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Kernel Inference on the Entropy and Parameters of the Inverse Weibull Distribution Using Dual Generalized Progressive Hybrid Censored Data

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Abstract: - In statistical inference, entropy quantifies the uncertainty associated with a random variable. This study introduces a novel numerical iteration technique specifically, a kernel estimation method for estimating both the entropy and the parameters of the inverse Weibull distribution. The proposed kernel method is implemented and its performance is compared with that of the Bayesian approach via Monte Carlo simulations. The simulation results indicate that the kernel method yields more accurate estimates and outperforms the Bayesian approach under the dual generalized progressive hybrid censored data. Finally, the practical applicability of the proposed method is illustrated and validated through the analysis of two real datasets.

Key-Words: - Bayesian estimation; Informative gamma prior; Kernel estimation method; kernel prior.

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1 Introduction

Entropy is a fundamental concept in statistical inference, where it quantifies the uncertainty associated with a random variable. Specifically, it represents the expected value of the information content inherent in that variable's possible outcomes. Consequently, measuring entropy is crucial across numerous scientific disciplines, including statistics, physics, chemistry, economics, insurance, financial analysis, and biology. In essence, outcomes with lower probability provide more information and thus contribute less to the overall entropy, whereas more probable, less informative outcomes increase the entropy measure.

One of the most widely adopted methods for estimating entropy was introduced by Shannon [34], now known as Shannon entropy. This measure has proven highly successful in a broad range of research applications. However, a key limitation arises with continuous distributions: the direct extension, known as differential entropy, can become negative. This violates the interpretation of entropy as a non-negative measure of uncertainty, rendering it less useful in such contexts. To address this, a redefinition was proposed by [15], as the differential entropy P(X) of the random variable X with distribution function F(x) and probability density function f(x) as follows:

$$P(X) = P(f) = -\int_{-\infty}^{\infty} log\, f\,(x) f(x) dx. \tag{1}$$

It is observed that a sharply peaked distribution corresponds to very low entropy, whereas a more spread-out probability distribution yields significantly higher entropy. In this context, P(x) serves as a measure of the uncertainty associated with the distribution f(x). Moreover, as entropy increases, the distribution f(x) tends toward uniformity. Thus, entropy can be interpreted as a measure of the uniformity of a probability distribution for various lifetime distributions. [5] derived the entropy of upper record values and established several upper and lower bounds using the hazard rate function. The entropy of the Weibull distribution under progressive censoring was studied in [14], while [1] proposed an efficient method for computing entropy based on progressive Type-II censored samples. In the context of Rayleigh distribution, [12] derived the entropy function using maximum likelihood estimators (MLEs), approximate MLEs, and Bayes estimators under doubly generalized Type-II hybrid censoring. [14] derived the entropy estimators for the Weibull distribution under a generalized progressive hybrid censoring scheme (GPHCS). Entropy expressions for the generalized exponential distribution based on record values were provided in [11], and for the double exponential distribution under multiple Type-II censoring in [20]. The entropy under hybrid censoring schemes was examined in [32].

The probability density function (pdf) and the cumulative distribution function (cdf) for the inverse Weibull distribution (IWD) are given, respectively, as follows:

$$f(x;\alpha,\beta)=\alpha\beta x^{-\alpha-1}\,exp(-\beta x^{-\alpha}),\ \ x>0 \quad (2)$$

$$F(x;\alpha,\beta)=exp(-\beta x^{-\alpha}), \quad x>0 \quad (3)$$

$\alpha,\beta>0$ are the shape and scale parameters, respectively.

For the pdf (2), the entropy P can be derived in simple form as follows:

$$\mathbf{P}(\boldsymbol{\alpha},\boldsymbol{\beta})=\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{\alpha}}\right)(\boldsymbol{\gamma}+\mathbf{log}(\boldsymbol{\beta})-\mathbf{log}(\boldsymbol{\alpha\beta})+\mathbf{1},$$

where $\boldsymbol{\gamma}=\mathbf{0.5772}$ is the Euler–Mascheroni constant, see Appendix A.

This model has been proposed as a useful tool in the analysis of life test data and has played an important role in numerous applications, including the reliability analysis of dynamic components in diesel engines and the modeling of various datasets such as the breakdown times of insulating fluids subjected to constant tension. Moreover, it is widely employed in reliability engineering to characterize different failure behaviors, including infant mortality and wear-out periods, and to support decisions regarding cost-effectiveness and maintenance scheduling in reliability-centered maintenance programs.

The inverse Weibull distribution (IWD), in particular, has been shown to provide a good fit to life testing data reported by [33]. [22], [23] derived this model based on physical considerations related to the failure mechanisms of mechanical components undergoing degradation. [7], [8]investigated the statistical properties of maximum likelihood estimators (MLEs) for the parameters and reliability function based on complete samples. [18] applied the least squares (LS) method to estimate the parameters and reliability. [8] derived both MLEs and LS estimators for the model parameters, while [9] developed the Bayes probability intervals in a load-strength model. [10] addressed the problem of predicting future observations. [24], [25] provided theoretical insights and derived MLEs for the IWD parameters. In recent decades, several authors have incorporated technical information about real systems into prior beliefs about model parameters that improved the accuracy of the estimators, see [10]. [26] extended the applicability of the IWD to new domains, including hydrological and meteorological phenomena such as floods, droughts, and rainfall, highlighting its usefulness in modeling extreme value data. [31] derived the least squares estimators for the three-parameter IWD.

As a continuation of these efforts, we estimated the entropy and parameter of the IWD based on the Dual generalized order statistics (DGOS). Originally the lower generalized order statistics (LGOS) introduced by [6], which provide a unified framework for modeling random variables arranged in descending order. This concept was developed as the inverse counterpart of the generalized order statistics (GOS) framework proposed by [21]. Several authors have employed the DGOS to develop estimation procedures for lifetime distributions under complete and progressively censored data, see [2], [3], [26], [30].

Recently, [12], [13], [4] introduced the generalized progressive hybrid censoring scheme, which has been widely applied in the context of various lifetime distributions, including the inverse Weibull distribution, see [26], [27], [28], [29], [30]. In this work, analogous to the concept of dual generalized order statistics, we introduce the dual generalized progressive hybrid censoring scheme (DGPHCS) for estimating the parameters of the inverse Weibull distribution. The estimation procedure is implemented using numerical iterative techniques such as the Runge–Kutta and Picard methods, see [27], [29]. The proposed scheme can be described as follows.

Consider a life-testing experiment in which n identical units $X_1,X_2,\ldots\ldots,X_n$ are placed on test. For $\mathrm{T}\in(0,\infty)$ and integers k and m are pre-fixed such that $\mathrm{k}<\mathrm{m}$ with $\mathrm{R}_1,\mathrm{R}_2,\ldots,\mathrm{R}_\mathrm{m}$ are the random removal units that are fixed at the beginning of the experiment such that $\mathrm{n}=\mathrm{m}+\sum_{\mathrm{i}=1}^{\mathrm{m}}\mathrm{R}_\mathrm{i}$. Generally, at the time of the i_th failure, R_i units are randomly removed from the remaining surviving units $\mathrm{S}_\mathrm{i}=\mathrm{n}-\mathrm{i}-\sum_{\mathrm{j}=1}^{\mathrm{i}-1}\mathrm{R}_\mathrm{j}$, where $\mathrm{i}=1,2,..\mathrm{m}$.This process continues until, immediately following the terminated time $\mathrm{T}^*=\min\{\mathrm{X}_{\mathrm{k:m:n}},\max\{\mathrm{X}_{\mathrm{m:n:n}},\mathrm{T}\}\}$, where at this time all the remaining surviving units are removed from the experiment according to the following cases. Let D denote the number of observed failures up to the timeT. Thus, we have one of the following types of observations, see Figure 1:

Case I: $\mathrm{X}_{1:m:n}\geq\ldots\geq\mathrm{X}_{k:m:n}>\mathrm{X}_{k+1:m:n}>\ldots>\mathrm{X}_{D:m:n}$ If $\mathrm{X}_{\mathrm{m:m:n}}\leq\mathrm{T}<\mathrm{X}_{\mathrm{k:m:n}}$.

Case II: $\mathrm{X}_{1:m:n}\geq\ldots\geq\mathrm{X}_{k:m:n}>\mathrm{X}_{k+1:m:n}>\ldots>\mathrm{X}_{\mathrm{m:m:n}}$ If $\mathrm{T}<\mathrm{X}_{\mathrm{m:m:n}}<X_{k:m:n}$.

Case III: $\mathrm{X}_{1:m:n}\geq\ldots\geq\mathrm{X}_{D:m:n}>\mathrm{X}_{D+1:m:n}>\ldots>\mathrm{X}_{k:m:n}$ If $\mathrm{X}_{\mathrm{m:m:n}}<\mathrm{X}_{\mathrm{k:m:n}}<T$.

Note that for Case I, $\mathrm{X}_{\mathrm{D:m:n}}>\mathrm{T}>\mathrm{X}_{\mathrm{D+1:m:n}}$ and $\mathrm{X}_{\mathrm{D+1:m:n}},\ldots\ldots,\mathrm{X}_{\mathrm{m:m:n}}$ are not observed.

For Case III, $\mathrm{T}>\mathrm{X}_{\mathrm{k:m:n}}>\mathrm{X}_{\mathrm{m:m:n}}$. and $\mathrm{X}_{\mathrm{k+1:m:n}},\ldots\ldots,\mathrm{X}_{\mathrm{m:m:n}}$ are not observed.

Thus, given the DGPHCS, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\theta;x)=C\prod_{i=1}^{n}f(x_{i,m,n})\,[F(x_{i,m,n})]^{R_i}[F(T)]^{\delta R_T^*}, \quad (4)$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$n = \begin{cases} D, & \delta = 1 \quad , \quad if \quad X_{m:m:n} \leq \quad T < \quad X_{k:m:n} \\ m, & \delta = 0 \quad , \quad if \quad T < \quad X_{m:m:n} \leq \quad X_{k:m:n} \\ k, & \delta = 0 \quad , \quad if \quad X_{m:m:n} < \quad X_{k:m:n} < \quad T \end{cases}$$

and

$$\underline{X} = \begin{cases} (X_{1:m:n}, X_{2:m:n}, \dots, X_{D:m:n}) & if \quad X_{m:m:n} \leq \quad T < \quad X_{k:m:n} \\ (X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}) & if \quad T < \quad X_{m:m:n} \leq \quad X_{k:m:n} \\ (X_{1:m:n}, \dots, , X_{D+1:m:n} \dots, X_{k:m:n}) & if \quad X_{m:m:n} < \quad X_{k:m:n} < T \end{cases},$$

For $F^{-1}(0) > x_{1:m:n} > x_{2:m:n} > .. , > x_{n:m:n} > F^{-1}(1)$ of R^n.

