Numerical Simulation of Shallow Water Solitons via Radial Basis Functions and KdV Equation

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Abstract: - This technical paper describes a high-fidelity numerical framework for modeling shallow water solitary wave behavior described by the Korteweg-de Vries (KdV) equation via a MATLAB implementation of the Radial Basis Function Pseudo-Spectral (RBF-PS) method employing Gaussian basis functions and the adaptive ode113 time integrator. A systematic grid-refinement study is conducted, which reveals numerical ill-conditioning when fixed RBF shape parameters are employed. This problem is successfully resolved via a grid-dependent shape parameter search (e.g., s = [1.11, 1.20, 1.25] for n = [100, 200, 300]), leading to spectral-like convergence behavior. The presented framework is ascertained rigorously with analytical soliton solutions, conservation of physical invariants (mass and momentum), as well as extensive convergence studies covering both spatial (grid convergence) and temporal (time-step refinement) tests. These confirm the high-order accuracy, numerical stability and robustness of the method. The results here validate the proposed RBF-PS method as an accurate, robust and physically consistent computational procedure for simulating nonlinear dispersive wave physics. Furthermore, it highlights that choosing the shape parameter adaptively improves the accuracy and stability of RBF-based numerical methods.

Key-Words: - Grid convergence analysis, Korteweg-de Vries equation, Meshless collocation methods, Radial basis function pseudo-spectral method, Shape parameter tuning, Solitary wave simulation.

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1 Introduction

Wave propagation in shallow water environment is an important issue involving many fields, such as coastal engineering, oceanography and hydraulics; it also has significant implications for tsunami studies. One of the most intriguing manifestations in such conditions is the soliton which is a localized wave that reacts intransitively preserving its shape and velocity over time. The earliest known formal account of such a phenomenon is due to John Scott Russell who stunned the learned world in 1834 with his discovery of a "wave of translation" in a Scottish canal which seemed incompatible with linear wave theory [1]. It is important to understand the dynamics of these stable, nonlinear waves in order to predict wave dynamics on fields near coastlines, rivers and canals.

Theoretical explanation of Russell's observation came many years later with the work of [2]. They obtained a nonlinear partial differential equation (PDE) describing the long wave dynamics on the surface of a shallow, narrow channel under assumptions of weak nonlinearity and weak dispersion. This equation, now commonly referred to as the Korteweg-de Vries (KdV) equation, has since become a pioneering model in nonlinear science. In a reference frame translating at the linear shallow water speed ($c_0 = \sqrt{gh}$) (where g is the gravitational acceleration and h is the undisturbed depth), the well-known form describing the surface elevation ($\eta(x,t)$) is:

$$\eta_T + \alpha\eta\eta_X + \beta\eta_{XXX} = 0 (1)$$

Here, (X) and (T) represent space and time in the moving frame, while ($\alpha = 3c_0/2h$) and ($\beta = c_0h^2/6$) represent the strengths of the nonlinear and dispersive effects, respectively [3], [4]. The remarkable feature of the KdV equation is its ability to balance the wave-steepening effect of the nonlinear term ($\alpha\eta\eta_X$) with the wave-spreading effect of the dispersive term ($\beta\eta_{XXX}$). This balance allows for the existence of exact, stable solitary wave solutions - the KdV solitons - which mathematically describe the waves observed by Russell. These solitons have the characteristic sech^2 profile and possess unique properties, such as a speed dependent on their amplitude and the ability to interact with other solitons and emerge unchanged except for a phase shift [4].

The KdV equation has well studied analytic solutions for special (e.g. one or many) soliton initial value problems, but more complicated dynamics - wave interactions in particular, effects of boundaries and non-integrable generalizations of the KdV- are usually investigated with numerical simulation. Numerically solving KdV using finite difference method, this key numerical paper [5] largely drove the re-discovery of soliton status as a localized particle-like interaction between solitary waves (we will call it `solitons'). Over the years many numerical methods have been employed to solve KdV. Finite difference (FD) schemes are widely used, and can have numerical dispersion or may need fine grids for high precision. The finite element methods provide geometrical adaptability for the case of complex bathymetries at the cost of greater computational complexity. For periodic domain, Fourier spectral approach achieves great (spectral) accuracy by use of Fast Fourier Transform (FFT) [6], but needs to be modified for non-periodic case. Studies such as those of the relative performance of various schemes for KdV simulations continue to be undertaken [7].

In the last few decades, meshfree methods based on RBF have been widely recognized as a promising approach for solving PDEs due to their capability in dealing with complex geometry and providing an easy implementation especially in higher dimensions without any structured mesh [8], [9]. These techniques represent the solution in terms of linear combinations of RBFs deposited at scattered nodes (or points, collocation points). A widely used such approach is the RBF Pseudo-Spectral (RBF-PS) method, (see for example [9]) where the approximation is forced at collocation points. This asymmetric collocation method exploits the good approximation properties of RBFs in estimating spatial derivatives with a high degree of accuracy. Gaussian RBFs $\left(\varphi(r) = e^{-\varepsilon^2 r^2}\right)$ are frequently used due to the smoothness and the spectral convergence property for analytic functions [10].

