Methods to Generate Pythagorean Triples

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Abstract: In this study, two different methods are presented for generating Pythagorean triples. Furthermore, based on the obtained results, it is mathematically demonstrated that one element of a Pythagorean triple is always a multiple of 4, another is always a multiple of 5, and in certain special cases, one of the elements is a multiple of 20.

Key-Words: Generation of Pythagorean triples, Pythagorean triples, Primitive Pythagorean triples

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1 Introduction

Throughout the historical development of mathematics, numerous methods have been proposed for generating Pythagorean triples. Among these, the most well-known is the technique referred to as the Euclidean method. According to this method, for $m, n \in \mathbb{N}^+$with $m > n$, a Pythagorean triple (a, b, c)can be obtained as follows:

$$a = m^2 - n^2 \,, \; b = 2mn, \;\; c = m^2 + n^2$$

These formulas are derived from the well-known identity

$$(x - y)^2 + 4xy = (x + y)^2$$

by taking $x = m^2$and $y = n^2$. By assigning appropriate natural number values to m and n, all primitive Pythagorean triples can be generated using this method.

One of the alternative methods is the technique attributed to Plato. In this method, if the first number is odd, the triple is given in the form

$$(a, a^2 - 1, a^2 + 1);$$

if it is even, it is generated as follows:

$$a_1 = 2a, b_1 = \left(\frac{a_1}{2}\right)^2 - 1, c_1 = \left(\frac{a_1}{2}\right)^2 + 1$$

It is observed that this method cannot generate all Pythagorean triples; for instance, the triple $(20, 21, 29)$ cannot be obtained using this approach.

The Fibonacci method, on the other hand, aims to generate Pythagorean triples by utilizing consecutive Fibonacci numbers [1], [2]. In addition, Michael Stifel, in 1544, produced triples through the use of compound number sequences [3], [4]. The Dickson method is based on constructing triples by selecting certain parameters in a systematic (or arbitrary) manner [5].

Another significant approach is the method proposed by Berggren, known as the “Berggren Tree.” In this technique, starting from the triple $(3, 4, 5)$, new primitive Pythagorean triples are systematically derived through specific matrix transformations [6–8].

In addition to these, various mathematicians have proposed alternative methods. Wader introduced a technique based on generating triples from certain positive integers [9]; other approaches include deriving triples via quadratic equations [10], generating primitive Pythagorean triples using half-angle tangents, and methods based on area proportions related to sums of squares [11]. Furthermore, Wade proposed methods grounded in the excess–height counting theorem [12], [13].

In this study, two novel methods for generating Pythagorean triples are presented, and the structural properties of these triples are examined in detail. In particular, it is shown through examples that, in every primitive Pythagorean triple, one element is necessarily a multiple of 4, while another is always a multiple of 5; moreover, in certain cases, one of the elements is a multiple of 20. This corresponds to the situation where a single element is simultaneously a multiple of both 4 and 5, thereby confirming the aforementioned observation. Furthermore, attention is drawn to the limitations of some existing methods, and it is demonstrated that the proposed approaches overcome these shortcomings.

The second method considered, unlike the Dickson method which relies on the arbitrary selection of parameters, takes as parameters the differences between the side lengths of a right triangle. Using these differences, the side lengths of the triangle are expressed, and by applying the Pythagorean Theorem, an algebraic relation among these differences is obtained. Through this relation, all primitive Pythagorean triples can be generated; subsequently, all Pythagorean triples are obtained as positive integer multiples of these primitive triples.

2 Problem Formulation

First, we introduce several definitions:

Definition 1. A Pythagorean triple $(a, b, c)$is called a *primitive Pythagorean triple* if a and b are coprime.

Definition 2. If $(a, b, c)$is a primitive Pythagorean triple, then for all $k \in \mathbb{N}^+$, the triple $(ka, kb, kc)$is also a Pythagorean triple.

Based on these definitions, in order to obtain all solutions of the equation $a^2 + b^2 = c^2$over the set of positive integers, it suffices to determine all primitive Pythagorean triples.

Method 1: Observation-Based Approach

This method, developed through observations on Pythagorean triples, considers the triples in increasing order and examines them under two distinct cases:

(i) When the first element of the Pythagorean triple is odd:

Examples:

(3,4,5) → $3^2 = 4 + 5$
(5,12,13) → $5^2 = 12 + 13$
(7,24,25) → $7^2 = 24 + 25$

Based on these examples, the following rule is obtained:

Rule 1. If the first element of a Pythagorean triple is an odd integer greater than one, then the square of this number can be expressed as the sum of two consecutive natural numbers. These two numbers constitute the second and third elements of the triple.

Examples:

$11 \Rightarrow$ (square is taken)$121 = 60 + 61$
$\Rightarrow 11^2 + 60^2 = 61^2$
$13 \Rightarrow$ (square is taken)$169 = 84 + 85$
$\Rightarrow 13^2 + 84^2 = 85^2$
$15 \Rightarrow$ (square is taken)$225 = 112 + 113$
$\Rightarrow 15^2 + 112^2 = 113^2$
etc.