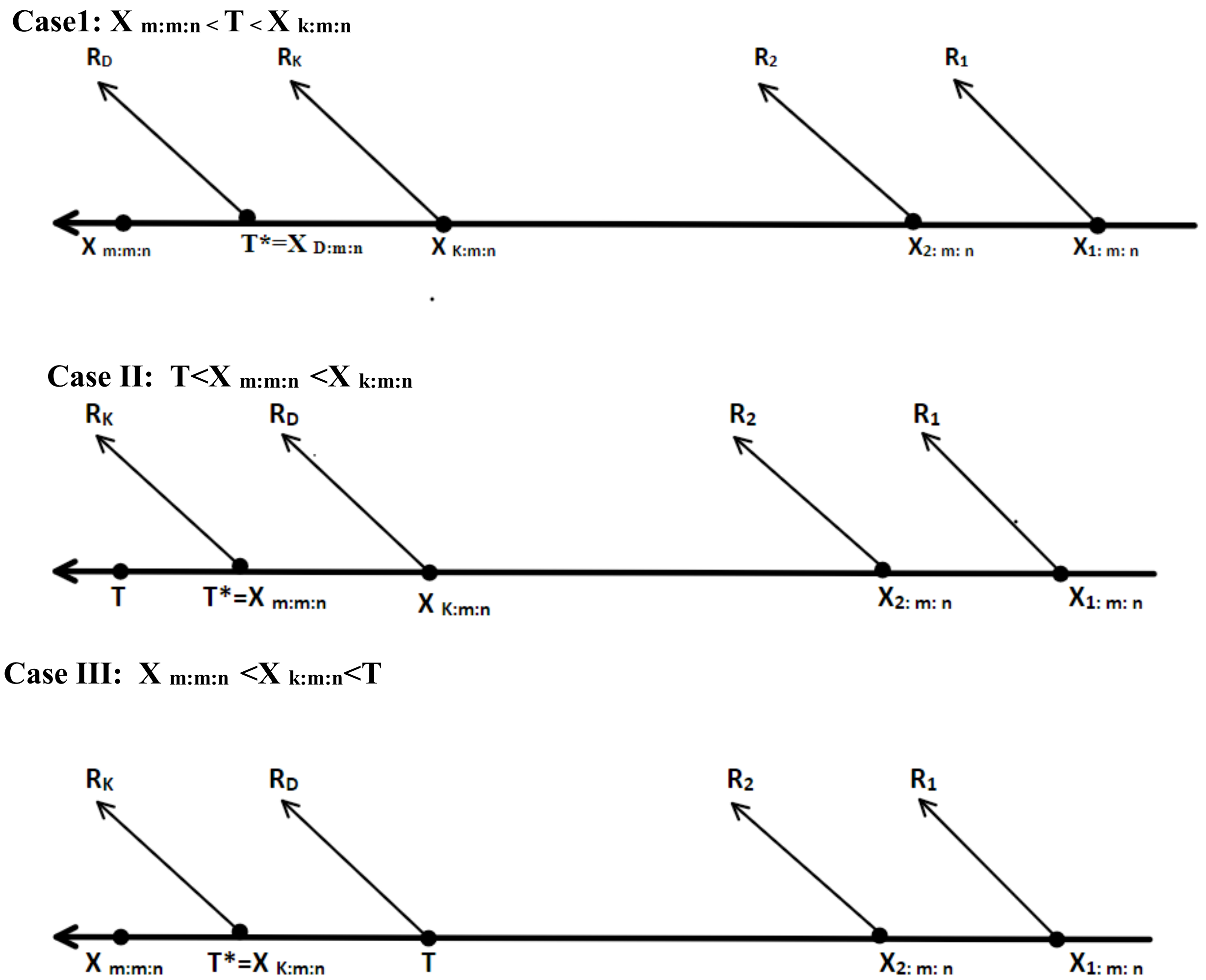

Fig. 1: Schematic representation of the dual generalized progressive hybrid censoring scheme

2 Estimation Methods

2.1 Kernel Method

We propose a simple and tractable algorithm for estimating the entropy and parameters of the IWD based on the kernel density estimate as the following:

The kernel estimates for the function $g(\alpha, \beta, P)$ can be derived by using the trivariate kernel density estimator for the unknown probability density function with support on $[0, \infty)$, which is defined as follows:

$$\hat{g}(\alpha, \beta, P) = \frac{1}{nh_1h_2h_3} \sum_{i=1}^{n} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1}, \frac{\beta-\hat{\beta}_i}{h_2}, \frac{P-\hat{P}_i}{h_3}\right), \quad (5)$$

$h_i, i = 1,2,3$are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i which minimizes the mean squared errors is $h_i = 1.06S_i n^{-0.2}$, S_i the sample standard deviations. The optimal choice for the kernel function $K(.,.)$ can be used as the trivariate standard normal distribution for the parameters α, β and P.

1. Generate a random sample $X = (x_1, x_2, \dots, x_n)$ from the inverse Weibull distribution.

2. Bootstrapping with replacement n samples from the random sample in step 1, as follows:
$X_1 = (X_{11},\ldots,X_{1n})$, $X_2 = (X_{21},\ldots,X_{2n})$,…,
$X_n = (X_{n1},\ldots,X_{nn})$.

3. For each sample in step 2, find the maximum likelihood estimator (MLE) for the parameters α, β, and P. Thus, we get the following random variables: $A = (\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_n)$, $B = (\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_n)$, and $\hat{P} = (\hat{P}_1,\hat{P}_2,\ldots,\hat{P}_n)$.

Integrate the density function (5) with respect to θ from θ_0 to θ_1 for $\theta = (\alpha,\beta,P)$ as follows:

$$\int_{\theta_0}^{\theta_1} g(\theta)d\theta = \frac{1}{nh}\sum_{i=1}^{n}\int_{\theta_0}^{\theta_1} K\left(\frac{\theta-\theta_i}{h}\right)d\theta$$

Thus,

$$\mathrm{G}(\theta_1) - \mathrm{G}(\theta_0) = \frac{1}{\mathrm{nh}}\sum_{\mathrm{i=1}}^{\mathrm{n}}\int_{\theta_0}^{\theta_1} \mathrm{K}\left(\frac{\theta-\theta_\mathrm{i}}{\mathrm{h}}\right)\mathrm{d}\theta$$

$$G(\theta_1) - G(\theta_0) = [\gamma - \frac{1}{nh}\sum_{i=1}^{n}\int_{0}^{\theta_0} K\left(\frac{\theta-\theta_i}{h}\right)d\theta]$$

$$= [\mathrm{n}\gamma - \sum_{\mathrm{i=1}}^{\mathrm{n}}\int_{\frac{-\theta_\mathrm{i}}{\mathrm{h}}}^{\frac{\theta_0-\theta_\mathrm{i}}{\mathrm{h}}} \mathrm{K(y)\,dy}]/\mathrm{n}$$

where $\gamma \in U(0,1)$.

Let $W(\frac{-\theta_i}{h},\frac{\theta_0-\theta_i}{h}) = \int_{\frac{-\theta_i}{h}}^{\frac{\theta_0-\theta_i}{h}} K(y)\,dy$.

Thus,

$$G(\theta_1) - G(\theta_0) = \frac{1}{n}\left[\mathrm{n}\gamma - \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{W}\left(\frac{-\theta_\mathrm{i}}{\mathrm{h}},\frac{\theta_0-\theta_\mathrm{i}}{\mathrm{h}}\right)\right]. \quad (6)$$

It is known that the second order approximation of the first derivative, which is defined by the centered differencing, can be written as follows:

$$\frac{\mathrm{dG}(\theta_m)}{\mathrm{d}\theta} = \frac{G(\theta_1)-G(\theta_0)}{\theta_1-\theta_0} = \mathrm{g}(\theta_\mathrm{m}), \quad (7)$$

for $\theta_0 < \theta_\mathrm{m} < \theta_1$.

From (6) and (7) we get the integral equation

$$\tilde{\theta}_1 = \tilde{\theta}_0 + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h},\frac{\theta_0-\theta_i}{h}\right)\right],$$

Thus, the iterative process for the kernel method can be derived as follows:

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h},\frac{\theta_n-\theta_i}{h}\right)\right], \quad (8)$$

for n =1,2,3,…

where $0 < \mathrm{C} \le \frac{2\mathrm{h}}{\mathrm{nL_1}}$, and $\mathrm{L_1} = \mathrm{K}(0)$.

The convergence of (8) continues until two consecutive numerical solutions are almost the same, that is if $|\hat{\theta}_{n+1} - \hat{\theta}_n| < 10^{-5}$.

2.2 Bayes Method

In this section, the Bayes estimations will be derived based on the informative gamma and the non-parametric kernel priors. The non-parametric prior distribution don't contain hyper parameters, which can mislead to subsequent inferences.

2.2.1 The Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with a joint probability density function defined as the following:

$$h(\alpha,\beta) \propto \alpha^{a-1}\beta^{c-1}e^{-\mathrm{b}\alpha-d\beta}. \quad (9)$$

The hyper-parameter a, b, c and d are assumed to be known and positive to reflect the prior belief about the unknown parameters.

2.2.2 The Informative Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function g(α, β) with support on (0, ∞), which is defined as follows:

$$\hat{\mathrm{g}}(\alpha,\beta) = \frac{1}{\mathrm{nh_1h_2}}\sum_{\mathrm{i=1}}^{n} \mathrm{K}\left(\frac{\alpha-\hat{\alpha}_\mathrm{i}}{\mathbf{h_1}},\frac{\beta-\hat{\beta}_\mathrm{i}}{\mathbf{h_2}}\right), \quad (10)$$

The optimal choice for □$_i$, for minimizing the mean squared errors is given by $\mathrm{h_i} = 1.06S_\mathrm{i}\mathrm{n}^{-0.2}$, where $\mathrm{S_i}$ is the sample standard deviation, and the optimal

choice for the kernel function $K(.,.)$ can be used as the bivariate standard normal distribution for the parameters α and β. Based on the properties of the MLEs of the parameters, which are converging in probability with the original parameters, the kernel prior estimate can be derived. It is worthwhile to mention that this kernel prior has been used for some distributions, see [4], [26].

