High Resolution ISAR Using Adaptive Filtering and Stochastic Calculus

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Abstract: - We study the case of two or more closely spaced reflectors of a moving target (the distance between the reflectors is less than the Fourier transform resolution). This is a common problem in synthetic aperture Radar (SAR) and inverse SAR (ISAR). We use adaptive noise canceling to find the sinusoid (representing small reflector) that is obscured by the sinusoid of the dominant reflector. Moreover the echo phase, of ISAR, is modeled to have an additive Wiener process that represents the error in the target Doppler shift equation. The maximum likelihood method of Girsanov theory is used to find the parameters of the recovered small signal.

Key-Words: - Inverse synthetic aperture Radar (ISAR), High resolution Radar, Adaptive noise cancelling, Stochastic calculus, Girsanov theory, Maximum likelihood estimation

Received: March 17, 2026. Revised: April 16, 2026. Accepted: May 22, 2026. Published: July 15, 2026.

1 Introduction

Pulse compression schemes based on conventional Fourier techniques are efficient and robust; however, they often suffer from relatively poor resolution [1]. The relative motion of a target to the radar can generally be decomposed into two parts: translation and rotation [2]. Achieving high-resolution inverse synthetic aperture Radar (ISAR) imaging requires precise compensation for the target's translational and rotation motions. However, accurately estimating the motion parameters of the target can be challenging [3] even if one uses super resolution methods. [4] introduced an ISAR motion compensation and imaging method, approximating the target's translational motion as uniform. However, this simplistic approximation leads to residual phase errors, making focusing a difficult task. An approach to mitigate this is integrating autofocusing into the ISAR image reconstruction process.

Almost all the proposed methods simultaneously search for all the reflectors; this is a daunting task. In this report, and from the Fourier transform peaks, we suggest to isolate the signal of each reflector one at a time (the peak represents one reflector or more). The Fourier transform has dominant peaks. Each peak is isolated through band pass filter, and inverse Fourier transform is used to return the echo to the time domain. The peak might represent one reflector or more. An adaptive line enhancement is used to reduce the noise level and, through hypothesis testing, we determine if we have more than one reflector. We then use adaptive noise canceling to remove the dominant peak since we approximately know its frequency but not its amplitude nor its phase. The dominant frequency of the remainder is then calculated using, for example, autoregressive (AR) model. This process is repeated till the remainder is just noise. Compared to other methods, this proposed approach drastically improves resolution beyond super resolution methods and even at relatively small signal to noise ratio (SNR)

This paper is organized as follows: in Section 2, problem formulation is presented and the conventional methods of Fourier transform and AR model are used to find the peaks/reflectors. In Section 3, we present the proposed approaches of adaptive line enhancement, adaptive noise canceling, and maximum likelihood method for frequency estimation. In Section 4, we present simulated data for both known sum of sinusoids and simulated ISAR of two reflectors.

2 Problem Formulation

The target is composed of point scatterers and the ISAR received signal is a superposition of each return from the individual scatterers. The target return from the kth Radar pulse is written as [3]:

$$S_R(t,k) = \sum_i a_i S(t - kT - \tau_i) \qquad (1)$$

Where T is the pulse width, a_i is the amplitude and τ_i is the time delay corresponding to the range of the ith scatterer. The time delay could be expressed as:

$$\tau_i = \frac{2R(t, x_i)}{C} \qquad (2)$$

Where $R(t, x_i)$ is the range to the ith scatterer, and C is the speed of light. Taking the Fourier transform of the reflected time domain signal, we get:

$$S_R(f,k) = S(f)\sum_i a_i e^{-j2\pi f\left(kT + \frac{2R(t,x_i)}{C}\right)} \qquad (3)$$

Where $S(f)$ is the Fourier transform of the transmitted signal.

When the object has both translational and rotational motion, the range of the ith scatterer can be expressed as

$$R(t,x_i) = R_o + v_R t + a_R t^2/2 + x_i \sin\theta(t) + y_i \cos\theta(t) \qquad (4)$$

Where R_o is the distance between the radar and the center of the target, v_R is the speed of the target, a_R is the acceleration of the target, $\theta(t) = \Omega t$ is the angular speed or rotational displacement, (x_i, y_i) is the coordinates of the ith scattering point on the target.

The reflected signal becomes:

$$S_R(f,k) = S(f) e^{-j2\pi f\left(kT + \frac{2R_0}{C}\right)} \sum_i a_i e^{-j\frac{4\pi}{C} f\left(v_R t + a_R t^2/2 + x_i \sin\theta(t) + y_i \cos\theta(t)\right)} \qquad (5)$$

For the ith scatterer, the phase angle becomes:

$$\varphi_i(t) = \frac{4\pi f R_i(t)}{C} = \frac{4\pi}{C} f\left(R_0 + v_R t + a_R t^2/2 + x_i \sin\theta(t) + y_i \cos\theta(t)\right) \qquad (6)$$

The Doppler shift, which is the derivative of the phase, becomes:

$$f_{Di}(t) = \frac{1}{2\pi}\frac{d\varphi_i(t)}{dt} = \frac{2}{C} f\left(v_R + a_R t + x_i \frac{d\theta(t)}{dt}\cos\theta(t) - y_i \frac{d\theta(t)}{dt}\sin\theta(t)\right) \qquad (7)$$

Where f is the carrier frequency. Using Taylor series expansion, around zero,

$$\sin x = x - \frac{x^3}{3!} + \ldots = \sum_n \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

$$\cos x = 1 - \frac{x^2}{2!} + \ldots = \sum_n \frac{(-1)^n}{(2n)!} x^{2n}$$

the angular speed is approximated as:

$$\sin\theta(t) \approx \Omega t + \frac{1}{2}\frac{d\Omega}{dt} t + \frac{1}{3}\frac{d^2\Omega}{dt^2} t^2 + \ldots \qquad (8)$$

Where Ω is rotational velocity of the scatterers. It is assumed that the translational part of the Doppler shift "$v_R + a_R t$" has been compensated for. Thus, we only deal with the rotational part "$x_i \frac{d\theta(t)}{dt}\cos\theta(t) - y_i \frac{d\theta(t)}{dt}\sin\theta(t)$".

