Statistical Analysis for Kumaraswamy Weibull Frechet Distribution under Type II Censored Samples

ABDALLAH M. M. BADR[1,2,a,*], SHAMS TAREK M.[1,2], AHMED ABOU ALMAATY[1],
ABD EL-HAMID EISA[1,b], HANEM MOHAMED[3]
[1]Department of Statistics, Faculty of Commerce,
Al-Azhar University, Cairo,
EGYPT

[2]College of Business,
King Khalid University,
SAUDI ARABIA

[3]Department of Statistics, Faculty of Commerce, Girls Campus,
Al-Azhar University, Cairo,
EGYPT

*Corresponding Author

[a]https://orcid.org/0000-0001-7270-6502
[b]https://orcid.org/0009-0007-5055-7930

Abstract: - This paper introduces a robust statistical framework for analyzing lifetime data using the Kumaraswamy Weibull Frechet [KWFR) distribution under Type II censoring. The KWFR distribution is a flexible composite model that integrates the Kumaraswamy, Weibull, and Frechet distributions, enabling the modeling of data with heavy tails and extreme events. Type II censoring, commonly employed in reliability and survival studies, involves observing the smallest failure times from a total of units, with the remaining observations censored. The study derives the likelihood function for the KWFR distribution based on Type II censored samples and applies the Maximum Likelihood Estimation (MLE) method to estimate its parameters. Analytical techniques are employed to handle the complexity of the model, supported by numerical optimization methods to ensure accurate parameter estimation. The properties of the MLE, including consistency and asymptotic efficiency, are discussed, emphasizing its effectiveness in capturing the distribution's underlying characteristics even with incomplete data. Additionally, Bayesian estimation methods are explored, employing gamma priors for unknown parameters within a Squared loss function and LINEX loss function. The Metropolis-Hasting algorithm is utilized as part of the Markov Chain Monte Carlo technique to obtain Bayesian estimates. The proposed methodology is validated through simulation studies, demonstrating the flexibility and robustness of the KWFR distribution in modeling lifetime data under various censoring scenarios. Additionally, the practical utility of the model is illustrated with real-world datasets, highlighting its potential applications in reliability engineering, risk assessment, and environmental studies. This research provides a significant contribution to the statistical modeling of censored data, offering insights into the behavior of complex lifetime distributions.

Key-Words: - Kumaraswamy Weibull Frechet distribution, Type II censoring, Maximum Likelihood Estimation, reliability analysis, survival studies, Bayesian Estimation.

Received: March 8, 2026. Revised: April 9, 2026. Accepted: May 16, 2026. Published: July 3, 2026.

1 Introduction

Statistical inference is an essential aspect of data analysis, particularly in fields such as reliability theory, survival analysis, and extreme value modeling. Recent advances have focused on establishing new families that expand well-known distributions providing tremendous flexibility in modeling data in practice. In literature many families of distributions have been introduced, such as the Kumaraswamy Marshall-Olkin family by [Alizadeh et al., 2015], [ShamsTarek.M, 2019] introduced the Kumaraswamy-generalized distribution as a general class ,the Kumaraswamy alpha power-G family by [Mead, Afify, Butt, & Research, 2020], the mixture of the exponentiated Kumaraswamy-G class by[Alotaibi et al., 2022], the exponentiated Kumaraswamy distribution and its log-transform[Lemonte, Barreto-Souza, & Cordeiro, 2013]. These families allow researchers to incorporate one or more additional shape parameters into the baseline distribution, enhancing its flexibility, particularly for analyzing tail behavior. The inclusion of these parameters also improves the goodness-of-fit of the proposed generalized family of distributions, despite the potential computational challenges in some cases.[Cordeiro, De Castro, & simulation, 2009], introduced Kumaraswamy-G family, they studied some of its mathematical properties and presented special sub-models. The cumulative distribution function cdf and probability density function pdf are given by

$$F(x; a, b) = 1 - \ [1 - G^{a}(x)]^{b} \quad (1)$$

$$f(x{:}\, a, b) = abg(x)G^{a}(x)[1 - G^{a}(x)]^{b-1} \quad (2)$$

$$, a, b > 0 \ are\ to\ shape\ parameters$$

Afify, et al. (2016], proposed the Weibull Frechet distribution The cdf and the pdf are given respectively by:

$$F(x; \alpha, \theta, \gamma, \emptyset) = 1 - e^{\left[-\alpha\left\{e^{\left[\left(\frac{\gamma}{x}\right)^{\emptyset}\right]}-1\right\}^{-\theta}\right]} \ ; \ x > 0\,, \alpha, \theta, \gamma, \emptyset > 0 \quad (3)$$