Thus, using the joint priors (9) and (10) with the likelihood function of the DGPHCS (4) the posterior density for the unknown parameters α and β can be written in a unified form as follows:

$$\mathrm{f}(\alpha,\beta|\underline{X}) = \mathrm{KQ}(\alpha,\beta)\mathrm{L}(\mathrm{X};\alpha,\beta)$$

where Q(α, β) is the general prior, which can be defined as follows:

$$\mathrm{Q}(\alpha,\beta) = \mathrm{g}(\alpha,\beta)\mathrm{h}(\alpha,\beta)$$
$$= \hat{\mathrm{g}}^{\mathrm{p_1}}(\alpha)\hat{\mathrm{g}}^{\mathrm{p_2}}(\beta)\alpha^{a-1}\beta^{c-1}e^{-\mathrm{b}\alpha-\mathrm{d}\beta},$$

which has the following special cases:

i) For the informative prior (9): $\mathrm{p_1} = \mathrm{p_2} = 0$

ii) For the kernel prior (10): $\mathrm{p_1} = \mathrm{p_2} = 1$ and $\mathrm{a} = \mathrm{c} = 1$, $\mathrm{b} = \mathrm{d} = 0$.

Thus, the posterior density can be written as follows:

$$f(\alpha,\beta|\mathrm{x}) = K\hat{\mathrm{g}}^{\mathrm{p_1}}(\alpha)\hat{\mathrm{g}}^{\mathrm{p_2}}(\beta)\alpha^{n+a-1}\beta^{n+c-1}$$
$$\times\, exp[-\alpha b - (\alpha+1)\sum_{i=1}^{n} ln(x_i)].$$
$$\times\, exp[-\beta(d + \sum_{i=1}^{n}(R_i+1)x_i^{-\alpha} + \delta R_T^* T^{-\alpha}). \quad (11)$$

Based on (11), we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [37] introduced an easily computable approximation for the posterior mean of a non-negative parameter or more generally, of a smooth function of the parameter. For detail, let $\mathbf{q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be a smooth, positive function in the parameter space. The posterior expectation of $\mathbf{q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be obtained as

$$\mathrm{q}^* = \mathrm{E}(\mathrm{q}(\alpha, \beta)|\mathrm{x}) = \frac{\int_0^\infty \int_0^\infty \mathrm{e}^{n\mathrm{H}^*(\alpha,\beta)} \mathrm{d}\alpha \mathrm{d}\beta}{\int_0^\infty \int_0^\infty e^{nH(\alpha,\beta)} d\alpha d\beta}, \quad (12)$$

where $\mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{ln\, f}\,(\boldsymbol{\alpha}, \boldsymbol{\beta}|\mathbf{x})/\mathbf{n}$, and $\mathbf{H}^*(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \mathbf{ln\, q}\,(\boldsymbol{\alpha}, \boldsymbol{\beta})/\mathbf{n}$.

Thus, the Bayes estimator for $\mathbf{q}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ using Tierney and Kadane approximation can be obtained as follows:

$$\mathrm{q}^* = \sqrt{|\sum^* | / |\sum |}\, \exp[\, \mathrm{n}[\mathrm{H}^*(\alpha, \beta) - \mathrm{H}(\alpha, \beta)],$$

where $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ and $(\widehat{\boldsymbol{\alpha}}^*, \widehat{\boldsymbol{\beta}}^*)$ maximize the $\mathbf{H}(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ and $\mathbf{H}^*(\widehat{\boldsymbol{\alpha}}^*, \widehat{\boldsymbol{\beta}}^*)$, respectively.

$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix}^{-1}$, and $|\Sigma*| = \begin{vmatrix} H_{11}^* & H_{12}^* \\ H_{21}^* & H_{22}^* \end{vmatrix}^{-1}$,

denote the minus of inverse of Hessians of $H(\alpha, \beta)$ and $H^*(\alpha, \beta)$ at $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\alpha}^*, \hat{\beta}^*)$ respectively.

Using (11) we can define the log posterior density $\mathrm{H} = \mathrm{H}(\alpha, \beta)$ as follows:

$$\mathrm{H}(\alpha, \beta|\underline{\mathrm{x}}) = [\mathrm{p}_1 \ln \hat{\mathrm{g}}_1(\alpha) + \mathrm{p}_2 \ln \hat{\mathrm{g}}_2(\beta) + (\mathrm{n} + \mathrm{a} - 1) \ln \alpha$$
$$+ (\mathrm{n} + \mathrm{c} - 1) \ln \beta - \alpha \mathrm{b} - (\alpha + 1) \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln(x_i)$$
$$\beta[d + \textstyle\sum_{\mathrm{i}=1}^{\mathrm{n}} (1 + R_i) x_i^{-\alpha} + \delta R_T^* T^{-\alpha})].(13)$$

The derivatives of $\boldsymbol{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathbf{H}^*(\boldsymbol{\alpha}, \boldsymbol{\beta})$ have been derived in Appendix B.

3 Simulation Study

The simulation studies are carried out for studying the performance of the kernel and Bayes methods through the absolute average bais (AVB) and the mean squared error (MSE), which are given respictively, as follows: $\mathrm{AVB} = \frac{1}{\mathrm{L}} \sum_{\mathrm{i}=1}^{\mathrm{L}} |\hat{\theta}_{\mathrm{i}} - \theta|$, and $\mathrm{MSE}(\theta^*) = \frac{1}{L} \sum_{\mathrm{i}=1}^{\mathrm{L}} (\hat{\theta}_{\mathrm{i}} - \theta)^2$ where θ^* is the point estimate of the unknown parameter $\boldsymbol{\theta}$ and $\boldsymbol{L}$ is the number of replications.

In our simulation study, we calculated the mean estimators for the hyper parameters of the gamma priors of α and β. We choose two values for the parameter $\alpha = (1, 2)$ and two values for the parameter $\beta = (2, 3)$. Using these values for generating different samples from the IWD with sizes $n = 20$, 40, and 60 to represent small, moderate and large sizes. We choose two values for the termination time T=(0.75, 1.5) for the parameters and T=(0.75, 1.5, 3, 4,5) for the entropy P. To assess the performance of these estimates, the AVB and the MSE for each sample were calculated using 1000 replications.

From the simulation results in Tables 3, 4, 5 and 6, some points are quite clear based on these estimates and the others have been summarized in the following main points:

1. It is clear that, in general the point estimates based on the kernel method have the smallest estimated values of AVB and MSE as compared to the estimates based on the Bayes method with the two different priors.
2. The estimated values of the AVB and MSE increase as the value of α increases and decreases as the value of β increases.
3. The estimated values of the AVB and MSE decrease with decreasing the hyperparameters of the informative priors and increasing the sample size and the termination time of the experiment T as expected.
4. The Bayes estimated values of AVB and MSE based on the kernel prior are less than the counterparts based on the informative gamma prior.
5. The kernel and Bayes estimates for the entropy decrease as increasing the values of T, which indicate that increasing the information contianed in the samples.

As a conclusion, it appears that the point estimates based on the kernel method compete and outperform the Bayes method based on the different priors.

4 Real Data Applications

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the IW model in practice and to illustrate that this distribution can be considered as a good lifetime model for some new area of applications, comparing to many known distributions such as the Weibull distribution. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Chi-Square (CH2) tests for significance level test equals 0.05.

4.1 Flood Data

Consider the data given by [17], which represent the maximum flood levels (in millions of cubic feet per second) of the Susquehanna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

[30] has fitted these data to the IWD.

We found the inverse Weibull model is a good fit for this dataset as shown in Table 1 and Figure (2 a). For studying the behavior of th flood levels based on this datset we find the estimates for the parameters that represent the shape and scale of the flood level for 20 four-year periods. We noticed that the kernel and Bayes estimates for α are close to 4, while for β is close to zero and for P are close to one that indicate the graph is approximately symmetric, see Figure (2 a). Moreover, decreasing entropy means more information contained in the sample as seen in Figure (2a). These estimates indicate that the maximum flood levels are more stable on this period of years.

4.2 Reactor Pumps Data

A real data set for secondary nuclear pumps has been analyzed to illustrate the proposed methods. An important aspect of nuclear energy is safety. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feed water pump that takes from the desecrator storage tank feed water pressured up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be high temperature and high-pressure pump since it requires the head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represent the times between the failures of the secondary reactor pumps. [35], [36] have been discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the inverse Weibull model is a good fit for this dataset as shown in Table 1 and Figure (3 a). For studying the reliability of these reactor pumps based on this datset we find the estimates for the parameters which represent the shape and scale of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that the kernel and Bayes estimates for α are close to 0.75, for β are close to 0.46, and for P are close to 1.5. These estimates indicate that the above dataset is heavily right-skewed and that means the failure rate decreases with increasing the time, see Figure (3 b). Moreover, increasing entropy means less information contained in the sample as seen in Figure (3a). Thus, we conclude that decreasing the reliability of safety mechanism with increasing the time.