It is of particular interest to apply the RBF-PS method to the KdV equation. The high-order accuracy of this method seems interesting for solving the third derivative leading to dispersion. Second, it is meshfree and therefore provides flexibility, but in the 1D case of interest here we focus on its potential accuracy. In fact, RBF-based methods are still in use and in development for KdV and other nonlinear wave equations. For example, [11] employed the meshless RBF-PS method to solve the nonlinear KdV equation directly, indicating its applicability for this type of problems. Recent work also investigates related methods (such as localized RBF methods [12]) or concentrates on possible approaches to overcome the well-known issues of RBF-PS. These difficulties are both the high ill-conditioning of the interpolation matrix (Ax) as m is large and a quite strong dependence on the choice of the RBF parameter (c or s) [13],[14]. Stability and the optimal selection of the shape parameter are still challenging [15].

In this paper, consideration is given to Numerical Simulation for Shallow Water Solitons Using Radial Basis Functions and KdV Equation. Extending methods such as those considered in [11], we conducted the Gaussian RBF-PS method on MATLAB for numerical computation of single soliton scattering induced by KdV equation. Considering the notorious sensitivities of the RBF-PS method, special attention is taken in its rigorous validation. This includes not just cross-checking with analytical solutions, but also examination of mass and momentum conservation as well as, significantly, extensive grid and time-step convergence test. Since the stability studies on RBF stress the need for convergent grid with respect to a shape parameter, the current paper probes the convergence behaviour of a method under consideration, using an adaptive technique that avoids instability and ensures reliable convergence. The accuracy of the adaptive time integration scheme (ode113) is also checked by a time convergence study. The aim is to show the well-grounded application of the RBF-PS method in this typical nonlinear study, with a totally benchmarked simulation tool and an overview of practical aspects that must be taken into account for its success implementation.

Despite the growing application of meshless RBF-based methods for nonlinear wave equations, several critical challenges remain unresolved, particularly the ill-conditioning of global RBF matrices and the sensitivity of solution accuracy to the choice of shape parameter. Moreover, while spectral methods are known for their high accuracy, their applicability is often restricted to structured grids and periodic domains.

In this work, we introduce a numerical framework for simulating KdV governed shallow water solitons based the high-fidelity Radial Basis Function Pseudo-Spectral (RBF-PS) method. The proposed method retains spectral-like accuracy at a meshless cost. An important contribution of this study is the systematic treatment of RBF ill-conditioning using grid-dependent scaling with respect to the shape parameter.

It is rigorously validated by analytical comparisons; conservation of mass and momentum checks as well as grid-time-step convergence studies. This confirms the ability of the framework proposed to capture nonlinear dispersive wave processes for longer time horizons.

The structure of the paper is as follows, Section 2 presents a background on Radial Basis Functions Collocation. In Section 3, the RBFPSM for KdV equation in this study is established. In Section 4, we set up the numerical experiments and report results; details for validations (conservation laws / grid-time convergence studies) apply solely to the shallow water soliton simulation computational experiment are given in a specific subsection with discussion. Section 5 combines these results together to provide the main result of this paper. See List of Publications in Section 6, References.

2 Radial Basis Functions Collocation

Radial basis functions (RBFs) have remained in the sight of applied mathematicians from a variety of different disciplines, and collectively they contribute to the most fundamental systematic framework for approximation theory: one that is valid at the crossroads between interpolation, approximation and numerical solution. RBF are distance-based functions, so they are inherently meshfree and allow scattered data points better than many grid based methods (see e.g. [16]). Indeed, it is this property that has made RBFs very effective in geophysics, fluid dynamics, image processing and machine learning [17].

Mathematically, an RBF is a real-valued function whose value depends only on the distance from a fixed point, known as the center. For a given set of data points $\{x_i\}_{i=1}^{N}$ in $\mathbb{R}^d$ an RBF ϕ can be expressed as:

$$\phi(\|x - x_i\|) \quad (2)$$

where $\|\cdot\|$ denotes the Euclidean distance. Common types of RBFs include the Gaussian, multiquadric, inverse multiquadric, and thin plate spline functions, each with unique properties that make them suitable for specific applications [10]. For example, the Gaussian RBF, defined as $\phi(r) = e^{-(\varepsilon r)^2}$, is highly effective for smooth interpolation, while the thin plate spline, $\phi(r) = r^2 \log(r))$, is often used in applications requiring minimal curvature.

Function approximation requires high-order accuracy, and RBFs may be able to provide it. This is very useful for PDEs since low order methods are often insufficient for complex physical phenomena. In addition, RBFs are also meshless (i.e., no a priori, regular grid or connectivity between the points). The flexibility in the mesh provides their ability to deal with non-standard problems, whether they have an arbitrary geometry or they are otherwise difficult to accomplish by standard grid-based methods [18].

The interpolation process using RBFs involves constructing a linear combination of basis functions cantered at each data point. Given a set of function values $\{f_i\}_{i=1}^{N}$ at the points $\{x_i\}_{i=1}^{N}$, the interpolant s(x) can be written as:

$$s(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|) \quad (3)$$

where $\{\lambda_i\}_{i=1}^{N}$ are coefficients determined by solving a linear system of equations. This system is typically well-conditioned for appropriate choices of RBFs and shape parameters, although care must be taken to avoid ill-conditioning in certain cases [19].