In summary, it is assumed that the target rotation outside of the plane between the radar and the target is small, and the standard range alignment has been applied to the data so that the coarse translation motion has been removed. Usually we retain the first order term of the rotational speed and thus the term "$\frac{1}{2}\frac{d\Omega}{dt} t + \frac{1}{3}\frac{d^2\Omega}{dt^2} t^2 + \ldots$" is ignored. This term among others is the source of error as it increases with time.

The Radar echo could be represented as a sum of sine waves with additive noise. Due to target rotation, target maneuvering or others, the phase of the sine wave is corrupted by additive noise. It is suggested that this could be modeled as an additive Wiener process. In Figure 1, we show the Radar echo of a simulated two rotating reflectors and that of a sum of two sinusoids where the phases have additive Wiener process.

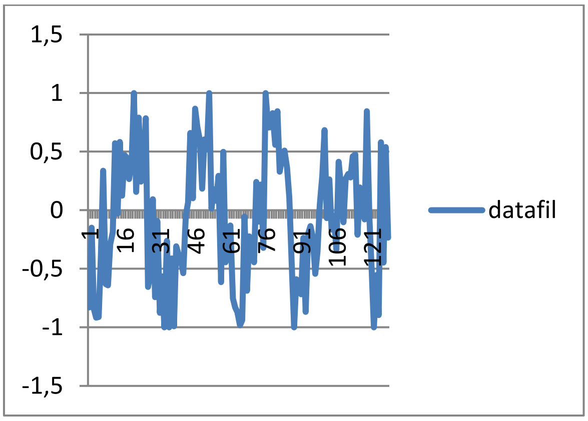

Fig.1a, Radar echo of simulated rotating two reflectors (datafil)

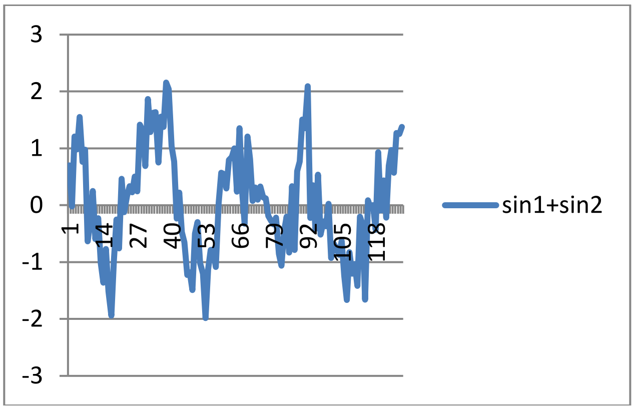

Fig. 1b, Simulated sum of two sinusoids with random phase (sin1+sin2)

The Fourier transform of the signals is shown in Figure 2.

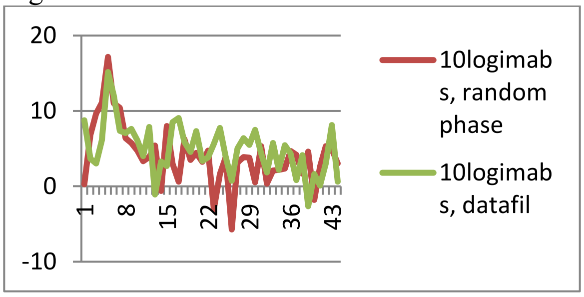

Fig. 2, Fourier transform of simulated two reflectors and sum of two sinusoids with random phase

There are two closely spaced targets but we could only sea one due to the Fourier transform resolution. There is a great similarity between the FT of the simulated data and that of the sum of two sinusoids with random phase. That is why we use the random phase model in this report.

2.1 An autoregressive model for the signal:

Usually the echo is modeled as sum of sinusoids with deterministic phase [2] which makes it easier to estimate than the random phase model. The price we pay is low resolution. In the classical approach the signal is modeled as a sum of sinusoids:

$$s(t) = \sum_i A_i \sin(2\pi f_i t + \phi_i) + n(t) \qquad (9)$$

For simple rotation, $\theta(t) \approx \Omega t$, $\sin\theta(t) \approx \Omega t$, and $\cos\theta(t) \approx 1$.

$$\frac{4\pi}{C} f\left(x_i \sin\theta(t) + y_i \cos\theta(t)\right)$$

Thus, $\approx \frac{4\pi}{C} f\left(x_i \Omega t + y_i\right) = 2\pi f_i t + \phi_i$

The autoregressive model of the signal is:

$$s(t) = \sum_i a_i s(t-i) \qquad (10)$$

where a_i is the unknown parameter to be estimated by minimizing the squared error between the observed signal s(t) and its estimated value through eqn. [10].

The spectrum at frequency f is given as:

$$P_{AR}(f) = \frac{\sigma^2}{\left|1 + \sum_k a_k e^{-j2\pi f k \Delta t}\right|^2} \qquad (11)$$

Where σ^2 is the additive noise variance.

In Figure 3, we show the estimated spectrum $P_{AR}(f)$ of the echo of two close reflectors at different frequencies. Still there is only one peak.

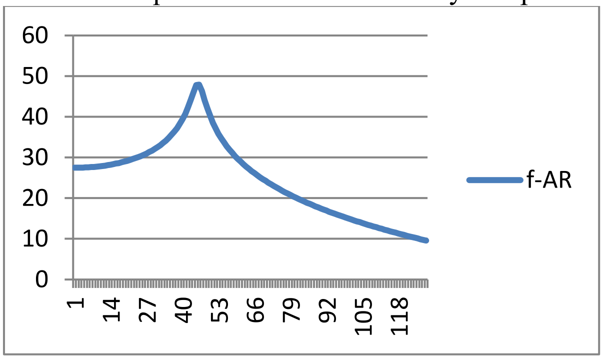

Fig. 3, Fourier trasform of an AR mode of the simulated reflectors (f-AR)

Notice that the location of the peak is different than that of the Fourier transform. This is because when we are plotting the $P_{AR}(f)$ we use sampling frequency that is much smaller than that of Fourier transform. The conventional methods do not incorporate the added Brownian motion in the phase. Instead they focus on high resolution methods to estimate the amplitudes and frequencies. Still in this example we are unable to find two peaks.