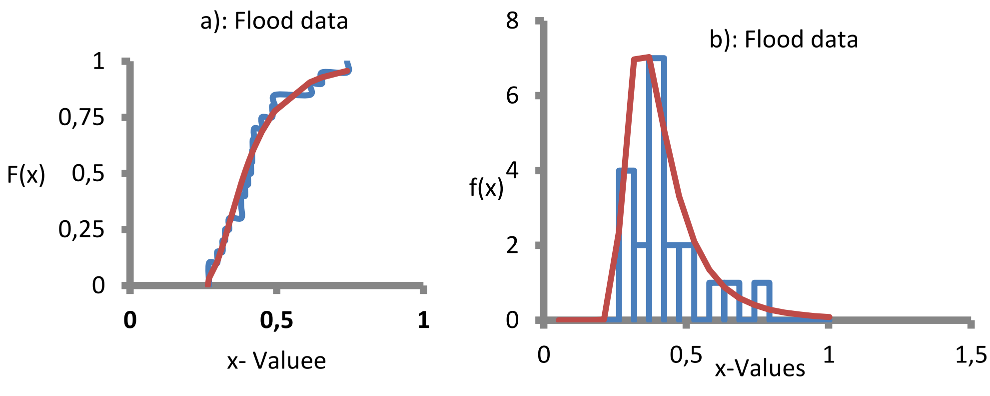

Fig. 2: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF

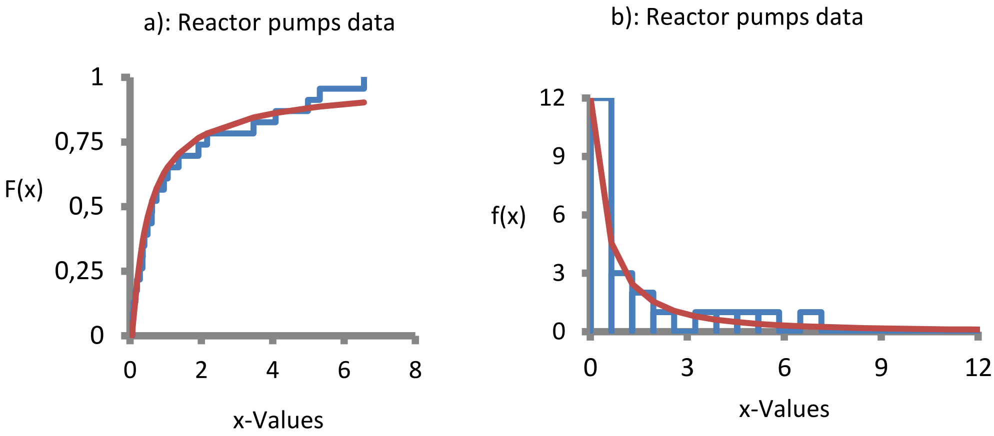

Fig. 3: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF

Table 1. The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values) for the inverse Weibull model. The MLE's for the parameters have been calculated.

Table 2. The estimate (Est.) and the mean square errors (MSEs) for the parameter α, β and H based on

Data	The Tests	Calculated value	Critical value	The p-values	MLES		
					α	β	P
Flood Data N=20	K-S	0.6976	0.8482	0.2138	4.3141	0.0119	0.7770
	A-D	0.3104	0.7414	0.5899			
	CH2	3.5552	31.1109	0.3294			
Reactor pumps N=23	K-S	0.4741	0.8528	0.8113	0.7832	0.4463	1.5285
	A-D	0.3443	0.7472	0.4915			
	CH2	10.270	31.5744	0.1223			

the Kernel and Bayes methods for the DGHPCS: for m = $3n/4$ and $k = 3m/4$.

Sam.	T	Par.	Kernel Estimate		Gamma Prior		Kernel Prior	
			Est.	MSE	Est.	MSE	Est.	MSE
Flood Data N=20	0.25	α	3.8824	0.1861	3.6424	0.4506	3.6424	0.4507
		β	0.0469	0.0012	0.0102	0.0003	0.0101	0.0004
		P	0.7285	0.0024	1.2636	0.2367	0.9099	0.0039
	0.65	α	3.8961	0.1744	3.6077	0.4985	3.6102	0.4949
		β	0.0843	0.0052	0.0101	0.0003	0.0098	0.0046
		P	0.6444	0.0176	1.0904	0.0982	0.5069	0.0729
Reactor Pumps Data N=23	0.15	α	0.7495	0.0011	0.6489	0.0180	0.6499	0.0178
		β	0.3559	0.0082	0.3702	0.0058	0.3705	0.0057
		P	1.3758	0.0233	1.3690	0.0254	1.3671	0.02605
	5.0	α	0.6449	0.0191	0.6441	0.0194	0.6449	0.0191
		β	0.4598	0.0018	0.3669	0.0063	0.3672	0.0063
		P	1.3428	0.0345	1.3426	0.0345	1.3407	0.0353

The results presented in Table 1 demonstrate that the IWD model fits these datasets well, as illustrated in Figure 2a and 3a. This is supported by the goodness-of-fit test statistics, which are below their respective critical values, and by the test power levels exceeding the significance level of the tests, which equals 0.05.

Table 2. shows that the mean squared error (MSE) estimates obtained using the kernel method are lower than those derived from Bayes' method with two different priors, particularly for large values of T, when considering the MLEs as the true parameter values. These findings align with and reinforce the simulation results

5 Conclusion

In this study, both kernel and Bayesian methods are employed to estimate the entropy and parameters of the inverse Weibull distribution based on the dual generalized progressive hybrid censored data. The results obtained from the simulation and real datasets indicate that kernel estimates exhibit unbiased, which remains noticeable even when the sample sizes are very small. Nevertheless, kernel estimates are highly unbiased and substantially more efficient than their Bayesian counterparts, even when informative priors are used. Moreover, kernel estimates for the entropy are lower than the Bayesian estimates, suggested that the information contained in the samples is sufficient to indicate a good fit for the flood dataset and is also adequate for the reactor pumps dataset, as illustrated in Figures (2a) and (3a).

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References:

[1] Abo-Eleneen, Z.A. The entropy of progressively censored samples. Entropy, Vol. 13, 2011, pp. 437–449.
[2] Ahsanullah, M. (2000 a) Generalized order statistics from exponential distribution, J. Statist. Planning Inference, Vol. 85, 2000 a, pp. 85–91.
[3] Ahsanullah, M. The generalized order statistics from two parameter uniform distribution, Commun. Statist. -Theory Methods, Vol. 25, No. 10, 2000 b, pp. 2311–2318.
[4] Ahsanullah, M., Maswadah, M. and Seham, M. Kernel inference on the generalized gamma distribution based on generalized order statistics. J. Stat. Theory Appl. Vol.12, 2013, pp. 152–172.
[5] Baratpour, S, Ahmadi, J., and Arghgmi, N. R. Entropy properties of record statistics, Statist Papers. Vol. 48, No. 2, 2007, pp. 197-213.
[6] Burkschat M, Cramer E, Kamps U Dual generalized order statistics. Metron Vol. 61, No. 1, 2003, pp.13–26.
[7] Calabria, R. and Pulcini, G. Confidence limits for reliability and tolerance limits in the inverse Weibull distribution. Reliability Engineering and system safety, Vol. 24, 1989, pp.77-85.
[8] Calabria, R. and Pulcini, G. On the maximum likelihood and least-squares estimation in the inverse Weibull distribution. Statistica Applicata, Vol. 2, 1990, pp 53-66.
[9] Calabria, R. and Pulcini, G. Bayes probability intervals in a load-strength model. Comm. In Statistics-Theory and Methods, Vol. 21, 1992, pp. 3393-3405.
[10] Calabria, R. and Pulcini, G. Bayes 2-Sample prediction for the Inverse Weibull Distribution. Commun. Statist. –Theory Meth, Vol. 23, No. 6, 1994, pp. 1811-1824.
[11] Chacko, M. and Asha, P. "Estimation of entropy for generalized exponential distribution based on record values," Journal of the Indian Society for Probability and Statistics, Vol. 19, No. 1, 2018, pp. 79–96.
[12] Cho, Y., Sun, H., Lee, K. An estimation of the entropy for a Rayleigh distribution based on doubly generalized Type-II hybrid censored samples. Entropy, Vol. 16, 2014, pp. 3655–3669.
[13] Cho, Y., Sun, H., Lee, K. Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme. Stat. Method, Vol. 23, 2015 a, pp. 18–34.
[14] Cho, Y., Sun, H. and Lee, K. Estimating the Entropy of a Weibull Distribution under Generalized Progressive Hybrid Censoring, Entropy, Vol. 17, 2015 b, pp. 102-122. doi:10.3390/e17010102.
[15] Cover, T.M.; Thomas, J.A. (2005). Elements of Information Theory; Wiley: Hoboken, NJ, USA.
[16] Cramer, E. Bagh, C. Minimum and maximum information censoring Plans in progressive censoring. Commun. Stat. Theory Methods. Vol. 14, 2011, pp. 2511-2627.
[17] Dumonceaux, R. and Antle, C.E. Discrimination between the Lognormal and Weibull distribution. Technometrics, Vol.15, 1973, pp. 923-926.
[18] Erto, P. Genesis, properties and identification of the inverse Weibull lifetime model. Statistica Applicata, Vol. 2, 1989, pp. 117-128.
[19] Jiheel, A. K.; Shanubhoque, A. Shrinkage estimation of the entropy function for the sample. Int. J. Math. Comput. Res. Vol. 2, 2014, pp. 394–402.
[20] Kang, S. B., Cho, Y. S., Han, J. T. and Kim, J. "An estimation of the entropy for a

double exponential distribution based on multiply type-II censored samples," Entropy, Vol. 14, No. 2, 2012, pp. 161–173.

[21] Kamps U. A concept of generalized order statistics. J Stat Plann .Inference Vol. 48, 1995, pp.1–23.

[22] Keller, A.Z. & Kamath, A.R.R. Alternative reliability models for mechanical systems. Paper presented at 3-rd international conference on Reliability and Maintainability, Toulouse, France, 1982.

[23] Keller, A. Z., Giblin, M. T. and Farnworth, N. R. Reliability analysis of commercial vehicle engines Reliability Engineering, Vol. 10, 1985, pp. 15-25.

[24] Khan, M.S., Pasha, G.R. and Pasha, A.H. Theoretical analysis of inverse Weibull Mathematics. Vol. 7, No. 2, 2008 a, pp. 30-38.

[25] Khan, M.S., Pasha, G.R. and Pasha, A.H. Fisher information matrix for the inverse Weibull distribution. International J. of. Math. Sci. & Eng. Appls. Vol. 2, No. 3, 2008b, pp. 257-262.

[26] Maswadah, M. Kernel inference on the inverse Weibull distribution. The Korean Communications in Statistics, Vol. 13, No. 3, 2006, pp. 503-512.

[27] Maswadah, M. An optimal point estimation method for the inverse Weibull model parameters using the Runge-Kutta method, Aligarh Journal of Statistics (AJS), Vol. 41, 2021, pp.1-21.

[28] Maswadah, M. Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme. Journal of Applied Statistics, Vol. 49, No. 11, 2022, pp. 4825-4849. DOI: 10.1080/02664763.2021.1924638.

[29] Maswadah, M. Numerical estimation for the three-parameter Burr-XII model parameters based on Picard's method. Pak. J. Statist. Vol. 40, No. 4, 2023b, pp.335-35.