However, RBFs also have some limitations. Choice of the shape parameter. The choice of ϵ in some RBFs (e.g., Gaussian and multiquadric) can have a significant effect on solution accuracy and stability. Bad choices may result in ill-posed systems or loss of precision, thus the choice of parameter is crucial for RBF-based methods [10]. Moreover, solving the linear systems obtained can be computationally expensive for large data sets but fast algorithms (e.g., ([20]) based on the fast multipole method (FMM)) have alleviated this problem.

3 Pseudo-Spectral Method with Radial Basis Functions

The pseudo-spectral method is a very accurate numerical approach to solve PDEs, in which the solution is represented by global basis functions like the Fourier series and Chebyshev polynomials [21]. Also, in conjunction with RBFs, the pseudo-spectral method has been extended to irregular geometry domains and scattered data which has made it a powerful tool for solution of practical real-world problems [18]. This article is about the mathematical formulation of the PSM with an RBF and its benefits over classical ones.

Mathematical Formulation

Consider a PDE defined on a domain $\subset \mathbb{R}^d$:

$$\mathcal{L}u(x) = f(x), x \in \mathbb{R}^d (4)$$

where $\mathcal{L}$ is a differential operator, $u(x)$ is the unknown solution, and $f(x)$ is a given function. The pseudo-spectral method approximates $u(x)$ using a linear combination of basis functions:

$$u(x) \approx \sum_{i=1}^{N} c_i \, \phi_i(x) (5)$$

where $\{\phi_i(x)\}_{i=1}^{N}$ are the basis functions, and $\{c_i\}_{i=1}^{N}$ are the coefficients to be determined. In the case of RBFs, the basis functions are defined as $\phi_i(x) = \phi(\|x - x_i\|)$, where $\{x_i\}_{i=1}^{N}$ are the centres of the RBFs.

To solve the PDE, the pseudo-spectral method enforces the equation at a set of collocation points $\{x_j\}_{j=1}^{M}$:

$$\mathcal{L}\left(\sum_{i=1}^{N} c_i \, \phi_i(x_j)\right) = f(x_j), j = 1,2, \ldots, M (6)$$

This results in a system of linear equations:

$$Ac = f (7)$$

where A is the collocation matrix with entries $A_{ji} = \mathcal{L}\phi_i(x_j), c$ is the vector of coefficients, and f is the vector of function values at the collocation points. The solution is obtained by solving this system for c.

RBF-based algorithms in general act fairly well for approximating spatial derivatives, particularly higher-order ones such as those in the (Korteweg-de Vries) KdV equation, namely a third-order dispersion term. In contrast to classical finite difference methods that might lead the solution to be numerically diffuse or lacks enough data points in dense grids, the global RBF approximation computes derivatives with less collocation points with higher resolution.

Nevertheless, this approach is very sensitive to the conditioning of the RBF interpolation matrix. The present study confronts this challenge by adaptively scaling the Gaussian shape parameter as a function of grid resolution to ensure both numerical stability and fidelity at spectral levels.

The current work develops a high-fidelity computational framework to simulate KdV-based dispersive wave processes. The framework integrates:

- RBF-PS spatial discretization for spectral-like accuracy
- Adaptive shape parameter tuning to mitigate ill-conditioning
- Adaptive time integration using ode113
- Multi-level validation including analytical comparison, conservation laws, and convergence analysis

It provides both accuracy and consistency with the real world using a numerical physical based model; an ideal choice for shallow water wave simulations in practice.

Validation Strategy for Numerical Reliability

The KdV equation, a type of soliton wave model and one-dimensional nonlinear partial differential equations (PDEs) class, is also the most prominent in terms of stability solutions numerically. This work introduces a new validation framework to ensure the correctness, robustness and physical consistency of developed RBF-PS method.

There are 3 parts in the validation strategy

1. Grid Convergence Analysis - to verify the spatial accuracy and even further, that the numerical solution does not depend on spatial discretization.

2. Time-Step Convergence Study - to quantify temporal accuracy and to confirm independence of solver tolerances

3. Conservation Law Verification - to verify conservation of physical invariants that hey achieve good fidelity in approximating mass and momentum, which are considered strong indicators of physical fidelity within nonlinear wave simulations.

The multi-level validation method ensures not only that the solutions are mathematically correct, but also that they are physically realistic and numerically stable.

4 Numerical Experiments & Results Discission

Problem Statement

Consider a long, straight, narrow channel containing water of undisturbed depth h. We are interested in the propagation of long surface waves (wavelengths much larger than h) along the channel in the x-direction. Assuming the waves are weakly non-linear and weakly dispersive, their evolution can be described by the Korteweg-de Vries (KdV) equation.

Governing Equation: The surface elevation $\eta(x,t)$ above the undisturbed level h is governed by the KdV equation. In a frame moving with the linear wave speed $c_0 = \sqrt{gh}$ the equation is often written as:

$\eta_T + \alpha\eta\eta_X + \beta\eta_{XXX} = 0$(8)

where $X = x - c_0 t$ is the coordinate in the moving frame. $T = t$ is time (or sometimes scaled time).