Based on these observations, we formulate the following rule:

Rule 2. Let a Pythagorean triple have its first element as a multiple of 4. If the square of this number is computed and half of this value is expressed as the sum of two consecutive odd integers, then these two odd integers correspond to the second and third elements of the Pythagorean triple.

$12 \Rightarrow$ (square) $144 \Rightarrow$ (half) 72
$= 35 + 37 \Rightarrow 12^2 + 35^2$
$= 37^2$
$16 \Rightarrow$ (square) $256 \Rightarrow$ (half) 128
$= 63 + 65 \Rightarrow 16^2 + 63^2$
$= 65^2$
$20 \Rightarrow$ (square) $400 \Rightarrow$ (half) 200
$= 99 + 101 \Rightarrow 20^2 + 99^2$
$= 101^2$
⋮

It has also been observed that when one element of a Pythagorean triple is a multiple of 20, there exist certain special triples that cannot be obtained using the previous rules. Such triples typically exhibit structures that are not generated by classical methods. Our analysis shows that these triples can be expressed in the following form:

Rule 3. Let (a, b, c)be a primitive Pythagorean triple defined as

$$a = x, \qquad b = 20k, \qquad c = x + 8,$$
$$k = 1,2,3, \ldots$$

Then, primitive Pythagorean triples that cannot be obtained from the first two rules can be generated.

Using these three rules, all primitive Pythagorean triples can be obtained.

The primitive triples generated by Rule 3 are derived from the following equation. The relationship between $x$and $k$is obtained from the Pythagorean theorem:

$$(x+8)^2 - x^2 = (20k)^2$$

which yields

$$x + 4 = 25k^2.$$

Here, by assigning arbitrary positive integer values to k, the corresponding values of x are determined, thereby generating primitive Pythagorean triples that cannot be obtained by the first two rules.

In this case, let us prove that we always obtain Pythagorean triples in which one element is a multiple of 20.

As we observed from the equation above:

$$x = 25k^2 - 4, x + 8 = 25k^2 + 4$$

Substituting these into the Pythagorean theorem, we get:

$$(25k^2 - 4)^2 + (20k)^2 = (25k^2 + 4)^2$$

This corresponds to the Euclid's parametrization with the choice $m = 5k$ and $n = 2$.

Examples:

$k = 1 \Rightarrow (a, b, c) = (21,20,29)$,
$k = 2 \Rightarrow (a, b, c) = (96,40,104)$,
$k = 3 \Rightarrow (a, b, c) = (221,60,229)$,
etc.

Method 2

Let the side lengths of a right triangle be $a, b, c \in \mathbb{N}^+$with $a < b < c$. Define the differences between the side lengths of the triangle as follows:

$$c - b = x, c - a = y, a + b - c = z \qquad (1)$$

Here, $x, y, z \in \mathbb{N}^+$, with $z > 0$and $0 < x < y$.

Using the relations in (1), the side lengths of the triangle can be expressed in terms of x, y, z as:

$$a = x + z \quad b = y + z, c = x + y + z \qquad (2)$$

Substituting these expressions into the Pythagorean theorem and simplifying, we obtain the following relation among x, y, z:

$$z^2 = 2xy \qquad (3)$$

Now, using equation (3), we attempt to construct primitive Pythagorean triples in the form:

$$(x + z, y + z, x + y + z) \qquad (4)$$

Theorem 1. If $x$and $y$are coprime in equation (3), then all primitive Pythagorean triples can be obtained.

Proof.

Assume that $x$and $y$are coprime. In the equation $z^2 = 2xy$, let $x = 2n^2$and $y = k^2$, where $n, k \in \mathbb{N}^+$. Then $z = 2nk$, and $2n$and $k$are coprime.

In this case, the elements $a$and $b$of the Pythagorean triple are given by:

$$a = x + z = 2n^2 + 2nk = 2n(n + k),$$
$$b = y + z = k^2 + 2nk = k(k + 2n).$$

To show that $a$and $b$are coprime, it suffices to prove that each factor of $a$is coprime with each factor of b. For this purpose, we express $k$and $2n$in terms of $(n+k)$and $(k+2n)$as follows:

$$k = 2(n + k) - (k + 2n),$$
$$2n = 2(k + 2n) - 2(n + k).$$

From these relations, any common divisor of $(n+k)$and $(k+2n)$must also divide both $k$and $2n$. Since $k$and $2n$are coprime by assumption, it follows that $(n+k)$and $(k+2n)$are also coprime. Consequently, $a$and $b$are coprime.

According to equation (3), the number $z$must be even. Hence, $\frac{z^2}{2}$is also an even natural number. For each even value of z, if $\frac{z^2}{2}$is factorized into two coprime factors x and y, then by using formulas (4), all primitive Pythagorean triples can be obtained. This completes the proof.