[30] Maswadah, M. and Alkhathami, Alia, A. Numerical Inference on the Inverse Weibull Model Parameters Based on Dual Generalized Hybrid Progressive Censoring Data. International Journal of Statistical Distributions and Applications. Vol. 11, No. 2, 2025, pp. 28-44. https://doi.org/10.11648/j.ijsda.20251102.12

[31] Miljenko, M., Darija, M. and Dragan, J. Least squares fitting of three-parameter inverse Weibull density. Math. Commun. Vol. 15, No. 2, 2010, pp. 539-553.

[32] Morabbi, H, Razmkhan, M. Entropy of hybrid censoring scheme. J. Stat. Res. Vol. 6, No. 2, 2010, p.161-176.

[33] Nelson W. Applied life analysis. John Wiley & Sons, New York, 1982.

[34] Shannon, C.E. A mathematical theory of communication. Bell Syst. Tech. J., Vol. 27, 1948, pp. 379–423.

[35] Singh, S.K., Singh, U. and Sharma, V.K. Bayesian estimation and prediction for flexible Weibull model under Type-II censoring scheme. Journal of Probability and Statistics, Vol.20, 2013, pp. 1-17.

[36] Singh, S.K., Singh, U. and Sharma, V.K. Estimation and prediction for Type-I hybrid censored data from generalized Lindley distribution. Journal of Statistics and Management Systems, Vol.19, 2016, pp. 367–396.

[37] Tierney, L. and Kadane, J. B. Accurate Approximations for Posterior Moments and Marginal Densities. Journal of the American Statistical Association; Vol. 81, No. 393, 1986, pp. 82-86.

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APPENDIX A:

$$ln\, f\,(x;\alpha,\beta) = ln(\,\alpha\beta) - (\alpha+1)\,ln\,x - \beta x^{-\alpha}$$

By using the entropy equation (1), we can calculate

$$\mathrm{P} = -\ln(\,\alpha\beta) + (\alpha+1)\mathrm{E}(\ln \mathrm{X}) + \beta \mathrm{E}(\mathrm{X}^{-\alpha})$$

$$E(X^r) = \int_0^\infty x^r f(x)dx$$

$$= \alpha\beta \int_0^\infty x^{r-\alpha-1}\, exp(\,-\beta x^{-\alpha})dx$$

Let $u = \beta x^{-\alpha} \Rightarrow du = -\alpha\beta x^{-\alpha-1}dx,$

$and \quad x = \beta^{1/\alpha} u^{-1/\alpha}$

$$E(X^r) = \beta^{\frac{r}{\alpha}} \int_0^\infty u^{-\frac{r}{\alpha}} e^{-u} du = \beta^{\frac{r}{\alpha}} \Gamma(1 - r/\alpha),$$

Firstly, by choosing $r = -\alpha$,

we found $E(X^{-\alpha}) = \frac{1}{\beta}$.

Secondly, by differentiating $E(X^r)$ with respect to r, we found

$$E(X^r\, ln\, X) = \frac{\beta^{\frac{r}{\alpha}}}{\alpha}[ln\,\beta\, \Gamma(1 - r/\alpha) - \Gamma'(1 - r/\alpha)],$$

by putting $r = 0$ in this equation we get

$$E(ln\,X) = \frac{1}{\alpha}[ln\,\beta\,\Gamma(1) - \Gamma'(1)] = \frac{ln\,\beta - \Psi(1)}{\alpha},$$

Thus, the entropy can be derived as follows:

$$\mathrm{P} = -\ln(\,\alpha\beta) + (1 + \frac{1}{\alpha})(\gamma + \ln \beta) + 1.$$

where $-\gamma = \Gamma'(1) = \Psi(1) = -0.5772.$

APPENDIX B:

The log of the posterior density function (13) can be derived as follows:

$$\mathrm{H}(\alpha,\beta|\underline{x}) = [p_1 \ln \hat{g}_1(\alpha) + p_2 \ln \hat{g}_2(\beta)$$

$$+(n + a - 1)\ln\alpha + (n + c - 1)\ln\beta$$

$$-\alpha b - (\alpha+1)\sum_{i=1}^{n} \ln(\,x_i)$$

$$-\beta[d + \sum_{i=1}^{n}(1 + R_i)x_i^{-\alpha} + \delta R_T^* T^{-\alpha})].$$

$$\mathrm{H}_1 = \frac{\partial \mathrm{H}}{\partial\alpha} = [p_1 \frac{\hat{g}_1'(\alpha)}{\hat{g}_1(\alpha)} + \frac{n + a - 1}{\alpha} - b - \sum_{i=1}^{n} \ln x_i$$

$$+\beta[\textstyle\sum_{i=1}^{n}(1 + R_i)x_i^{-\alpha} \ln x_i + \delta R_T^* T^{-\alpha} \ln T],$$

$$\mathrm{H}_{11} = \frac{\partial^2 \mathrm{H}}{\partial\alpha^2} = [p_1 \frac{\hat{g}_1(\alpha)\hat{g}_1''(\alpha) - \hat{g}'^2_1(\alpha)}{\hat{g}_1^2(\alpha)} - \frac{n + a - 1}{\alpha^2}$$

$$-\beta[\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}(\ln x_i)^2+\delta R_T^* T^{-\alpha}(\ln T)^2],$$

$$H_2=\frac{\partial H}{\partial \beta}=[p_2\frac{\hat{g}_2'(\beta)}{\hat{g}_2(\beta)}+\frac{n+c-1}{\beta}-d$$

$$-(\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}+\delta R_T^* T^{-\alpha})],$$

$$H_{22}=\frac{\partial^2 H}{\partial \beta^2}=[p_2\frac{\hat{g}_2(\beta)g_2''(\beta)-\hat{g}_2'^{\,2}(\beta)}{\hat{g}_2^{\,2}(\beta)}-\frac{N+c-1}{\beta^2},$$

$$H_{12}=\frac{\partial^2 H}{\partial \beta \partial \alpha}=\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha}\ln(x_i)+\delta R_T^* T^{-\alpha}\ln T$$

$$\frac{\partial^2 H}{\partial \beta \partial x}=\alpha\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha-1}$$

$$\frac{\partial^2 H}{\partial \alpha \partial x}=-\sum_{i=1}^{n}\frac{1}{x_i}-\alpha\beta\sum_{i=1}^{n}(1+R_i)x_i^{-\alpha-1}[\ln x_i+1]$$

where the r^{th} derivative of the kernel density estimation can be defined as follows:

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r}=\hat{g}_1{}^r(\alpha)=\frac{1}{nh_1^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right), \quad (14)$$ where $r=0,1,2,3,$ -----.

Using the Gaussian kernel and (14), we have

$$\hat{g}_1(\alpha)=\frac{1}{nh_1\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha)=-\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1''(\alpha)=\frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2-1]\,e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$.

Table 3: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the Weibull distribution parameter α using the kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), et T=0.75 and T=1.5.