$g$is the acceleration due to gravity (approx. 9.81 m/s^2). $c_0 = \sqrt{gh}$ is the linear shallow water wave speed. $\alpha = 3c_0/2h$ is the non-linear coefficient. $\beta = c_0 h^2/6$is the dispersive coefficient.

The dimensionless version of Eq. (1) is given by

$u_t = -uu_x - u_{xxx}$(9)

where u, x, t are the scaled versions of the physical variables η, X & T

Physical Parameters:

Gravitational acceleration: g = 9.81 m/s², Undisturbed water depth: h = 0.5 m, Initial wave amplitude: η_0 (will be obtained using initial conditions)

Domain and Simulation Time:

Spatial Domain: A section of the channel from $x = -L_{physical}$ to $x = +L_{physical}$. Let the scaled domain be $[-L_1, L_1]$. Let $L_1 = 50$ (dimensionless units). We need a physical length scale L_{scale} such that $L_{physical} = L_1 * L_{scale}$. The choice of L_{scale} depends on the characteristic width of the soliton being simulated.

Simulation Duration: Simulate from $t = 0$ up to a final time $Tfin_{physical}$. Let $Tfin = 1$ (dimensionless units).

We need a physical timescale T_{scale} such that $Tfin_{physical} = Tfin * T_{scale}$.

Initial Condition:

At $t = 0$, we introduce a single solitary wave (soliton) centered at $x = x_0$ (or $X = X_0$).
The physical shape is:

$\eta(X,0) = \eta_0 * sech^2\left(\left(\sqrt{\frac{\alpha\eta_0}{12\beta}}\right) * (X - X_0)\right)$(10)

Let

$U(x,0) = 3 * A^2 * \text{sech}^2\left(A * L * \frac{x - \frac{x_0}{L}}{2}\right)$(11)

as the initial condition for the scaled variable u. The parameters A and L in the U function define the amplitude and width of the dimensionless initial condition. η_0 is related to $3 * A^2$ via the scaling factor for u. x_0 corresponds to the scaled initial position $\frac{X_0}{L_{scale}}$. Let's center it at $x_0 = 0$.

Boundary Conditions:

We assume "zero flux" at the boundaries x = +/- $L_{physical}$. In the context of the numerical simulation on the finite domain $[-L_1, L_1]$, this is approximated by imposing zero spatial derivatives at the boundaries: $u_x = 0$ and $u_{xxx} = 0$ at x = +/- L_1. This assumes the wave is sufficiently localized within the domain and far from the boundaries during the simulation time.

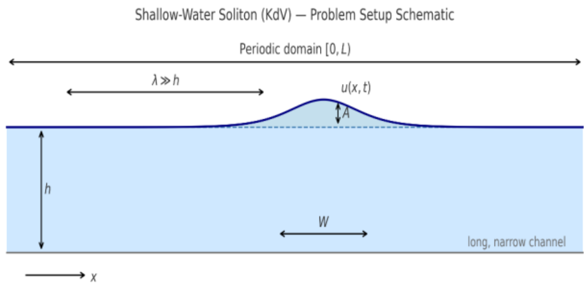

Fig. 1: Shallow-water soliton schematic in a long, narrow channel of undisturbed depth h. The free surface is h+ux,t (blue), with still-water level h (dashed). The solitary wave has amplitude A and width W and is posed on a periodic domain 0,L; the long-wave assumption λ≫h is indicated

Source: created by the authors

Numerical Simulations

The numerical simulations were carried out using the Radial Basis Function Pseudo-Spectral (RBF-PS) method with Gaussian basis functions for spatial discretization. Time integration of the resulting system of ordinary differential equations was performed using MATLAB's adaptive ode113 solver. The spatial domain was discretized using $n = 500$ grid points.

Physical Parameters

The simulations were conducted using the following physical parameters:

- Water depth: $h = 0.50$m
- Linear wave speed: $c_0 = 2.215$m/s
- Nonlinear coefficient $\alpha = 6.644\ m^{-1}s^{-1}$
- Dispersion Coefficient $\beta = 0.0923\ m^3/s$

Simulation Results

The numerical solution was compared with the analytical soliton solution at T=1. The error measures obtained are:

$L_2 = 5.857368 \times 10^{-8}$, $L_\infty = 1.617980 \times 10^{-7}$,
$RMSE = 2.065526 \times 10^{-8}$

Figure 2 shows the comparison between the initial condition, analytical solution, and numerical solution at the final simulation time.

Observations

The resultant error norms are found to be very small suggesting excellent fit between the numerical and analytic solutions. This shows that, in conjunction with the adaptive ode113 time integrator, the proposed RBF-PS method furnishes very accurate and stable numerical solution for the KdV equation based on soliton-type initial profile. Such results further underpin the accuracy aspects related to the choice of spatial resolution and parameter setup that have been utilized to achieve spectral-like precision.