Examples:

Example1.

For $z = 2$, we have

$$\frac{z^2}{2} = 2 = 1 \cdot 2.$$

Taking $x = 1, y = 2$, we obtain:

$$a = x + z = 1 + 2 = 3,$$
$$b = y + z = 2 + 2 = 4,$$
$$c = x + y + z = 1 + 2 + 2 = 5.$$

Thus, the primitive triple $(3, 4, 5)$ is obtained.

Example2.

For $z = 4$, we have

$$\frac{z^2}{2} = 8 = 1 \cdot 8.$$

Taking $x = 1, y = 8$, we obtain:

$$a = 5, b = 12, c = 13.$$

Example3.

For $z = 6$, we have

$$\frac{z^2}{2} = 18.$$

This can be factorized into coprime pairs in two ways:

(i) $18 = 1 \cdot 18$

$$(a, b, c) = (7,24,25)$$

(ii) $18 = 2 \cdot 9$

$$(a, b, c) = (8,15,17)$$

These examples clearly demonstrate that, as the value of z varies and $\frac{z^2}{2}$ is decomposed into different pairs of coprime factors, various primitive Pythagorean triples can be obtained.

Theorem 2. In any primitive Pythagorean triple, one of the elements is always a multiple of 4.

Proof.

Starting from equation (3), from which primitive Pythagorean triples are derived, we observe that $z$must be even.

Since $z$is an even number, it can be written as $z = 2k$, where $k \in \mathbb{N}^+$. In this case,

$$\frac{z^2}{2} = \frac{(2k)^2}{2} = 2k^2 = x \cdot y,$$

which can be decomposed into coprime factors.

Case1: Let $x = 1$ and $y = 2k^2$. Substituting into formulas (4), we obtain:

$$a = x + z = 1 + 2k,$$
$$b = y + z = 2k^2 + 2k = 2k(k + 1),$$
$$c = x + y + z = 1 + 2k^2 + 2k.$$

In this case, b is clearly a multiple of 4.

Case2: Let $x = 2$ and $y = k^2$. For $x$and $y$to be coprime, $k$must be odd. Hence, $k = 2n + 1$or $k = 2n - 1$, where $n \in \mathbb{N}^+$.

Subcase 2(i): If $k = 2n + 1$, then

$$a = 2 + 2(2n + 1) = 4n + 4 = 4(n + 1),$$

so a is a multiple of 4.

Subcase 2(ii): If $k = 2n - 1$, then

$$a = 2 + 2(2n - 1) = 4n,$$

so again a is a multiple of 4.

In both cases, one element of the primitive Pythagorean triple is a multiple of 4. This completes the proof.

Theorem 3. One element of every Pythagorean triple is always a multiple of 5.

In all cases, based on Euclid's parametrization,

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2,$$

it follows that at least one of m, n, or $m^2 + n^2$is necessarily a multiple of 5.

Now, let us observe through examples that one element of every primitive Pythagorean triple is always divisible by 5.

From the equation $z^2 = 2xy$, taking $x = 2n^2$and $y = k^2$, we obtain $z = 2nk$, where $n, k \in \mathbb{N}^+$and $2n$and $k$are coprime. In this case, all primitive Pythagorean triples can be expressed as:

$$a = x + z = 2n^2 + 2nk = 2n(n + k),$$
$$b = y + z = k^2 + 2nk = k(k + 2n),$$
$$c = x + y + z = 2n^2 + k^2 + 2nk$$
$$= (n + k)^2 + n^2.$$

By choosing n and k to be coprime, and examining the values of a, b, c, it can be observed that in every primitive Pythagorean triple, one element is always a multiple of 4, another is always a multiple of 5, and in some cases, one element is simultaneously a multiple of both 4 and 5, i.e., a multiple of 20.

The following table presents primitive Pythagorean triples corresponding to various values of $n$and k (chosen such that $2n$ and k are coprime):

The following table presents primitive Pythagorean triples corresponding to various values of $n$and k(chosen such that $2n$and $k$are coprime):

n	k	a	b	c

1	3	8	15	17	$a$multiple of 4, $b$multiple of 5
2	1	12	5	13	$a$multiple of 4, $b$multiple of 5
1	7	16	63	65	$a$multiple of 4, $c$multiple of 5
3	1	20	7	21	$a$multiple of 20
2	3	20	21	29	$a$multiple of 20
4	1	40	9	41	$a$multiple of 20
1	9	20	99	101	$a$multiple of 20
3	5	48	55	73	$a$multiple of 4, $b$multiple of 5
5	1	60	11	61	$a$multiple of 20
1	11	24	7	25	$a$multiple of 4, $c$multiple of 5
4	3	44	33	65	$a$multiple of 4, $c$multiple of 5
6	1	84	13	85	$a$multiple of 4, $c$multiple of 5
5	3	56	33	65	$a$multiple of 4, $c$multiple of 5
⋮	⋮	⋮	⋮	⋮	⋮

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