3 Problem Solution; the Proposed Approach:

The received echo is usually corrupted with additive noise. We use adaptive line enhancement (ALE) to improve the SNR. From the improved signal, we use adaptive noise canceling (ANC) to separate the first dominant echo. The remainder signal is then tested to find if we have another sinusoid or just noise. If we have another sinusoid, we use the random phase model and the Girsanov theory to find the maximum likelihood estimate of the frequency. We then apply ANC again to separate the second dominant reflector. This process is repeated several times till we get just noise.

3.1 How do we know there are more than one reflector

The FT resolution does not allow for the recognition of closely spaced reflectors. If we band pass the FT around different peaks and then use inverse Fourier transform, we could observe in the time domain the presence of more than one reflector. A single reflector shows as one sine wave with some noise. The amplitude and the frequency (represented by the zero crossing points) are almost constants. For more than one reflector, the amplitude is changing and the zero crossing points change as well. Due to noise and due to processing problems, the amplitude seems to be constant while actually it is changing. We use adaptive line enhancement (ALE) to reduce the noise level and detect the periodic signal.

3.2 Adaptive Line Enhancement (ALE):

ALE is an adaptive self-tuning filter capable of separating the periodic and stochastic components in a signal. The ALE could detect extremely low-level sine waves in noise, and is applied in many areas with noisy environment. The ALE simply uses a single source and is therefore easy to control. Furthermore, ALEs do not require direct access to the noise nor a way of isolating noise from the useful signal. In ALE it is assumed that a delayed version of the signal is correlated with itself. The source of correlation is the desired signal while the noise part is independent or uncorrelated. In the literature, several ALE methods have been proposed. These methods mainly focus on improving the convergence rate of the adaptive algorithms using modified filter designs, realized as transversal Finite Impulse Response, recursive Infinite Impulse Response, lattice, and sub-band filters and others [5].

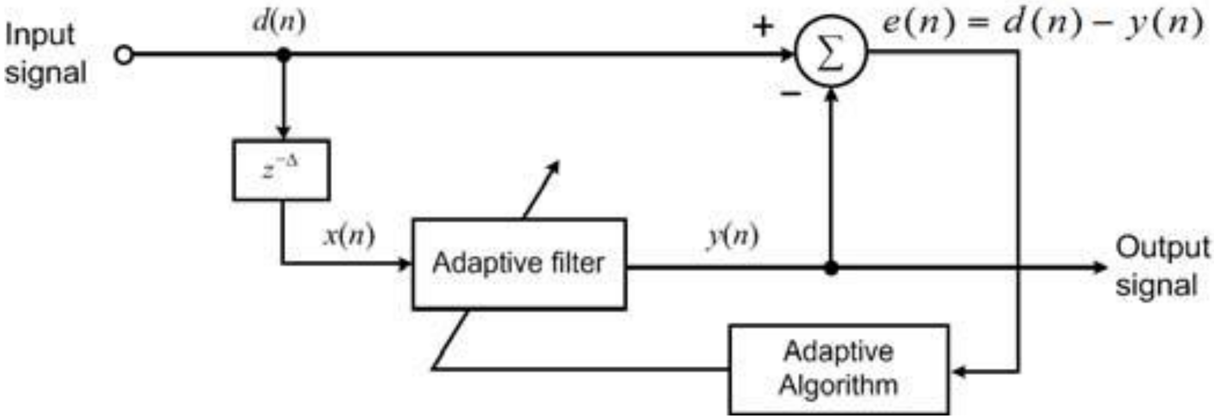

Fig. 4: Block diagram of a daptive line enhancer

The observed signal d(n)=s(n)+n(n) has two components ; the real signal s(n) and the additive noise n(n).

The estimated noise $\hat{n}(n) = y(n)$ is given as:

$$\hat{n}(n) = y(n) = \sum_{j}^{J} h_j d(n - j\Delta) \qquad (12)$$

Where h_j are the adaptive filter parameters, and Δ is the unit delay or the sampling interval. The estimated signal

$$\hat{s}(n) = e(n) = d(n) - y(n) \qquad (13)$$

The coefficient h_j is adaptively updated according to the equation:

$$h_j(n+1) = h_j(n) + \mu\,\hat{s}(n) d(n - j\Delta) \qquad (14)$$

where μ is a tuning parameter. Other algorithms could also be used.

3.3 Adaptive noise canceling to remove the known dominant reflector

In the case that we know the dominant frequency form the FT, as in ISAR, one could use adaptive noise canceling to find an estimate of the second sinusoid. The second sinusoid represents the echo from the smaller reflector that is close to the bigger reflector. In the next subsection we present the adaptive noise canceling algorithm to get the other reflectors. We then present the maximum likelihood estimate of the parameters of the echo of a single reflector.

In adaptive noise canceling, we have two signals. The first is a known component (in our case the sinusoid of the dominant reflector with known frequency obtained from FT). The second component is the sum of the known component (or function of it) and the other components (reflectors).

