpha|z)$,

where $k_i(\theta)$ -are the roots of characteristic equation of the differential equation,

$\tilde{\tilde{R}}_{\mathrm{sp}}(\theta,\alpha|z)$-is the special solution of the differential equation equation :

$$\tilde{\tilde{R}}_{\mathrm{sp}}(\theta,\alpha|z) = \mathrm{A}\,\mathrm{e}^{-\alpha z},\quad A = \frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^m - \mu^m]}{(\alpha+\theta)\,\prod_{i=1}^{m}[\alpha+k_i(\theta)]} \quad (7)$$

By finding $C_i(\theta,\alpha)$ from equation (6) we will get the system of algebraic equations. We can proof linear dependence of this algebraic system. If to consider some substitutions, we will get

$$C_1(\theta,\alpha) =$$
$$= \frac{\alpha\mu^m\,[(\alpha+\mu+\theta)^m - \mu^m]}{(\alpha+\theta)([\mu+\theta-k_1(\theta)]^m - \mu^m)[\lambda\mu^m - (\lambda-\alpha)(\mu+\theta+\alpha)^m]}$$

Then the general solution of integral equation (6) will be as follows:

$$\tilde{\tilde{R}}(\theta,\alpha|z) = \frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^m - \mu^m]}{(\alpha+\theta)\,[[\lambda\mu^m - (\lambda-\alpha)(\mu+\theta+\alpha)^m]]}\,\mathrm{e}^{-\alpha z}$$

This expression is the Laplace transform on time, Laplace-Stieltjes transform on phase for conditional distribution of the process X(t). We will need to find Laplace transform on time, Laplace-Stieltjes transform on phase for unconditional distribution of the process X(t) .
From construction process X(t) is seen that

$$X(0) = X_z(0) = \xi_k(\omega)$$

Then we can find

$$\tilde{\tilde{R}}(\theta,\alpha) = \int_0^{\infty} \tilde{\tilde{R}}(\theta,\alpha|z)\,\mathrm{dP}\{X(0) < z\}$$

This expression is Laplace transform on time, Laplace-Stieltjes transform on phase for unconditional distribution of the process X(t). Now, we will find Laplace-Stieltjes

transform for ergodic distribution of the process X(t).

In [14] proved a general theorem on the ergodicity of the process semi-Markov

random walk. The process described in this article a special case of this process. Process

X(t) will be ergodic, if $E\xi_1 < E\zeta_1$ or $\frac{n}{\lambda} < \frac{1}{\mu} \Rightarrow \lambda > m\mu$

If process X(t) ergodic, then we can use Tauber's theorem [1]

$$\tilde{R}(\alpha) = \lim_{\theta\to 0} \theta \overset{\approx}{R}(\theta,\alpha) = \frac{1}{(n-1)!} \frac{[(\alpha+\mu)^m - \mu^m]\mu}{[\lambda\mu^m + (\alpha-\lambda)(\alpha+\mu)^m]} \quad (8)$$

Expression (8) is Laplace-Stieltjes transform for ergodic distribution of the process X(t).

Respectively, we will get the following characteristics:

$$\tilde{R}'(0) = -EX(\omega) = \frac{m}{\lambda - m\mu},\ \lambda > m\mu$$

$$\tilde{R}''(0) - [\tilde{R}'(0)]^2 = DX(\omega) = \frac{2m}{(\lambda - m\mu)^m},$$

$\lambda > m\mu$

3.2 Laplace transformation of the distribution of the time of system sojourns within a band

One of the important problems in probability theory is finding the distribution of the time of the sojourn of a system (a process) within a specified band. With this purpose, we will investigate the semi-Markov random processes with positive tendency, negative jumps and delaying boundary at zero in this article. The Laplace transformation of the distribution of the time of the system sojourn within a given band found. Introduce a random variable $\tau(\omega)$ denoting the duration of the sojourn of the process X(t) within a band (a,b).We can find an explicit form of the Laplace transformation of the conditional distributions of the random variable $\tau(\omega)$.

We denote

$K(t|z) = P\{\tau > t|X(0) = z\}$

$\tilde{K}(\theta,a,b|z) = \int_{t=0}^{\infty} e^{-\theta t} K(t,a,b|z)dt,\ \ \theta > 0$

Then

$K(t,a,b|z) = P\{\tau > t|z\}$=

$$= P\left\{\inf_{0\le s\le t} X(s) > b\,; \sup_{0\le s\le t} X(s) < a\ |X(0) = z\right\}$$

By the formula of the composite probability, we have

$K(t,a,b|z) =$

$$=P\left\{\inf_{0\le s\le t} X(s) > b\,; \sup_{0\le s\le t} X(s) < a\,;\ \xi_1 > t\ |X(0) = z\right\} +$$

$$+P\left\{\inf_{0\le s\le t} X(s) > b\,; \sup_{0\le s\le t} X(s) < a\,;\ \xi_1 > t\ |X(0) = z\right\}$$

We get an integral equation for $\breve{K}(\theta,a,b|z)$:

$$\tilde{K}(\theta,a,b|z) = \int_{t=0}^{a-z} e^{-\theta t} P\{\xi_1 > t\}\, dt +$$

$$+ \int_{y=b}^{a} \tilde{K}(\theta,a,b|y) \int_{t=y-z}^{a-z} e^{-\theta t} d_t\, P\times$$

$$\times \{\xi_1 < t\} d_y\, P\{\zeta_1 < z + t - y\} \quad (9)$$

This integral equation can be solved by the method of successive approximations, yet the resulting solution is unfit for applications. We can solve this integral equation in special case. In the case where the random variables have an exponential distribution with the parameters λ and μ respectively,

$$P\{\xi_1(\omega) < t\} = [1 - e^{-\mu t}]\,\varepsilon(t), \qquad \mu > 0,$$

$$P\{\zeta_1(\omega) < t\} = [1 - e^{-\lambda t}]\,\varepsilon(t)\,, \qquad \lambda > 0$$

$$\text{where } \varepsilon(t) = \begin{cases} 0\,, & t < 0, \\ 1\,, & t > 0. \end{cases}$$