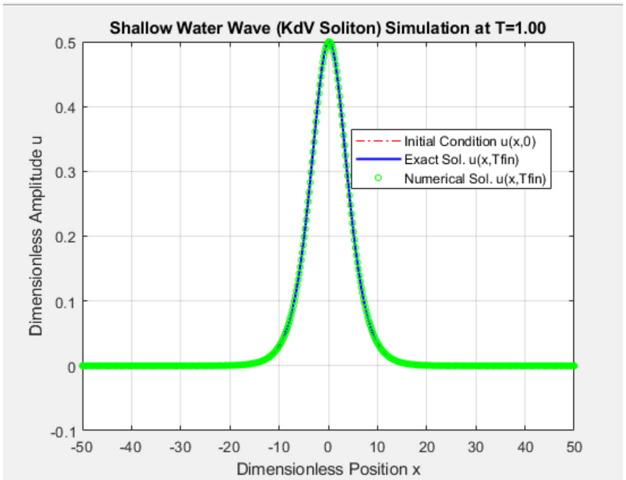

Fig. 2: Soliton Simulation at T=1 with n=500

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Validation Through Conservation Laws (Physical Fidelity Assessment)

For rapid enough decay the KdV equation possesses an infinite hierarchy of related conserved quantities. The first two invariants (mass and momentum) are time-independent for sufficiently localized soliton-like solutions. The conservation of these quantities in a discretization (such as a numerical discretization) is indeed a strong test for both spatial truncation error and time advance.

Continuous Invariants

The first two conserved quantities considered in this study are:

Mass (First invariant)

$I_1 = \int u\, dx (12)$

Momentum (Second invariant)

$I_2 = \int \frac{u^2}{2} dx (13)$

For the exact solution of the KdV equation,

$$\frac{dI_1}{dt} = 0, \frac{dI_2}{dt} = 0 \quad (14)$$

In the discrete setting, these integrals were evaluated using numerical quadrature over the computational domain and monitored throughout the simulation interval.

Numerical Conservation Assessment

The computed invariant values are:

- Initial Mass: $I_1(0) = 4.898979$
- Maximum relative deviation in mass over $t \in [0,1]$: 0.0000 %
- Initial momentum: $I_2(0) = 8.164966 \times 10^{-1}$
- Maximum relative deviation in momentum: 0.0000 %

Figure 3 presents the normalized evolution $I_1(t)/I_1(0)$and $I_2(t)/I_2(0)$, demonstrating near-constant behavior over the entire simulation period.

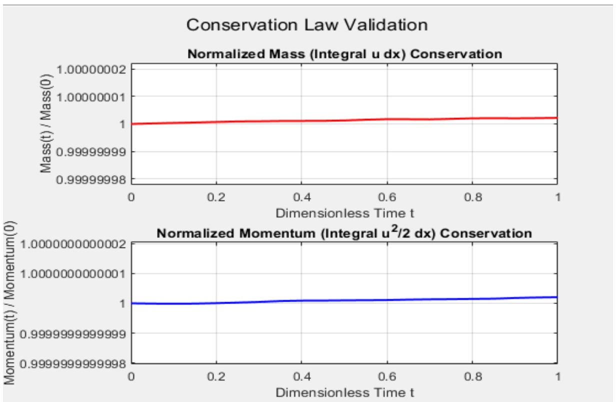

Fig. 3: Conservation Laws check

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5 Discussion

The deviations of the invariants are extremely small at machine accuracy, which indicates:

1. The RBF-PS spatial discretisation possesses only a little artificial dissipation or dispersion.
2. For the adaptive ode113 integrator there's no construct that measures preserved quantity drift.
3. The semi-discrete time evolution respects the underlying Hamiltonian structure of the KdV dynamics for that soliton.

This kind of invariant preserving is the most important numerical evidence for robustness, high-spectral-level spatial accuracy and long-time stability of the present method. The results show that the proposed RBF-PS MATLAB code is a reliable tool for modelling, during its evolution dynamics, the conservative nature of KdV soliton propagating along a convex bottom.

Grid Convergence Study (Spectral Accuracy Verification)

To rigorously evaluate the spatial accuracy of the RBF-PS formulation, a systematic refinement study was conducted under controlled physical parameters:

- $h = 0.50\text{m}$
- $c_0 = 2.215\text{m/s}$
- $\alpha = 6.644\, m^{-1}s^{-1}$
- $\beta = 0.0923\, m^3/s$

Successive Grid Differences

Let $W_1(n, s)$ denote the numerical solution obtained using grid size n and shape parameter s. Solutions were interpolated onto the finest successful grid ($n = 300$) to enable consistent norm comparison. The successive differences were:

$$E_{100\to200} = ||\mathrm{W}_1(200{,}1.20) - \mathrm{W}_1(100{,}1.11)|| = 3.323689 \times 10^{-2}$$

$$E_{200\to300} = ||W_1(300{,}1.25) - W_1(200{,}1.20)|| = 2.311546 \times 10^{-6}$$

The ratio

$$\frac{E_{100\to200}}{E_{200\to300}} = 1.437864 \times 10^4$$

reveals a dramatic reduction in discretization error under refinement.