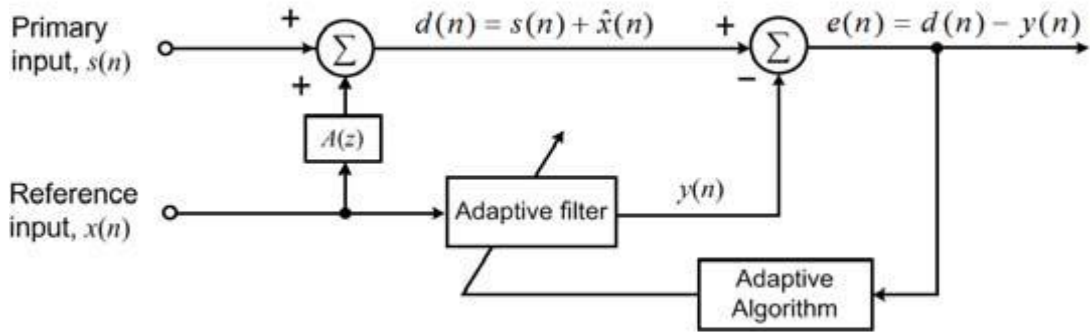

Fig. 5: Block diagram of a daptive noise cancellation system

Specifically;

$$d(t) = s(t) + \mathscr{B}\sin(2\pi f t + \phi) \qquad (15)$$

Where $x(n) = \mathscr{B}\sin(2\pi f_1 t + \phi)$ is defined as the reference signal, $\mathscr{B}$ is unknown amplitude, and ϕ is unknown phase. The frequency f_1 is known from the Fourier transform. In the figure, we set A(z)=1, and thus $\hat{x}(n) = x(n)$.

Using the least mean square algorithm (LMS), the estimate of the unknown sinusoid s(n), which is also the error signal, is obtained as:

$$\hat{s}(n) = e(n)$$
$$= d(n) - \sum_{j=1}^{J} h_j(n)\sin(2\pi f n\Delta t + 2\pi j / J) \qquad (16)$$

Where Δt is the sampling interval,

$$y(n) = \sum_{j=1}^{J} h_j(n)\sin(2\pi f n\Delta t + 2\pi j / J) \qquad (17)$$

The coefficient h_j is adaptively updated according to the equation:

$$h_j(n+1) = h_j(n) + \mu\,\hat{s}(n)\sin(2\pi f n\Delta t + 2\pi j / J) \qquad (18)$$

where μ is a tuning parameter. Other algorithms could also be used.

We passed the estimated second component through a moving average filter (MA) to reduce the noise. We also estimated, for this second sinusoid, the amplitude and the frequency using conventional least square methods assuming deterministic phase.

It is this component that we shall try to find its amplitude and frequency. This is shown next.

3.4 The random additive phase model and its estimation

From the real data of a rotating reflector, it was noticed that the Fourier transform of the echo, Figure 2, is similar to that when the phase has an additive Wiener process component. This is expected because the ignored phase term " $\frac{1}{2}\frac{d\Omega}{dt}t + \frac{1}{3}\frac{d^2\Omega}{dt^2}t^2 + \ldots$" increases with time. Recall that the variance of a Wiener process increases with time. Thus, we develop a model for a single rotating reflector and find its SDE. We then find the maximum likelihood of the unknown parameters that represent the frequency and the amplitude.

3.4.1 Estimation of the amplitude A:

If the signal is the outcome of ANC, it must be smoothed first. A moving average filter is used or ALE. The amplitude could be estimated from the observed process $\hat{s}(t)$ as:

$$A = \frac{\max(\hat{s}(t)) + abs(\min(\hat{s}(t)))}{2} \qquad (19)$$

Where $\max(\hat{s}(t))$ is the maximum value of the observed signal, and abs(min($\hat{s}(t)$)) is the absolute value of the minimum value of $\hat{s}(t)$.Squaring the observations we get:

$$\hat{s}^2(t) = A^2 - A^2\cos^2(2\pi f_1 t + \gamma B(t)) \qquad (20)$$

The DC value of $\hat{s}^2(t)$ is A^2.

3.4.2 Estimation of the γ

Using Ito table dtdt=0, dtdB=0 and dBdB=dt, we square the SDE of the observations to obtain:

$$ds(t)ds(t) = \gamma^2\left(A^2 - s^2(t)\right)dt$$

i.e. $$\gamma^2 = \frac{ds(t)ds(t)}{\left(A^2 - s^2(t)\right)dt} \qquad (21)$$

Effectively we take the average of γ^2 as our estimate.

3.4.3 Maximum likelihood estimate of the frequencies

Having estimated the amplitude and the parameter γ, the diffusion term of the SDE ofsz(t) is completely known. After some manipulations (see Appendix A) we obtain the maximum likelihood estimate of the frequency $\hat{f}_2$, for observation period T, as:

$$2\pi\hat{f}_2 T = \int_0^T \frac{-ds(t)}{\sqrt{\left(A^2 - s^2(t)\right)}} - \frac{1}{2}\gamma^2 \int_0^T \frac{s(t)}{\sqrt{\left(A^2 - s^2(t)\right)}} dt \tag{22}$$

This is the desired result. In this analysis, one must be careful about the sign of $\sqrt{\left(A^2 - s^2(t)\right)}$.

4 Results and conclusions

In this section we use simulated data and real data. The simulated data is the sum of two sinusoids representing the reflection from two closely spaced scatterers where the phases are deterministic but unknown. We then simulate the sum of two sinusoids with random phase. Finally we use the real data that represent the echo of two closely space scatterers and find their values. In all cases we apply ALE and ANC to separate and estimate the different frequencies.

4.1 Sum of two sinusoids with deterministic phases

The signal is given as:

$$s(t) = \sum_{i=1}^{2} A_i \sin(2\pi f_i t + \phi_i) + n(t) \tag{9}$$

$A_1 = 0.5$, $A_2 = 0.2$, $f_1 = 3.0$, $f_2 = 3.4$, $\phi_1 = 0$, $\phi_2 = \pi/3$, sampling interval $\Delta t = 0.01$, number of data points is N=128. Fourier resolution = $1/(N\Delta t)$=0.78

The simulated data is shown in Figure 6

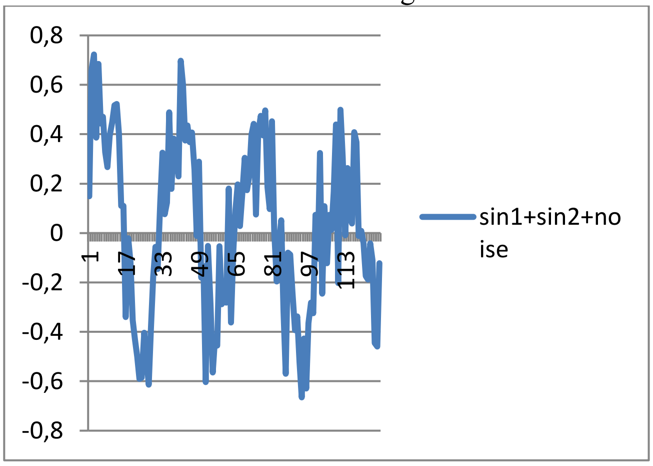

Fig. 6, Sum of two sinsusoids (sin1+sin2+noise)

and the Fourier transform is shown in Figure 7. Notice that we can separate the two sinusoids.