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (3) by $e^{\mu z}$ and derive on z. Then we will multiply both sides of last equation by $e^{-(\lambda+\mu+\theta)z}$ and derive on z .Then we have following differential equation**:**

$$\tilde{K}''(\theta,a,b\,|z) - (\lambda - \mu + \theta)\tilde{K}'(\theta,a,b\,|z) -$$
$$-\mu\theta\tilde{K}(\theta,a,b\,|z) = -\mu \quad (10)$$

The general solution of this differential equation will be:

$$\tilde{K}(\theta,a,b\,|z) = c_1(\theta)e^{k_1(\theta)z} +$$
$$+c_2(\theta)e^{k_2(\theta)z} + \tilde{K}_{sp}(\theta,a,b\,|z)$$

where $k_i(\theta)$ -are the roots of characteristic equation of $k^2(\theta) - (\lambda - \mu + \theta)k(\theta) - \mu\theta = 0$

$\tilde{K}_{sp}(\theta,a,b\,|z)$-is the special solution of the equation (11).

$$\tilde{K}_{sp}(\theta,a,b\,|z) = -\frac{\mu}{k_2(\theta) - k_1(\theta)}$$

By finding $c_1(\theta)$and $c_1(\theta)$ we will get the system of algebraic equations. Then we can find general solution of differential equation. This expression is the Laplace transform of the distribution of the duration of the sojourn of a system within a band.

4 Conclusion

This article is dedicated to investigation of the semi-Markov random processes with positive tendency and negative jump and at the some time lower and upper bounded semi-markov model of resource control theory.The following results have been obtained in the this article:

- Have been construction of the semi-Markov random processes with positive tendency and negative jump;
- The integral equation of Laplace transforms for the distribution of the first passage of the zero level of the process is defined.
- This equation has been determined within the Erlangen distribution class;
- Laplace transform with respect to time, Laplace - Stieltjes transforms with respect to phase have been found for conditional and unconditional distributions of bounded processes described by semi-Markov models;
- Have been found Laplace transform of the distribution of duration of the system in the given strip;

References:

[1] Levy P. Proceesus semi-markoviens. Proceedings of the International Congress of Mathematicians. Amsterdam. 1954; 3: 416-426.

[2] Smith W.L. Regenerative stochastic processes. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences. 1955; 232(1188): 6-31.

[3] Takacs L. Some investigations concerning recurrent stochastic processes of a certain type. Magyar Tudom_anyos Akad_emia Matematikai Kutat_o Int_ezet. 1954; 3: 115-128.

[4] Pyke R. Markov renewal processes: de_nitions and preliminary properties. The Annals of Mathematical Statistics. 1961; 32(4): 1231-1242.

[5] Pyke R. Markov renewal processes with _nitely many states. The Annals of Mathematical Statistics. 1961; 32(4): 1243-1259.

[6] Pyke R., Schaufele R. Limit theorems for Markov renewal processes. The Annals of Mathematical Statistics. 1964; 35(4):1746-1764.

[7] Çinlar E. Markov renewal theory. Advances in Applied Probability. 1969; 1 (2): 123-187.

[8] Korolyuk V. S., Turbin A. F. Semi-Markov Processes and Their Applications. Naukova Dumka: Kiev, 1976.(In Russian)

[9] Limnios N., Oprisan G. Semi-Markov Processes and Reliability. Birkhauser: Boston, 2001.

[10] Lotov, V. I., On random walks in a band. Probability Theory and its Application, 1991

[11] Silvestrov D. C. Semi-Markov Processes With a Discrete State Space. Sovetskoe Radio: Moscow 1980. (in Russian).

[12] Grabski F. Semi-Markov Processes: Applications in Systems Reliability and Main- tenance. Elsevier: Amsterdam, 2014.

[13] Harlamov B.P. Continuous Semi-Markov Processes. Nauka: St. Petersburg, 2001. (In Russian).

[14] Borovkov A.A. Probability Theory. Gordon and Breach Science Publishers: Amsterdam, 1998.

[15] Busarov, V. A. On asymptotic behavior of random wanderings in random medium with delaying screen, Vest. Mos. Gos. Univ., 2004

[16] Nasirova T.I. The Processes of Semi-Markov Random Walk. Elm: Baku, 1984. (In Russian).

[17] Nasirova T.I. Complex Semi-Markov Walk Processes With Delaying Screen. Elm: Baku, 1988. (In Russian).

[18] Nasirova T.H., Kerimova U.Y., Definition of Laplace transform of the first passage of zero level of the semi-Markov random process with positive tendency and negative jump, Applied Mathematics, 2(7), 908-911, 2011.

[19] Nasirova, T. I., Kerimova. U.Y. 2011. Definition of Laplace -Stieltjes transforms for distribution of the first passage of zero level of the semi-Markov random process with positive tendency, negative jump, Applied Mathematics, 2(1): 908-911.

[20] Nasirova, T. I., Shamilova, B. G., Kerimova. U.Y. 2011. Definition of Laplace transforms for ergodic distribution of the semi-Markov random process, SYLWAN, 158(6): 50-58. 69

[21] Nasirova, T. I., Kerimova. U.Y., Shamilova, B. G. 2014. Laplace transformation of the time of system sojourns within a band, Mathematica Aeterna, 4(5): 437 444.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.