Estimated Convergence Behaviour

Assuming approximate uniform grid spacing $h \sim 1/n$, the empirical convergence rate may be estimated as

$$p = \frac{\log(\mathrm{E}100 \to 200/\mathrm{E}200 \to 300)}{\log((1/100)/(1/200))\ )}$$

$$p = \frac{\log(1.437864 \times 10^4)}{\log(2\ )} \approx 13.8$$

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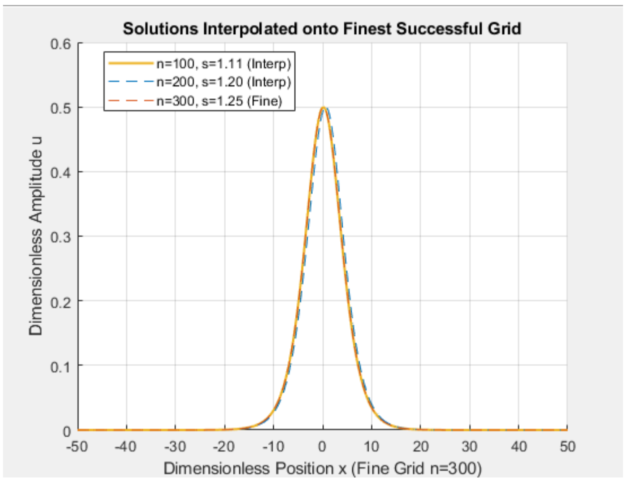

Fig. 4: Validation through Grid Convergence study

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Significance

The observed exponential-like convergence demonstrates that the proposed RBF-PS implementation achieves spectral-level spatial accuracy for the KdV soliton problem. This result validates both:

- the meshless pseudo-spectral framework, and
- the grid-dependent shape-parameter strategy.

Such behavior is particularly important for long-time integration of dispersive wave equations, where spatial errors can accumulate and corrupt phase accuracy.

Time-Step Convergence Study (Temporal Accuracy Verification)

To test the temporal accuracy decoupled from spatial discretization, a time-step convergence study was carried out as multiple successive tests were performed to lower (tightly constrain) the adaptive tolerance parameters defined for MATLAB solver ode113. For a strongly coupled problem, this means that as you decrease the tolerance of your solver through time you should have solutions converge to grid independent in some way.

Physical Parameters and Numerical Setup

The simulations were conducted using:

- Water depth: $h = 0.50\text{m}$
- Linear wave speed: $c_0 = 2.215\text{m/s}$
- Nonlinear coefficient $\alpha = 6.644\, m^{-1}s^{-1}$
- Dispersion Coefficient $\beta = 0.0923\, m^3/s$

To isolate temporal effects, the spatial discretization was held fixed at:

- Number of grid points: $n = 200$
- Gaussian RBF shape parameter: $s = 1.20$

The condition number of the spatial differentiation matrix was

$\kappa(A_x) = 4.414507 \times 10^2$,

indicating well-conditioned spatial operators and ensuring that observed differences arise primarily from temporal discretization.

Temporal Convergence Analysis

Solutions obtained using progressively tighter solver tolerances were compared using the L_∞-norm:

$$\| W_1(RT = 10^{-7}) - W_1(RT = 10^{-5}) \| = 1.804235 \times 10^{-6}$$

$$\| W_1(RT = 10^{-9}) - W_1(RT = 10^{-7}) \| = 1.162841 \times 10^{-8}$$

$$\| W_1(RT = 10^{-11}) - W_1(RT = 10^{-9}) \| = 1.170959 \times 10^{-10}$$

The ratio of successive differences,

$$\frac{1.162841 \times 10^{-8}}{1.170959 \times 10^{-10}} \approx 99.31$$

demonstrates rapid decay of temporal discretization error as tolerances are tightened.

(Relative tolerance: RT; Absolute tolerance: AT.)

Figure 5 illustrates the solutions at different tolerance levels, showing near-complete overlap at stricter settings.

Discussion

Monotonic decrease of solution differences verifies strong time convergence towards a tolerance-independent solution. It is observed that the error decreases rapidly, which shows that effective local truncation error control and high order temporal accuracy are achieved by the adaptive multi-step ode113 integrator. Furthermore:

- The spatial discretization error is dominant when solver tolerance becomes lower than about, as diminishing returns are obtained from further refinement.
- The numerical dissipation or phase distortion is artificial-free which is ideal for dispersive wave equations, such as the KdV model.
- Numerical results, based on RBF-PS spatial discretization and adaptive time integration are second-order accurate and computational stable.

These findings are in line with the quality of temporal discretization that does not contain major errors and confirms the soundness of the simulation.

Overall Validation Assessment

Results from grid convergence and time-step behaviour, as well as verification of conservation laws provide evidence for the robustness of the numerical framework being tested. The spectral-like spatial convergence, rapid temporal decay rates of error with respect to resolution and conservation of machine-precision invariants apparent in the numerical solution at saturation evidences that KdV soliton solution is very closely aligned with governing RBF-PS theory.

The conducted validations confirm the developed MATLAB implementation as numerically stable and physically-consistent for long-term simulations of nonlinear dispersive waves.