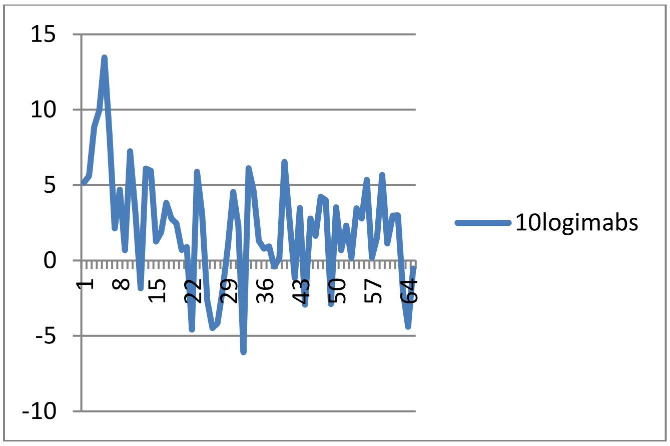

Fig. 7, Fourier transform of sum of two sinusoids

Applying the ALE to the observed noisy signal, we get the results shown in Figure 8.

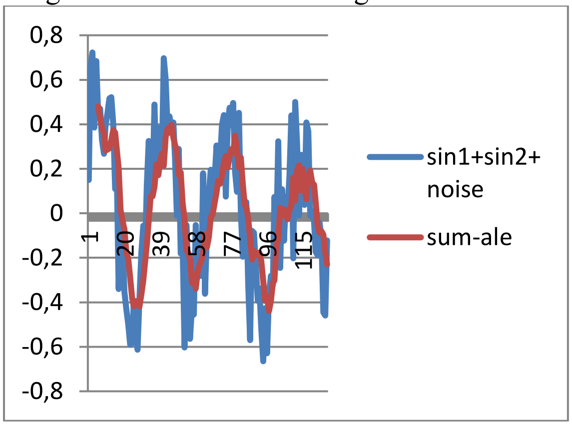

Fig. 8, Sum of sinusoids, and sum of sinusoids after ALE (sum_ale)

It is this filtered signal that we will be using in the analysis to follow. We then apply the ANC to remove the dominant peak, which is the first sinusoidal, and retain the smaller peak, which is the second sinusoidal, for further analysis. The result of the ANC is then passed to a moving average (MA) filter to reduce the noise. The estimate of the second frequency is obtained by minimizing the error between the estimated signal and the outcome of the ANC+MA. The estimated frequency is $\hat{f}_2 = 3.48$. The results are shown in Figure 9.

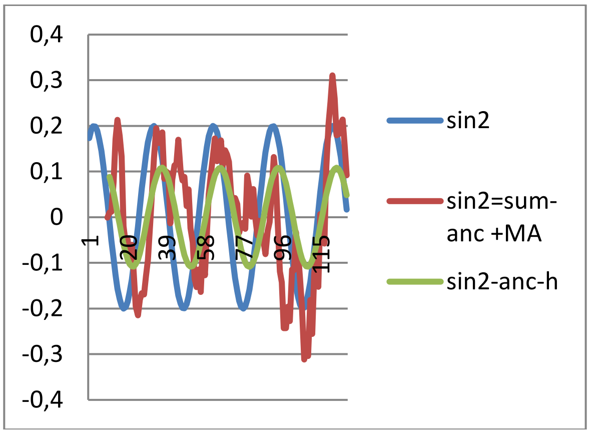

Fig. 9, True sin (sin2), sin after ANC and Moving Average (sum_anc+MA), estimated sin2 (anc-h)

4.2 Simulated data of ISAR:

As before, the data is the echo of the sum of two closely spaced scatterers

$$s(t)=\sum_{i=1}^{2} A_i \sin(\phi_i(t))+n(t)$$

Here the phase of the sine wave, after range correction, becomes:

$$\phi_i(t) = \frac{4\pi}{C} f\left(x_i \sin\theta(t)+y_i \cos\theta(t)\right)$$

$\theta(t) = \Omega t$, $\Omega = 0.07$, $x_1 = 2.0$,

$x_2 = 2.2$, $y_1 = y_2 = 5.0$, $f/c = 10$

The observed signal is shown in Figure 10

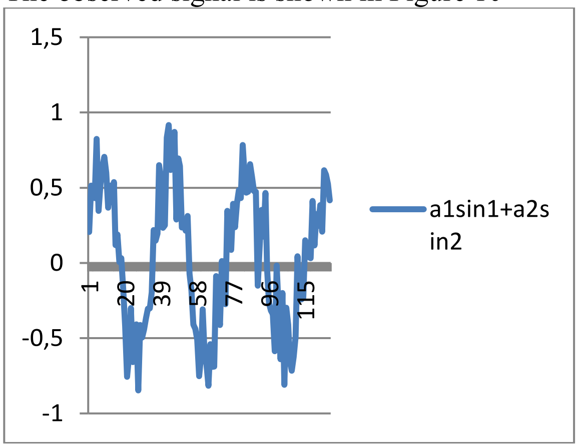

Fig. 10, The reflection of two closely spaced scatterers (a1sin1+a2sin2)

The Fourier transform of the echo (sum of two sinusoids with time-varying phase) is shown in Figure 11. You could only see one peak.