n	m	k	α	β	T=0.75			T=1.5		
					Kernel Estimations	Bayes Estimations		Kernel estimations	Bayes Estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1115(0.0139)	0.1649(0.0272)	0.1735(0.0301)	0.1090(0.0134)	0.1669(0.0283)	0.1936(0.0627)
				3	0.1111(0.0138)	0.1631(0.0266)	0.1669(0.0279)	0.1127(0.0146)	0.1669(0.0300)	0.1822(0.0341)
			2	2	0.1873(0.0366)	0.3242(0.1052)	0.3480(0.1212)	0.1933(0.0392)	0.3162(0.1006)	0.4050(0.1706)
				3	0.1908(0.0379)	0.3191(0.1019)	0.3354(0.1125)	0.1936(0.0393)	0.3132(0.0983)	0.3590(0.1325)
		8	1	2	0.1127(0.0142)	0.1649(0.0272)	0.1735(0.0301)	0.1062(0.0130)	0.1669(0.0281)	0.1866(0.0349)
				3	0.1124(0.0141)	0.1633(0.0267)	0.1671(0.0279)	0.1054(0.0127)	0.1708(0.0336)	0.1744(0.0305)
			2	2	0.1869(0.0364)	0.3243(0.1052)	0.3476(0.1209)	0.1901(0.0377)	0.3202(0.1028)	0.3625(0.1316)
				3	0.1880(0.0368)	0.3193(0.1021)	0.3356(0.1126)	0.1867(0.0365)	0.3169(0.1008)	0.3444(0.1188)
	15	8	1	2	0.1163(0.0149)	0.1640(0.0269)	0.1689(0.0285)	0.1085(0.0134)	0.1658(0.0276)	0.1826(0.0334)
				3	0.1121(0.0140)	0.1614(0.0261)	0.1639(0.0269)	0.1098(0.0137)	0.1717(0.0346)	0.1717(0.0295)
			2	2	0.1850(0.0355)	0.3243(0.1052)	0.3374(0.1138)	0.1887(0.0372)	0.3195(0.1022)	0.3581(0.1284)
				3	0.1852(0.0356)	0.3183(0.1014)	0.3272(0.1071)	0.1868(0.0365)	0.3164(0.1006)	0.3408(0.1162)
		11	1	2	0.1116(0.0137)	0.1643(0.0270)	0.1686(0.0284)	0.1124(0.0140)	0.1657(0.0275)	0.1725(0.0297)
				3	0.1070(0.0129)	0.1607(0.0258)	0.1638(0.0268)	0.1077(0.0130)	0.1635(0.0268)	0.1666(0.0278)
			2	2	0.1850(0.0357)	0.3226(0.1042)	0.3360(0.1129)	0.1854(0.0358)	0.3258(0.1063)	0.3465(0.1201)
				3	0.1846(0.0354)	0.3164(0.1001)	0.3258(0.1062)	0.1831(0.0349)	0.3204(0.1027)	0.3355(0.1126)
40	20	10	1	2	0.1092(0.0130)	0.1608(0.0259)	0.1619(0.0262)	0.1079(0.0129)	0.1683(0.0288)	0.1690(0.0286)
				3	0.1130(0.0139)	0.1588(0.0252)	0.1586(0.0251)	0.1076(0.0128)	0.1857(0.1120)	0.1641(0.0270)
			2	2	0.1740(0.0315)	0.3212(0.1032)	0.3268(0.1068)	0.1833(0.0350)	0.3237(0.1187)	0.5188(0.0352)
				3	0.1731(0.0311)	0.3156(0.0996)	0.3191(0.1019)	0.1844(0.0354)	0.3203(0.1160)	0.4002(0.2226)
		15	1	2	0.1098(0.0132)	0.1606(0.0258)	0.1618(0.0262)	0.1080(0.0128)	0.1685(0.0288)	0.1685(0.0284)
				3	0.1095(0.0131)	0.1587(0.0252)	0.1585(0.0251)	0.1080(0.0129)	0.1765(0.0414)	0.1646(0.0271)
			2	2	0.1763(0.0322)	0.3211(0.1031)	0.3266(0.1067)	0.1725(0.0310)	0.3215(0.1039)	0.3415(0.1167)
				3	0.1735(0.0312)	0.3157(0.0997)	0.3192(0.1019)	0.1714(0.0306)	0.3240(0.1329)	0.3315(0.1100)
	30	15	1	2	0.1106(0.0134)	0.1606(0.0258)	0.1617(0.0262)	0.1070(0.0127)	0.1679(0.0284)	0.1689(0.0286)
				3	0.1089(0.0130)	0.1587(0.0252)	0.1585(0.0251)	0.1074(0.0128)	0.1806(0.0509)	0.1637(0.0268)
			2	2	0.1739(0.0314)	0.3210(0.1030)	0.3265(0.1066)	0.1767(0.0325)	0.3220(0.1039)	0.3377(0.1141)
				3	0.1743(0.0315)	0.3156(0.0996)	0.3191(0.1018)	0.1764(0.0324)	0.3161(0.1000)	0.3236(0.1047)
		23	1	2	0.1123(0.0137)	0.1605(0.0258)	0.1612(0.0260)	0.1120(0.0136)	0.1615(0.0261)	0.1621(0.0263)
				3	0.1126(0.0138)	0.1589(0.0253)	0.1585(0.0251)	0.1117(0.0136)	0.1604(0.0257)	0.1594(0.0254)
			2	2	0.1715(0.0305)	0.3231(0.1044)	0.3279(0.1075)	0.1724(0.0308)	0.3235(0.1047)	0.3302(0.1090)
				3	0.1729(0.0309)	0.3175(0.1008)	0.3206(0.1028)	0.1718(0.0306)	0.3199(0.1024)	0.3243(0.1052)
60	30	15	1	2	0.1120(0.0136)	0.1589(0.0253)	0.1589(0.0253)	0.1097(0.0131)	0.1766(0.0333)	0.1677(0.0282)
				3	0.1110(0.0133)	0.1576(0.0248)	0.1565(0.0245)	0.1112(0.0134)	0.1733(0.0674)	0.1635(0.0268)
			2	2	0.1685(0.0294)	0.3203(0.1026)	0.3235(0.1047)	0.1683(0.0295)	0.3332(0.1802)	0.4673(0.7317)
				3	0.1691(0.0295)	0.3148(0.0991)	0.3168(0.1004)	0.1682(0.0294)	0.3246(0.1381)	0.4091(0.3069)
		23	1	2	0.1110(0.0133)	0.1589(0.0252)	0.1589(0.0253)	0.1112(0.0134)	0.1732(0.0332)	0.1657(0.0275)
				3	0.1117(0.0134)	0.1576(0.0249)	0.1566(0.0245)	0.1110(0.0134)	0.2011(0.6367)	0.1618(0.0262)
			2	2	0.1694(0.0297)	0.3203(0.1026)	0.3235(0.1047)	0.1713(0.0304)	0.3213(0.1039)	0.3356(0.1127)
				3	0.1674(0.0290)	0.3149(0.0992)	0.3168(0.1004)	0.1697(0.0299)	0.3404(0.2915)	0.3272(0.1071)
	45	23	1	2	0.1065(0.0122)	0.1580(0.0250)	0.1580(0.0250)	0.1071(0.0125)	0.1657(0.0275)	0.1638(0.0268)
				3	0.1060(0.0121)	0.1567(0.0246)	0.1561(0.0244)	0.1101(0.0132)	0.2249(0.6730)	0.1605(0.0258)
			2	2	0.1646(0.0280)	0.3216(0.1034)	0.3236(0.1047)	0.1699(0.0300)	0.3194(0.1021)	0.3271(0.1070)
				3	0.1631(0.0275)	0.3160(0.0998)	0.3173(0.1007)	0.1726(0.0309)	0.3153(0.0994)	0.3192(0.1019)
		34	1	2	0.1099(0.0130)	0.1582(0.0250)	0.1582(0.0250)	0.1093(0.0129)	0.1590(0.0253)	0.1590(0.0253)
				3	0.1081(0.0126)	0.1568(0.0246)	0.1561(0.0244)	0.1110(0.0133)	0.1582(0.0250)	0.1569(0.0246)
			2	2	0.1648(0.0281)	0.3213(0.1033)	0.3234(0.1046)	0.1681(0.0292)	0.3215(0.1034)	0.3249(0.1056)
				3	0.1646(0.0280)	0.3157(0.0997)	0.3171(0.1005)	0.1677(0.0290)	0.3181(0.1012)	0.3201(0.1025)

Table 4: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the Weibull distribution parameter β using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), at T=0.75 and T=1.5.

n	m	k	α	β	T=0.75			T=1.5		
					Kernel Estimations	Bayes estimations		Kernel Estimations	Bayes Estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1870(0.0365)	0.3427(0.1181)	0.3626(0.1315)	0.1846(0.0357)	0.3560(0.1323)	0.4285(0.1907)
				3	0.2308(0.0547)	0.5259(0.2799)	0.5140(0.2642)	0.2379(0.0583)	0.6328(0.1251)	0.6297(0.5249)
			2	2	0.1864(0.0362)	0.3072(0.0946)	0.3536(0.1250)	0.1949(0.0399)	0.3096(0.0980)	0.5185(0.2817)
				3	0.2280(0.0534)	0.4656(0.2175)	0.5030(0.2530)	0.2386(0.0588)	0.4659(0.2202)	0.6157(0.3845)
		8	1	2	0.1855(0.0359)	0.3423(0.1178)	0.3627(0.1315)	0.1847(0.0357)	0.3564(0.1305)	0.4195(0.1765)
				3	0.2289(0.0539)	0.5265(0.2804)	0.5140(0.2642)	0.2324(0.0556)	0.6154(0.5582)	0.5617(0.3161)
			2	2	0.1833(0.0350)	0.3078(0.0950)	0.3538(0.1252)	0.1844(0.0356)	0.3104(0.0973)	0.4085(0.1671)
				3	0.2294(0.0541)	0.4660(0.2179)	0.5031(0.2531)	0.2294(0.0543)	0.4778(0.2447)	0.5527(0.3058)
	15	8	1	2	0.1829(0.0349)	0.3361(0.1133)	0.3439(0.1183)	0.1871(0.0366)	0.3513(0.1258)	0.4052(0.1645)
				3	0.2254(0.0522)	0.5107(0.2616)	0.4967(0.2467)	0.2323(0.0557)	0.5759(0.3952)	0.5488(0.3014)
			2	2	0.1827(0.0348)	0.3066(0.0942)	0.3343(0.1118)	0.1846(0.0357)	0.3076(0.0950)	0.3954(0.1564)
				3	0.2229(0.0511)	0.4627(0.2144)	0.4834(0.2337)	0.2310(0.0549)	0.4683(0.2232)	0.5398(0.2915)
		11	1	2	0.1795(0.0336)	0.3362(0.1133)	0.3416(0.1167)	0.1829(0.0348)	0.3415(0.1172)	0.3539(0.1252)
				3	0.2244(0.0518)	0.5092(0.2602)	0.4995(0.2495)	0.2258(0.0523)	0.5242(0.2771)	0.5084(0.2584)
			2	2	0.1817(0.0344)	0.3070(0.0944)	0.3373(0.1138)	0.1800(0.0338)	0.3088(0.0956)	0.3476(0.1209)
				3	0.2224(0.0509)	0.4613(0.2132)	0.4860(0.2363)	0.2243(0.0518)	0.4653(0.2171)	0.4978(0.2478)
40	20	10	1	2	0.1704(0.0302)	0.3346(0.1121)	0.3325(0.1105)	0.1727(0.0310)	0.3734(0.1453)	0.3665(0.1345)
				3	0.2108(0.0455)	0.5104(0.2609)	0.4866(0.2368)	0.2156(0.0477)	0.8885(0.0153)	0.5278(0.2788)
			2	2	0.1693(0.0298)	0.3075(0.0947)	0.3225(0.1040)	0.1803(0.0339)	0.3240(0.1124)	0.6876(0.0537)
				3	0.2092(0.0449)	0.4633(0.2149)	0.4728(0.2236)	0.2236(0.0515)	0.6206(0/213	0.8241(02532)
		15	1	2	0.1702(0.0301)	0.3339(0.1116)	0.3323(0.1104)	0.1722(0.0309)	0.3719(0.1439)	0.3632(0.1321)
				3	0.2119(0.0461)	0.5105(0.2611)	0.4866(0.2368)	0.2172(0.0484)	0.7750(0.0253)	0.5309(0.2820)
			2	2	0.1670(0.0290)	0.3075(0.0946)	0.3225(0.1040)	0.1698(0.0301)	0.3170(0.1022)	0.3693(0.1364)
				3	0.2095(0.0450)	0.4637(0.2152)	0.4728(0.2235)	0.2152(0.0475)	0.5247(0.5471)	0.5277(0.2787)
	30	15	1	2	0.1731(0.0311)	0.3329(0.1110)	0.3316(0.1100)	0.1710(0.0304)	0.3730(0.1430)	0.3675(0.1352)
				3	0.2112(0.0457)	0.5102(0.2607)	0.4865(0.2367)	0.2174(0.0485)	0.7303(0.653)	0.5252(0.2760)
			2	2	0.1696(0.0299)	0.3077(0.0947)	0.3226(0.1041)	0.1725(0.0310)	0.3161(0.1012)	0.3578(0.1281)
				3	0.2094(0.0450)	0.4641(0.2156)	0.4729(0.2237)	0.2220(0.0505)	0.4737(0.2266)	0.4975(0.2475)
		23	1	2	0.1679(0.0293)	0.3292(0.1084)	0.3261(0.1063)	0.1678(0.0292)	0.3329(0.1109)	0.3291(0.1083)
				3	0.2100(0.0451)	0.5048(0.2551)	0.4815(0.2318)	0.2120(0.0460)	0.5139(0.2646)	0.4848(0.2350)
			2	2	0.1681(0.0293)	0.3064(0.0939)	0.3182(0.1013)	0.1651(0.0283)	0.3098(0.0961)	0.3245(0.1053)
				3	0.2096(0.0450)	0.4631(0.2146)	0.4690(0.2200)	0.2098(0.0451)	0.4678(0.2192)	0.4756(0.2262)
60	30	15	1	2	0.1651(0.0282)	0.3291(0.1083)	0.3243(0.1052)	0.1704(0.0301)	0.4289(0.2262)	0.3698(0.1371)
				3	0.2091(0.0447)	0.5057(0.2560)	0.4796(0.2301)	0.2126(0.0463)	0.8368(0.3251)	0.5365(0.2888)
			2	2	0.1654(0.0283)	0.3070(0.0943)	0.3158(0.0997)	0.1675(0.0292)	0.3492(0.2414)	0.7039(0.0437)
				3	0.2069(0.0438)	0.4630(0.2145)	0.4672(0.2182)	0.2122(0.0462)	0.5603(0.0634)	0.8928(0.1435)
		23	1	2	0.1674(0.0289)	0.3289(0.1082)	0.3243(0.1052)	0.1671(0.0289)	0.4008(0.1812)	0.3603(0.1300)
				3	0.2096(0.0450)	0.5056(0.2559)	0.4796(0.2301)	0.2122(0.0461)	0.8240(0.253)	0.5251(0.2759)
			2	2	0.1654(0.0283)	0.3069(0.0943)	0.3158(0.0997)	0.1668(0.0289)	0.3286(0.1300)	0.3608(0.1303)
				3	0.2089(0.0446)	0.4635(0.2149)	0.4671(0.2182)	0.2100(0.0452)	0.6170(0.1426)	0.5242(0.2751)
	45	23	1	2	0.1593(0.0262)	0.3205(0.1027)	0.3167(0.1003)	0.1635(0.0278)	0.3674(0.1365)	0.3500(0.1225)
				3	0.2067(0.0436)	0.4896(0.2398)	0.4719(0.2227)	0.2119(0.0459)	0.8337(0.1362)	0.5116(0.2618)
			2	2	0.1587(0.0261)	0.3043(0.0926)	0.3096(0.0958)	0.1677(0.0293)	0.3119(0.0975)	0.3339(0.1115)
				3	0.2052(0.0430)	0.4590(0.2107)	0.4604(0.2119)	0.2106(0.0454)	0.4746(0.2263)	0.4863(0.2365)
		34	1	2	0.1627(0.0274)	0.3214(0.1033)	0.3175(0.1008)	0.1641(0.0278)	0.3258(0.1062)	0.3210(0.1031)
				3	0.2005(0.0411)	0.4916(0.2417)	0.4729(0.2237)	0.2075(0.0440)	0.5032(0.2534)	0.4771(0.2277)
			2	2	0.1616(0.0270)	0.3048(0.0930)	0.3104(0.0963)	0.1660(0.0285)	0.3089(0.0954)	0.3169(0.1004)
				3	0.2012(0.0414)	0.4599(0.2116)	0.4612(0.2127)	0.2039(0.0425)	0.4660(0.2173)	0.4679(0.2189)