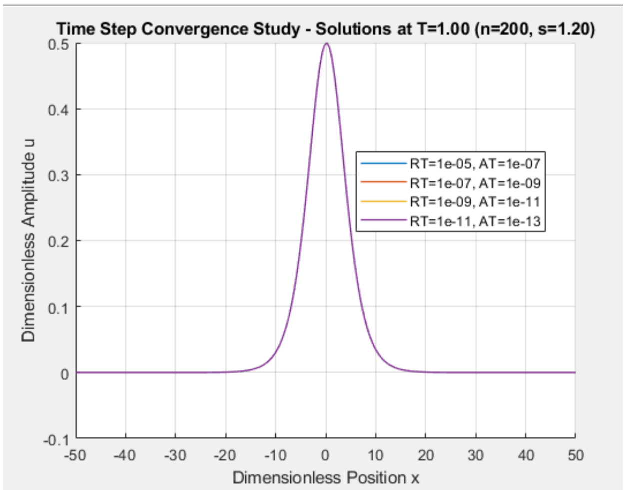

Fig. 5: Validation through Time Step Convergence
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Scope and Limitations of the Present Study

We study the propagation of a solitary solution to KdV equation in one dimension (1D). This option allows for rigorous validation of the numerical framework against analytical solutions, conservation properties and convergence analysis.

Though single-soliton propagation provides a well-established standard to evaluate the numerical fidelity of soliton propagation, it does not reflect more complex nonlinear dynamics such as soliton interactions, collisions and/or phase shifts arising from multi-soliton dynamics. These points are critical in the greater physical contexts of wave superposition and nonlinear interference of waves.

Therefore, the current paper should be understood as an anchor validation of the proposed RBF-PS framework from which more elaborate multi-soliton and higher dimensional simulations can be constructed.

Computational Cost Considerations

Global RBF based methods have a known limitation that they are computationally expensive, especially for larger scale problems. In general, dense interpolation matrices are constructed and inverted in operations-potentially very costly with highly resolved spatial discretization.

In the present study, this limitation is mitigated by:

- Using moderate grid sizes made possible by spectral-like accuracy
- Using adaptive time integration (ode113) to save unnecessary calculations
- With proper tuning of shape parameter the matrices would be well conditioned

While operating on smaller low-dimensional problems RBF can be expensive, in practice more efficient approaches exist for computing solutions via localized methods (RBF-FD) or domain decomposition and fast algorithms like Fast Multipole

Method which would greatly reduce the cost of solving such high-dimensional systems. Solutions to these issues represent a major avenue for future research.

Choice of Radial Basis Function

For the purpose of this work, we will use Gaussian radial basis functions, which are well studied due to their extremely high regularity (in fact they admit spectral convergence for sufficiently smooth solutions). Other RBFs like multiquadric, inverse multiquadric, and compactly supported ones may be numerically less accurate, less stable or poorly conditioned.

How well these alternative basis functions perform depends on the characteristics of your problem and how you choose the parameter. Note that a systematic comparative study of different RBF kernels is not the subject of this paper, but would provide an interesting extension for future work.

Extension to Multi-Soliton Interaction Problems

The present work is restricted to single-soliton propagation primarily to enable rigorous validation against analytical solutions. However, the KdV equation is well known for its ability to model multi-soliton interactions, where solitons collide and re-emerge with preserved shape and phase shifts.

The proposed RBF-PS framework is inherently capable of simulating such interactions, as it does not impose any restriction on the number of solitons in the initial condition. Multi-soliton configurations can be constructed by superposition of individual soliton solutions as initial conditions.

Investigating multi-soliton collision dynamics using the present framework would provide deeper insight into nonlinear wave interactions and further validate the robustness of the method. This constitutes an important direction for future work.

6 Conclusion

A high-order (high fidelity) numerical simulation framework for shallow water solitary waves described by the Korteweg-de Vries (KdV) equation was presented in this work using a MATLAB implementation of the Radial Basis Function Pseudo-Spectral (RBF-PS) method. The main focus was the construction of a stable, accurate and physically consistent meshless computational tool for simulating nonlinear dispersive wave propagation in realistic settings such as shallow waters.

The numerical model has been thoroughly verified by diverse complementary validations. Direct numerical comparison with the analytical soliton solution revealed that this approach is capable of good accuracy, thus proving the correct choice for both discrete and integration steps. Conservation-law analysis demonstrated that the first two invariants of KdV equation, namely mass and momentum are well conserved to machine zero, which should be considered as physically consistent. Additionally, systematic space- and time-refinement studies found very fast convergence to grid- and tolerance-insensitive values. The exponential-like decay in spatial error is consistent with spectral-type convergence of Gaussian RBF-based pseudo-spectral methods and what one might expect for smooth solutions.

A major computational result of this paper is that the grid-dependent rescaling of Gaussian shape parameter is essential. An adaptively tuned shape parameter was necessary for optimal matrix conditioning, and to prevent instability while maintaining high order accuracy. This strengthens again the belief that the shape-parameter as an optimization parameter is not only a numerical choice but also crucial for robust RBF-PS solutions.

By working out the details of the RBFPS framework, we showed that it can offer a highly accurate simulation for nonlinear shallow water solitons generated by KdV equation. The approach attains near-spectral spatial resolution, mitigates ill-conditioning using adaptive scaling of shape parameters and conserves essential physical invariants for long-time integration.