We assume that the first scatterer is given by a pure sine wave with unknown phase and unknown amplitude as:

$A_1 \sin(2\pi f_1 t+\psi_1)$

Where $f_1 = 3.0$ (obtained from the Fourier transform), A_1 is unknown amplitude, and ψ_1 is unknown phase. This is the reference signal in the adaptive noise canceling i.e. $x(t)=A_1 \sin(2\pi f_1 t+\psi_1)$

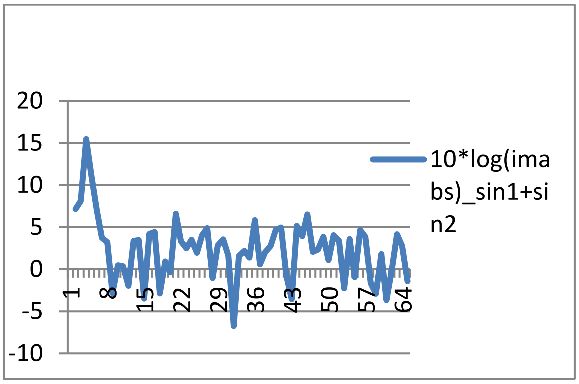

Fig. 11, Fourier Transform of the echo (10*log(imabs)_sin1+sin2)

We apply the ALE and the ANC to the above signal followed by a moving average filter and we get the estimate shown in Figure 12.

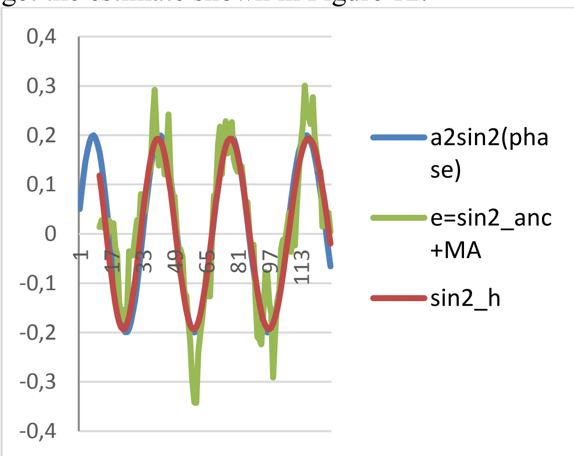

Fig. 12, True echo (a2sin2(phase)), estimated echo after ALE and ANC and MA (e=sin2_anc+MA), and estimated sine (sin2_h)

The parameters of the second scatterer are estimated as:

$\Omega = 0.069$, $x_2 = 2.16$, $y_2 = 5.0$, $amp_2 = 0.19$

4.3 Real data of ISAR:

As before, the data is the echo of the sum of two closely spaced scatterers

$$s(t)=\sum_{i=1}^{2} A_i \sin(\phi_i(t))+n(t)$$

Here the phase of the sine wave, after range correction, becomes:

$$\phi_i(t) = \frac{4\pi}{C} f\left(x_i \sin\theta(t) + y_i \cos\theta(t)\right)$$

We apply ALE to the data and we get the results in Figure 13.

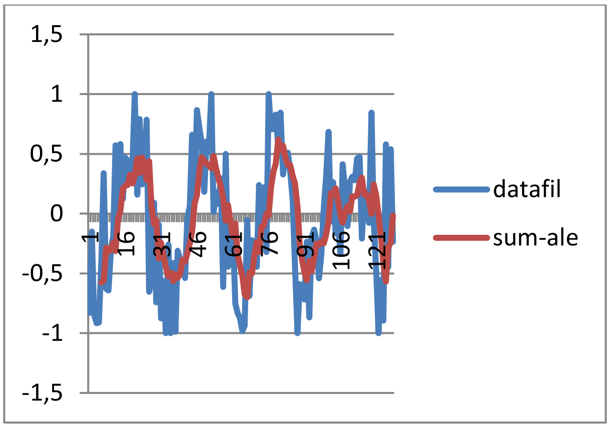

Fig. 13, Reflected echo before (datafil) and after adaptive line enhancement (sum-ale)

We assume that the first scatterer is given by a pure sine wave with unknown phase and unknown amplitude as:

$A_1 \sin(2\pi f_1 t + \psi_1)$

Where $f_1 = 3.0$ (obtained from the Fourier transform), A_1 is unknown amplitude, and ψ_1 is unknown phase. This is the reference signal used in the adaptive noise canceling i.e. $x(t) = A_1 \sin(2\pi f_1 t + \psi_1)$

The resultant estimated second component is shown in Figure 14.

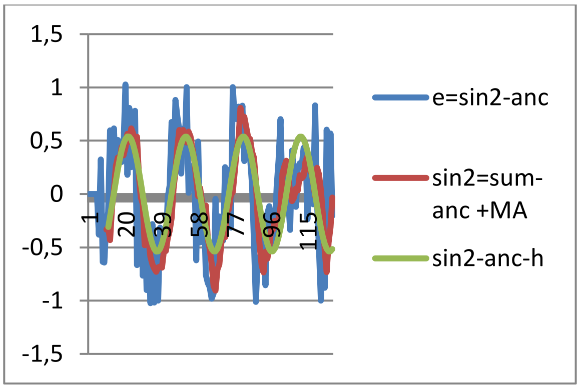

Fig. 14, The second reflector using adaptive noise canceling (sin2-anc), after using moving average (sum-anc+MA), and fitted sinusoid (sin2-anc-h)

In Figure 14, we fitted a pure sine wave and we estimated the frequency.

4.4 Frequency Estimation Using Ito Calculus:

Assuming that the signal is modeled as a sine wave with random phase and using the signal after ALE and ANC (s(t)), we apply the maximum likelihood method to find the estimate of the amplitude and the frequency.

$$2\pi\hat{f}_2 T = \int_0^T \frac{-ds(t)}{\sqrt{(A^2 - s^2(t))}} - \frac{1}{2}\gamma^2 \int_0^T \frac{s(t)}{\sqrt{(A^2 - s^2(t))}} dt$$

Where s(t) is the signal obtained after applying ALE and ANC.

The estimated frequency of the second scatterer was found to be 3.96 Hz.