Table 5: The estimate (EST) and Mean Squared Errors (MSE) in parentheses for the entropy (P) of Weibull distribution using the kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), et T=0.75 and T=1.5.

n	m	k	α	β	T=0.75			T=1.5		
					Kernel Estimations	Bayes Estimations		Kernel Estimations	Bayes Estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1370(0.0202)	0.1633(0.0277)	0.1427(0.0204)	0.1407(0.0214)	0.1595(0.0321)	0.0711(0.0243)
				3	0.1421(0.0217)	0.2262(0.0533)	0.2457(0.0604)	0.1515(0.0247)	0.2986(0.3172)	0.1653(0.0410)
			2	2	0.1098(0.0136)	0.1888(0.0357)	0.1596(0.0255)	0.1102(0.0141)	0.1826(0.0349)	0.0706(0.0122)
				3	0.1048(0.0124)	0.2210(0.0489)	0.2090(0.0437)	0.1189(0.0160)	0.2187(0.0631)	0.1468(0.0230)
		8	1	2	0.1411(0.0214)	0.1638(0.0278)	0.1427(0.0204)	0.1433(0.0222)	0.1507(0.0257)	0.0515(0.0034)
				3	0.1431(0.0219)	0.2263(0.0533)	0.2459(0.0605)	0.1496(0.0240)	0.2933(0.2904)	0.1986(0.0399)
			2	2	0.1073(0.0129)	0.1884(0.0356)	0.1593(0.0254)	0.1121(0.0141)	0.1819(0.0334)	0.1041(0.0110)
				3	0.1039(0.0123)	0.2209(0.0489)	0.2089(0.0437)	0.1094(0.0136)	0.2164(0.0518)	0.1782(0.0318)
	15	8	1	2	0.1332(0.0191)	0.1788(0.0322)	0.1758(0.0310)	0.1422(0.0219)	0.1519(0.0257)	0.0712(0.0058)
				3	0.1352(0.0196)	0.2449(0.0603)	0.2628(0.0691)	0.1457(0.0229)	0.2796(0.3819)	0.2112(0.0448)
			2	2	0.0995(0.0114)	0.1916(0.0367)	0.1815(0.0330)	0.1126(0.0142)	0.1839(0.0340)	0.1173(0.0139)
				3	0.1016(0.0117)	0.2247(0.0505)	0.2217(0.0492)	0.1091(0.0136)	0.2171(0.0477)	0.1858(0.0346)
		11	1	2	0.1354(0.0197)	0.1789(0.0323)	0.1782(0.0318)	0.1364(0.0200)	0.1655(0.0282)	0.1546(0.0240)
				3	0.1418(0.0215)	0.2457(0.0608)	0.2608(0.0680)	0.1437(0.0221)	0.2291(0.0540)	0.2510(0.0630)
			2	2	0.1012(0.0117)	0.1903(0.0363)	0.1799(0.0324)	0.1026(0.0119)	0.1886(0.0357)	0.1650(0.0272)
				3	0.1044(0.0123)	0.2240(0.0502)	0.2206(0.0487)	0.1041(0.0123)	0.2223(0.0495)	0.2122(0.0451)
40	20	10	1	2	0.1243(0.0166)	0.1670(0.0281)	0.1711(0.0293)	0.1297(0.0181)	0.1313(0.0260)	0.1135(0.0133)
				3	0.1286(0.0177)	0.2324(0.0543)	0.2544(0.0647)	0.1297(0.0181)	0.4975(0.0643)	0.2142(0.0461)
			2	2	0.0950(0.0102)	0.1874(0.0352)	0.1793(0.0322)	0.1026(0.0120)	0.1811(0.0370)	0.3924(0.0543)
				3	0.0957(0.0103)	0.2210(0.0489)	0.2184(0.0477)	0.1042(0.0123)	0.2213(0.0697)	0.2718(0.7863)
		15	1	2	0.1256(0.0169)	0.1680(0.0284)	0.1714(0.0294)	0.1290(0.0178)	0.1330(0.0256)	0.1186(0.0146)
				3	0.1296(0.0180)	0.2323(0.0543)	0.2544(0.0647)	0.1292(0.0179)	0.5635(0.0537)	0.2113(0.0448)
			2	2	0.0940(0.0099)	0.1875(0.0352)	0.1793(0.0322)	0.1017(0.0116)	0.1772(0.0326)	0.1323(0.0176)
				3	0.0953(0.0102)	0.2209(0.0488)	0.2185(0.0477)	0.1004(0.0114)	0.2199(0.0597)	0.1861(0.0347)
	30	15	1	2	0.1255(0.0168)	0.1694(0.0289)	0.1724(0.0297)	0.1296(0.0181)	0.1250(0.0204)	0.1119(0.0128)
				3	0.1317(0.0185)	0.2326(0.0544)	0.2545(0.0648)	0.1300(0.0181)	0.3566(0.5026)	0.2168(0.0471)
			2	2	0.0924(0.0096)	0.1873(0.0351)	0.1792(0.0321)	0.0982(0.0109)	0.1779(0.0320)	0.1438(0.0208)
				3	0.0962(0.0104)	0.2207(0.0487)	0.2184(0.0477)	0.1016(0.0116)	0.2136(0.0459)	0.2037(0.0415)
		23	1	2	0.1235(0.0163)	0.1755(0.0309)	0.1806(0.0327)	0.1253(0.0168)	0.1686(0.0286)	0.1748(0.0306)
				3	0.1271(0.0172)	0.2379(0.0567)	0.2591(0.0671)	0.1272(0.0172)	0.2285(0.0526)	0.2557(0.0654)
			2	2	0.0963(0.0103)	0.1897(0.0360)	0.1835(0.0337)	0.0968(0.0104)	0.1859(0.0346)	0.1760(0.0310)
				3	0.0949(0.0101)	0.2225(0.0495)	0.2211(0.0489)	0.0941(0.0099)	0.2199(0.0484)	0.2173(0.0472)
60	30	15	1	2	0.1197(0.0153)	0.1683(0.0284)	0.1754(0.0308)	0.1216(0.0158)	0.1808(0.2114)	0.1011(0.0110)
				3	0.1218(0.0158)	0.2307(0.0534)	0.2542(0.0646)	0.1268(0.0172)	0.8019(0.0452)	0.2009(0.0409)
			2	2	0.0876(0.0086)	0.1863(0.0347)	0.1807(0.0327)	0.0992(0.0109)	0.2022(0.1216)	0.3780(0.0536)
				3	0.0913(0.0093)	0.2196(0.0483)	0.2182(0.0476)	0.0983(0.0108)	0.2408(0.1027)	0.2845(0.4492)
		23	1	2	0.1199(0.0153)	0.1686(0.0285)	0.1754(0.0308)	0.1214(0.0158)	0.1709(0.8315)	0.1163(0.0140)
				3	0.1226(0.0160)	0.2308(0.0535)	0.2543(0.0647)	0.1263(0.0171)	0.5981(0.0472	0.2110(0.0447)
			2	2	0.0870(0.0085)	0.1864(0.0348)	0.1807(0.0327)	0.0963(0.0103)	0.1793(0.0450)	0.1368(0.0188)
				3	0.0931(0.0096)	0.2195(0.0482)	0.2183(0.0476)	0.0978(0.0106)	0.2369(0.1374)	0.1858(0.0346)
	45	23	1	2	0.1150(0.0142)	0.1820(0.0332)	0.1873(0.0351)	0.1234(0.0162)	0.1127(0.0148)	0.1327(0.0177)
				3	0.1181(0.0148)	0.2460(0.0606)	0.2616(0.0685)	0.1258(0.0168)	0.5415(0.0425)	0.2237(0.0501)
			2	2	0.0823(0.0076)	0.1902(0.0362)	0.1873(0.0351)	0.0953(0.0101)	0.1797(0.0324)	0.1629(0.0265)
				3	0.0859(0.0083)	0.2232(0.0498)	0.2229(0.0497)	0.0959(0.0102)	0.2117(0.0451)	0.2069(0.0428)
		34	1	2	0.1190(0.0151)	0.1806(0.0326)	0.1861(0.0346)	0.1175(0.0147)	0.1728(0.0299)	0.1800(0.0324)
				3	0.1209(0.0155)	0.2442(0.0597)	0.2607(0.0680)	0.1213(0.0156)	0.2328(0.0543)	0.2565(0.0658)
			2	2	0.0862(0.0083)	0.1896(0.0360)	0.1864(0.0348)	0.0888(0.0088)	0.1849(0.0342)	0.1792(0.0321)
				3	0.0904(0.0091)	0.2225(0.0495)	0.2223(0.0494)	0.0853(0.0082)	0.2192(0.0481)	0.2186(0.0478)