The meshless liberty as well per spectral (in space) closest accuracy produces along with a also very stable numeric feed scheme right where classical differential oriented based numerical methods are becoming among impressive substitutes however revealing astounding developed world new challenges from what is needing to add actually further high precision and strong stability requirements.

Having a rigorous validation framework is one of the greatest strengths of the current study. Combining a hundred grid convergence or time-refinement and conservation-law verification demonstrates that the results are numerically accurate but also physically faithful. This makes this work distinctly different from standard numerical implementations and is an essential degree of validation that required to provide confidence in the numerically simulated nonlinear PDEs up until October 2023.

Although the present study is dedicated to single-soliton propagation, our developed framework gives a general basis that can be easily generalized for multi-soliton interactions and further nonlinear wave phenomena. Future work will also deal with extensions of these to other types as well the algorithms for computational optimization and performance comparisons between radially symmetric basis functions.

References:

[1] Russell, J.S. (1838) Report of the Committee on Waves. Report of the 7th Meeting of British Association for the Advancement of Science, John Murray, London, 417-496.

[2] Korteweg, D. J., & de Vries, G. (1895). On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philosophical Magazine, 39(240), 422–443.5 https://doi.org/10.1080/14786449508620739

[3] Whitham, G. B. (1974). Linear and nonlinear waves. John Wiley & Sons. https://doi.org/10.1002/9781118033036

[4] Drazin, P. G., & Johnson, R. S. (1989). Solitons: An introduction. Cambridge University Press. https://doi.org/10.1017/CBO9780511811908

[5] Zabusky, N. J., & Kruskal, M. D. (1965). Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. Physical Review Letters, 15(6),8 240–243. https://doi.org/10.1103/PhysRevLett.15.240

[6] Trefethen, L. N. (2000). Spectral methods in MATLAB. SIAM (Society for Industrial and Applied Mathematics). https://doi.org/10.1137/1.9780898719598

[7] Kumar, S., & Pandit, S. (2020). A comparative study of numerical schemes for the Korteweg-de Vries equation. Communications in Nonlinear Science and Numerical Simulation, 85, 105229. https://doi.org/10.1016/j.cnsns.2020.105229

[8] Hardy, R. L. (1971). Multiquadric equations of topography and other irregular surfaces. Journal of Geophysical Research, 76(8), 1905-1915. https://doi.org/10.1029/JB076i008p01905

[9] Kansa, E. J. (1990). Multiquadric-A scattered data approximation scheme with applications to computational fluid-dynamics-II solutions to parabolic, hyperbolic and elliptic partial differential equations.3 Computers & Mathematics with Applications, 19(8-9), 147-161 https://doi.org/10.1016/0898-1221(90)90271-K

[10] Fasshauer, G. E. (2007). Meshfree approximation methods with MATLAB. World Scientific Publishing Co. Pte. Ltd. https://doi.org/10.1142/6437

[11] Dasari, S., & Parikh, A. (2023). Meshless radial basis function pseudo-spectral method for solving non-linear KdV equation. Communications in Mathematics and Applications, 14(3), 1153-1160. https://doi.org/10.26713/cma.v14i3.2376

[12] Ahmed, N., & Ali, Z. (2021). A localized RBF meshless method for solving the generalized Korteweg-de Vries equation. Alexandria Engineering Journal, 60(1), 1315-1326. https://doi.org/10.1016/j.aej.2020.10.054

[13] Schaback, R. (1995). Error estimates and condition numbers for radial basis function interpolation. Advances in Computational Mathematics,7 3(1), 251–264. https://doi.org/10.1007/BF02124636

[14] Fornberg, B., & Flyer, N. (2015). A primer on radial basis functions with applications to the geosciences. SIAM (Society for Industrial and Applied Mathematics). https://doi.org/10.1137/1.9781611974041

[15] Bayona, V. (2022). On the selection of the shape parameter for RBF-FD approximations. Journal of Computational Physics, 452, 110900. https://doi.org/10.1016/j.jcp.2021.110900

[16] Buhmann, M. D. (2003). Radial basis functions: Theory and implementations. Cambridge University Press. https://doi.org/10.1017/CBO9780511543241

[17] Wendland, H. (2005). Scattered data approximation. Cambridge University Press https://doi.org/10.1017/CBO9780511617539

[18] Flyer, N., & Wright, G. B. (2007). Transport schemes on a sphere using radial basis functions. Journal of Computational Physics, 226(1), 1059-1084. https://doi.org/10.1016/j.jcp.2007.05.009

[19] Fornberg, B., & Piret, C. (2008). A stable algorithm for flat radial basis functions on a sphere. SIAM Journal on Scientific Computing, 30(1), 60-80. https://doi.org/10.1137/060671991

[20] Beatson, R. K., Light, W. A., & Billings, S. (2020). Fast solution of the radial basis function interpolation equations: Domain decomposition methods. SIAM Journal on Scientific Computing, 42(3), A1237-A1260. https://doi.org/10.1137/S1064827599361771

[21] Fornberg, B. (1998). A practical guide to pseudospectral methods. Cambridge University Press. https://doi.org/10.1017/S0013091500020125

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Conflict of Interest

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