4.5 Conclusion and Future Work:

In this report we presented techniques from adaptive filtering to handle the echo coming from closely spaced multiple reflectors on the target. We introduced a method based on adaptive noise canceling to detect closely spaced sinusoids that could not be detected by Fourier transform. Compared to autoregressive based methods, the proposed approach has higher resolution. We also presented a method to estimate just the frequency when the phase is a random quantity. This random quantity represents the ignored parts of the moving reflectors. We then applied the proposed methods to simulated data and real data obtained for ISAR. In future work, we shall focus on ISAR with multiple reflectors and 3D ISAR.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article

Appendix A

In this appendix we present the derivation for the maximum likelihood estimate of the frequency. We also present the statistical properties of the estimate.

A.1 Derivation of the Stochastic Differential Equation (SDE) for the echo of the second reflector

Assume that the received signal is modeled as:

$$s(t) = A\sin(2\pi f_2 t + \gamma B(t)) \quad (A.1)$$

where the amplitude A is unknown, the frequency f_2 is unknown, and it is assumed that the additive error is embedded into the random phase (Wiener process or Brownian motion) γ B(t). The parameter γ is also unknown.

Let x(t) = B(t)

Thus $dx(t) = dB(t)$ (A.2)

where dx(t) is the stochastic differential for x(t), and dB(t) is what is known as white Gaussian noise with zero mean and variance dt. We need to find a stochastic differential equation (SDE) for ds(t). In order to do that, we first introduce the general forms for SDE and for nonlinear transformation.

The general form for a stochastic differential equation is:

$$dx(t) = a(t, x(t))\, dt + b(t, x(t))\, dB(t) \quad (A.3)$$

In our case, a(t, x(t)) = 0, and b(t, x(t)) = 1.

Let $U_1(t, x(t))$ be a nonlinear transformation of x(t) i.e. s(t)= $U_1(t, x(t))$, using Ito Lemma we obtain:

$$ds(t) = [\frac{\partial U_1}{\partial t} + a(t, x(t))\frac{\partial U_1}{\partial x} + \frac{1}{2}b^2(t, x(t))\frac{\partial^2 U_1}{\partial x^2}]dt + b(t, x(t))\frac{\partial U_1}{\partial x}dB(t) \quad (A.4)$$

In our case,

$$s(t) = U_1(t, x(t)) = A\sin(2\pi f_2 t + \gamma\, x(t)) \quad (A.5a)$$

x(t) = B(t), a(t, x(t))=0, b(t, x(t))=1, (A.5b)

The elements of the SDE of s(t) are obtained as:

$$\frac{\partial U_1}{\partial t} = 2\pi f_1 A\cos(2\pi f_2 + \gamma\, x(t)) = 2\pi f_2\sqrt{A^2 - s^2(t)} \quad (A.6a)$$

$$\frac{\partial U_1}{\partial x} = \gamma A\cos(2\pi f_2 t + \gamma\, x(t)) = \gamma\sqrt{A^2 - s^2(t)} \quad (A.6b)$$

$$\frac{\partial^2 U_1}{\partial x^2} = -\gamma^2\sin(2\pi f_2 t + \gamma\, x(t)) = -\gamma^2 s(t) \quad (A.6c)$$

Substituting the above expressions we get:

$$ds(t) = \{2\pi f_2\sqrt{A^2 - s^2(t)} - \frac{1}{2}\gamma^2 s(t)\}\, dt + \gamma\sqrt{A^2 - s^2(t)}\, dB(t) \quad (A.7)$$

where we made use of the fact that $A\cos x = \pm\sqrt{A^2 - A^2(\sin x)^2}$. The sign of the square root depends on which quadrant lies x.

Notice that the SDE of s(t) is linear in the unknown frequency component, f_1.

A.2 Parameter Estimation and the Cramer-Rao Lower Bound:

In the general vector case the observation SDE has the form:

$$ds(t) = a(s; \underline{\theta})dt + b(t, s)dB(t) \quad (A.8)$$

According to Girsanov Theory, the Radon-Nikodym derivative which is the likelihood function, L, is given by [6]:

$$L(t,s;\underline{\theta}) = \exp\{\frac{-1}{2}\int_0^t a(u,s;\underline{\theta})[b(u,s)b^T(u,s)]^{-1}\, \mathrm{a}(u,s);\underline{\theta})\, du + \int_0^t a(u,s;\underline{\theta})[b(u,s)b^T(u,s]^{-1}\, ds\}$$

(A.9)

and the log likelihood function $\lambda = \ln L$ becomes:

$$\lambda(t,s;\underline{\theta}) = \frac{-1}{2}\int_0^t a(u,s;\underline{\theta})[b(u,s)b^T(u,s)]^{-1}\, \mathrm{a}(u,s);\underline{\theta})\, du + \int_0^t a(u,s;\underline{\theta})[b(u,s)b^T(u,s]^{-1}\, ds$$

(A.10)

The maximum likelihood estimate for the unknown parameters is obtained by maximizing equation (A.10) w.r.t $\underline{\theta}$.

To obtain the Cramer-Rao lower bound on the variance of the estimates, it is easier to work with an SDE of the form:

$$dY(t) = g(Y;\underline{\theta})dt + dB(t) \qquad \text{(A.11)}$$

Then according to [6], the Cramer-Rao lower bound for the unbiased estimate of $\hat{\underline{\theta}}$ is:

$$E\left\{(\underline{\theta} - \hat{\underline{\theta}})(\underline{\theta} - \hat{\underline{\theta}})^T\right\} \geq \frac{1}{E\left\{\int_0^t \left[\frac{\partial g(Y(u);\underline{\theta})}{\partial \underline{\theta}}\right]^2 du\right\}} \qquad \text{(A.12a)}$$

Let $U_2(t,s(t))$ be a nonlinear transformation of s(t) i.e. Y(t)= $U_2(t,s(t))$, using Ito Lemma we obtain:

$$dY(t) = [\frac{\partial U_2}{\partial t} + a(t,s(t))\frac{\partial U_2}{\partial s}$$
$$+\frac{1}{2}b^2(t,s(t))\frac{\partial^2 U_2}{\partial s^2}]dt + b(t,s(t))\frac{\partial U_2}{\partial s}dB(t)$$