Table 6: The estimate (EST) and Mean Squared Errors (MSE) in parentheses for the entropy (P) of Weibull distribution using the kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), et T=3.0 and T=4.5.

n	m	k	α	β	T=3			T=4.5		
					Kernel Estimations	Bayes Estimations		Kernel Estimations	Bayes Estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1487(0.0239)	0.2923(0.0532)	0.3251(0.0452)	0.1520(0.0250)	0.1692(0.0301)	0.0574(0.0045)
				3	0.1515(0.0247)	0.5302(0.0623)	0.3283(0.0452)	0.1595(0.0274)	0.2396(0.0594)	0.2087(0.0440)
			2	2	0.1197(0.0162)	0.1827(0.0335)	0.1078(0.0119)	0.1204(0.0164)	0.1807(0.0327)	0.1222(0.0150)
				3	0.1183(0.0160)	0.2181(0.0476)	0.1866(0.0349)	0.1196(0.0163)	0.2170(0.0471)	0.1894(0.0359)
		8	1	2	0.1433(0.0222)	0.1610(0.0277)	0.1130(0.0129)	0.1388(0.0208)	0.1605(0.0274)	0.1135(0.0130)
				3	0.1492(0.0239)	0.2300(0.0651)	0.2332(0.0544)	0.1498(0.0241)	0.2292(0.0644)	0.2332(0.0544)
			2	2	0.1110(0.0139)	0.1820(0.0332)	0.1436(0.0206)	0.1137(0.0145)	0.1804(0.0326)	0.1481(0.0219)
				3	0.1096(0.0137)	0.2183(0.0478)	0.2007(0.0403)	0.1109(0.0138)	0.2166(0.0470)	0.2017(0.0407)
	15	8	1	2	0.1409(0.0215)	0.1594(0.0273)	0.1131(0.0129)	0.1403(0.0213)	0.1573(0.0265)	0.1138(0.0131)
				3	0.1478(0.0235)	0.2319(0.0981)	0.2338(0.0547)	0.1486(0.0237)	0.2263(0.0540)	0.2334(0.0545)
			2	2	0.1136(0.0145)	0.1821(0.0332)	0.1434(0.0206)	0.1150(0.0149)	0.1812(0.0329)	0.1453(0.0211)
				3	0.1095(0.0136)	0.2183(0.0477)	0.2007(0.0403)	0.1085(0.0133)	0.2171(0.0472)	0.2013(0.0405)
		11	1	2	0.1383(0.0206)	0.1644(0.0279)	0.1507(0.0228)	0.1402(0.0211)	0.1638(0.0277)	0.1488(0.0222)
				3	0.1430(0.0219)	0.2267(0.0530)	0.2508(0.0629)	0.1412(0.0213)	0.2258(0.0526)	0.2499(0.0625)
			2	2	0.1012(0.0116)	0.1828(0.0335)	0.1584(0.0251)	0.1048(0.0124)	0.1809(0.0328)	0.1600(0.0256)
				3	0.1046(0.0124)	0.2181(0.0476)	0.2079(0.0432)	0.1049(0.0124)	0.2167(0.0470)	0.2075(0.0430)
40	20	10	1	2	0.1392(0.0208)	0.5568(0.0352)	0.4262(0.0352	0.1390(0.0208)	0.4923(0.0352)	0.2135(0.0534)
				3	0.1420(0.0216)	0.5496(0.0532)	0.5782(0.0357)	0.1432(0.0220)	0.2857(0.0268)	0.2137(0.0459)
			2	2	0.1102(0.0136)	0.1775(0.0317)	0.1302(0.0170)	0.1107(0.0138)	0.1776(0.0316)	0.1435(0.0206)
				3	0.1050(0.0125)	0.2125(0.0454)	0.1917(0.0368)	0.1052(0.0126)	0.2130(0.0455)	0.1957(0.0383)
		15	1	2	0.1284(0.0177)	0.1357(0.0204)	0.1322(0.0175)	0.1309(0.0184)	0.1377(0.0206)	0.1333(0.0178)
				3	0.1348(0.0194)	0.1979(0.0457)	0.2342(0.0549)	0.1342(0.0193)	0.2080(0.1429)	0.2341(0.0548)
			2	2	0.0977(0.0108)	0.1783(0.0319)	0.1559(0.0243)	0.0982(0.0109)	0.1788(0.0320)	0.1578(0.0249)
				3	0.1056(0.0124)	0.2135(0.0457)	0.2034(0.0414)	0.1049(0.0123)	0.2136(0.0457)	0.2037(0.0415)
	30	15	1	2	0.1302(0.0182)	0.1377(0.0209)	0.1326(0.0177)	0.1294(0.0180)	0.1399(0.0213)	0.1332(0.0178)
				3	0.1361(0.0198)	0.2019(0.0557)	0.2343(0.0549)	0.1371(0.0200)	0.2009(0.0529)	0.2345(0.0550)
			2	2	0.0975(0.0108)	0.1781(0.0318)	0.1553(0.0241)	0.0966(0.0106)	0.1777(0.0316)	0.1568(0.0246)
				3	0.1045(0.0122)	0.2128(0.0455)	0.2034(0.0414)	0.1056(0.0124)	0.2135(0.0456)	0.2041(0.0417)
		23	1	2	0.1224(0.0161)	0.1641(0.0271)	0.1696(0.0288)	0.1224(0.0160)	0.1632(0.0268)	0.1688(0.0285)
				3	0.1265(0.0171)	0.2253(0.0512)	0.2541(0.0646)	0.1241(0.0165)	0.2268(0.0518)	0.2527(0.0638)
			2	2	0.0966(0.0104)	0.1800(0.0324)	0.1685(0.0284)	0.0977(0.0106)	0.1776(0.0316)	0.1693(0.0287)
				3	0.0950(0.0101)	0.2151(0.0463)	0.2106(0.0443)	0.0975(0.0106)	0.2155(0.0464)	0.2103(0.0442)
60	30	15	1	2	0.1298(0.0181)	0.5541(0.0342)	0.7627(0.0572)	0.1312(0.0184)	0.2060(0.2542)	0.3237(0*****)
				3	0.1371(0.0201)	0.4058(0.3859)	0.8261(0.0573)	0.1368(0.0200)	0.2834(0.6102)	0.2183(0.0478)
			2	2	0.0969(0.0107)	0.1756(0.0312)	0.1407(0.0198)	0.0990(0.0111)	0.1763(0.0312)	0.1509(0.0228)
				3	0.1051(0.0123)	0.2089(0.0442)	0.1948(0.0379)	0.1027(0.0117)	0.2106(0.0445)	0.1993(0.0397)
		23	1	2	0.1245(0.0166)	0.1415(0.0282)	0.1461(0.0215)	0.1227(0.0161)	0.1407(0.0205)	0.1490(0.0222)
				3	0.1255(0.0168)	0.1930(0.0534)	0.2386(0.0569)	0.1229(0.0162)	0.1891(0.0450)	0.2383(0.0568)
			2	2	0.0949(0.0101)	0.1774(0.0316)	0.1625(0.0264)	0.0970(0.0104)	0.1775(0.0315)	0.1633(0.0267)
				3	0.0961(0.0103)	0.2115(0.0448)	0.2064(0.0426)	0.0960(0.0103)	0.2119(0.0450)	0.2065(0.0427)
	45	23	1	2	0.1239(0.0165)	0.1455(0.0341)	0.1456(0.0215)	0.1230(0.0162)	0.1383(0.0200)	0.1486(0.0221)
				3	0.1265(0.0170)	0.1904(0.0506)	0.2388(0.0571)	0.1250(0.0167)	0.1909(0.0442)	0.2387(0.0570)
			2	2	0.0989(0.0109)	0.1774(0.0315)	0.1631(0.0266)	0.0945(0.0100)	0.1770(0.0314)	0.1624(0.0264)
				3	0.0971(0.0105)	0.2120(0.0451)	0.2064(0.0426)	0.0975(0.0106)	0.2119(0.0450)	0.2065(0.0426)
		34	1	2	0.1159(0.0143)	0.1682(0.0284)	0.1756(0.0308)	0.1185(0.0150)	0.1686(0.0285)	0.1764(0.0311)
				3	0.1236(0.0162)	0.2313(0.0537)	0.2555(0.0653)	0.1200(0.0153)	0.2312(0.0536)	0.2541(0.0646)
			2	2	0.0890(0.0088)	0.1797(0.0323)	0.1722(0.0297)	0.0888(0.0088)	0.1792(0.0321)	0.1721(0.0296)
				3	0.0871(0.0085)	0.2145(0.0461)	0.2127(0.0452)	0.0880(0.0086)	0.2151(0.0463)	0.2115(0.0447)