Recall that:
$$ds(t) = \{2\pi f_2\sqrt{A^2 - s^2(t)} - \frac{1}{2}\gamma^2 s(t)\}\, dt$$
$$+\gamma\sqrt{A^2 - s^2(t)}\, dB(t)$$

Choose $U_2(t,s(t))$ such that: $b(t,s(t))\frac{\partial U_2}{\partial s} = 1 = \gamma\sqrt{A^2 - s^2(t)}\frac{\partial U_2}{\partial s}$

Thus, $\frac{\partial U_2}{\partial s} = \frac{1}{\gamma\sqrt{A^2 - s^2(t)}}$, $\frac{\partial^2 U_2}{\partial s^2} = \frac{-\gamma\left(\frac{1}{2}\right)(-2s(t))(A^2 - s^2(t))^{-1/2}}{\gamma^2(A^2 - s^2(t))} = \frac{s(t)}{\gamma(A^2 - s^2(t))^{3/2}}$

and $\frac{\partial U_2}{\partial t} = 0$

Then:
$$\frac{\partial g(Y(u);\underline{\theta})}{\partial \underline{\theta}} = \frac{\partial}{\partial \underline{\theta}}\left[a(t,s(t))\frac{\partial U_2}{\partial s}\right] = \frac{\partial U_2}{\partial s}\frac{\partial a(t,s(t))}{\partial \underline{\theta}}$$
$$= \frac{1}{\gamma\sqrt{A^2 - s^2(t)}}(2\pi)\sqrt{A^2 - s^2(t)} = \frac{2\pi}{\gamma}$$

The Cramer-Rao lower bound, on the variance of the frequency estimate, becomes:

$$E\left\{(f_2 - \hat{f}_2)^2\right\} \geq \left(\frac{\gamma}{2\pi}\right)^2 \frac{1}{T} \qquad \text{(A.12b)}$$

A.3 Maximum likelihood estimate of the frequencies

Having estimated the amplitude and the parameter γ, the diffusion term of the SDE of s(t) is completely known.

The stochastic process s(t) has the following SDE form:

$$ds(t) = c(t,s(t))\, dt + e(t,s(t)\, dB(t) \qquad \text{(A.12)}$$

where

$$c(t,s(t))=\{2\pi f_2\sqrt{A^2-s^2(t)}-\frac{1}{2}\gamma^2 s(t)\} \quad \text{(A.13a)}$$

and

$$e(t,s(t)=\gamma\sqrt{A^2-s^2(t)} \quad \text{(A.13b)}$$

Maximizing the likelihood function w.r.t the unknowns $\underline{\theta}$ yields the equation:

$$\int_0^T \frac{\partial c(s(t),\underline{\theta})/\partial\underline{\theta}}{e^2(s(t))}\left[ds(t)-c(s(t),\underline{\theta})dt\right]=0 \quad \text{(A.14)}$$

Where $\underline{\theta}=f_1$ is the unknown parameter and is scalar.

$$\partial c(t,s(t))/\partial f_2=2\pi\sqrt{A^2-s^2(t)}$$

Thus,

$$\int_0^T \frac{2\pi\sqrt{A^2-s^2(t)}}{\gamma^2\left(A^2-s^2(t)\right)}\left[ds(t)-\left(2\pi f_2\sqrt{A^2-s^2(t)}-\frac{1}{2}\gamma^2 s(t)\right)dt\right]=0$$

$$\int_0^T \frac{1}{\sqrt{\left(A^2-s^2(t)\right)}}\left[ds(t)-\left(2\pi f_2\sqrt{A^2-s^2(t)}-\frac{1}{2}\gamma^2 s(t)\right)dt\right]=0$$

$$\int_0^T \frac{ds(t)}{\sqrt{\left(A^2-s^2(t)\right)}}$$

$$=2\pi f_2\int_0^T \frac{\sqrt{A^2-s^2(t)}}{\sqrt{\left(A^2-s^2(t)\right)}}dt-\frac{1}{2}\gamma^2\int_0^T \frac{s(t)}{\sqrt{\left(A^2-s^2(t)\right)}}dt$$

Which is reduced to:

$$2\pi\hat{f}_2 T=\int_0^T \frac{ds(t)}{\sqrt{\left(A^2-s^2(t)\right)}}+\frac{1}{2}\gamma^2\int_0^T \frac{s(t)}{\sqrt{\left(A^2-s^2(t)\right)}}dt \quad \text{(A.15)}$$

This is the desired result. In this analysis, one must be careful about the sign of $\sqrt{\left(A^2-s^2(t)\right)}$.

Substitute " $\begin{aligned}ds(t)&=\{2\pi f_2\sqrt{A^2-s^2(t)}-\frac{1}{2}\gamma^2 s(t)\}\,dt\\&+\gamma\sqrt{A^2-s^2(t)}\,dB(t)\end{aligned}$ " Into the above equation we get:

$$2\pi\hat{f}_2 T=\int_0^T \frac{\{2\pi f_2\sqrt{A^2-s^2(t)}-\frac{1}{2}\gamma^2 s(t)\}dt}{\sqrt{\left(A^2-s^2(t)\right)}}$$

$$+\int_0^T \frac{\gamma\sqrt{A^2-s^2(t)}\,dB(t)}{\sqrt{\left(A^2-s^2(t)\right)}}+\frac{1}{2}\gamma^2\int_0^T \frac{s(t)}{\sqrt{\left(A^2-s^2(t)\right)}}dt$$

Which is reduced to:

$$2\pi\hat{f}_2 T=2\pi f_2 T-\gamma\int_0^T dB(t) \quad \text{(A.16)}$$

Taking the expectation of both sides we obtain:

$$2\pi E\left\{\hat{f}_2\right\}T=2\pi f_2 T-\gamma E\left\{\int_0^T dB(t)\right\}$$

$$=2\pi f_2 T \quad \text{(A.17)}$$

i.e. the frequency estimate is unbiased.