$$f(x; \alpha, \theta, \gamma, \emptyset) = \theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{\left[-\theta\left(\frac{\gamma}{x}\right)^{\emptyset}\right]}\left\{1 - e^{\left[-\left(\frac{\gamma}{x}\right)^{\emptyset}\right]}\right\}^{-\emptyset-1} e^{\left[-\alpha\left\{e^{\left[\left(\frac{\gamma}{x}\right)^{\emptyset}\right]}-1\right\}^{-\theta}\right]} \ ; \ x > 0\,, \alpha, \theta, \gamma, \emptyset > 0 \quad (4)$$

where α is the scale parameter, θ is the shape parameter for the Weibull distribution, and γ and ∅ are the parameters for the Frechet distribution. This composite distribution combines the properties of both distributions to model phenomena characterized by extreme behavior and large deviations in the tails of the data. In recent years, Researchers have become more interested in developing these distributions and finding different ways to estimate their parameters.

One of the most robust methods of statistical estimation is Maximum Likelihood Estimation (MLE), which estimates the parameters of a statistical model by maximizing the likelihood function, given the observed data [Lindsay, 1988]. MLE is widely recognized for its desirable properties, including asymptotic efficiency and consistency [Cox & Hinkley, 1974], making it an ideal choice for parameter estimation, especially in complex distributions. Another approach for estimating the parameter is Bayesian estimation method which is often used to estimate population parameters, and it treats parameters as random or unknown variables. Bayesian estimation uses prior data to estimate the value of an unknown parameter. This reduces the difference between the estimator and the actual value of that parameter. [Mohammad, EL-Helbawy, AL-Dayian, & EL Gabrey, 2019] Introduced the Bayes estimators of the unknown parameters for the truncated modified Weibull distribution under squared error loss function as a symmetric loss function and linear exponential loss function as an asymmetric loss function. The Kumaraswamy Weibull Frechet distribution is a flexible model that combines the Kumaraswamy, Weibull, and Frechet distributions. It has become increasingly popular in modeling data with heavy tails and skewness, characteristics often observed in extreme value analysis [Genest, Rémillard, Beaudoin, & economics, 2009]. The Kumaraswamy Weibull Frechet distribution offers versatility in capturing the behavior of extreme events and is useful in areas such as reliability engineering, environmental studies, and risk assessment [Mudholkar & Srivastava, 1993]. However, real-world data often comes with limitations, such as incomplete or censored observations. Type II censoring is a common type of censoring in which data is truncated after a specific number of events or failures have occurred, and only a subset of observations is available for analysis [Kaplan & Meier, 1958].[Abbas et al., 2020] Introduced Bayesian Estimation of Gumbel Type-II Distribution under Type-II Censoring with Medical Applications, [Abu-Moussa, El-Din, Mosilhy, & Sciences, 2021] studied the estimation problem under adaptive progressive Type-II censoring scheme. [N Salem, R AL-Dayian, A EL-Helbawy, & E Abd EL-Kader, 2022] proposed a four-parameter competing risks model called the additive flexible Weibull extension-Lomax distribution. They estimated model parameters, reliability and hazard rate functions using maximum likelihood method based on Type II censored samples. [Wang & Gui, 2021] studied the estimation of the parameters for Gompertz distribution and prediction using general progressive Type-II censoring. Finally, [Laila A, Al-Duais, Aydi, & AL-Rezami, 2024] studied Bayesian estimation of the Pareto model based on type-II censoring data by employing non-linear programming. This type of censoring is frequently encountered in life-testing experiments, clinical trials, and reliability testing, where the experiment ends after a predefined number of failures [Butler et al., 2003]. In this context, Maximum Likelihood Estimation (MLE) is applied to estimate the parameters of the Kumaraswamy Weibull Frechet distribution from Type II censored samples. MLE's efficiency in handling censored data makes it particularly suitable for estimating parameters in situations where only partial information is available. Studies have shown that MLE provides robust parameter estimates under censoring, making it an ideal approach for censored data modeling[Fisher & sensing, 1996]. In Type II censoring, the observed sample consists of k ordered failure times $x_1, x_1, \ldots, x_k$ from a total of n units. The likelihood function for Type II censored samples incorporates the exact values of the observed failure times and accounts for the censored observations through the survival function $S(x; \varphi)$, The likelihood function in this case is described by

$$L(\varphi) = \frac{n!}{(n-k)!}\prod_{i=1}^{k} g(x; \varphi)\ [1 - G(x; \varphi)]^{n-k} \quad (5)$$

This paper is outlined as follows. Section 2, discusses the KWFR model and its properties. Section 3, discusses the estimate of parameters for the new model using the maximum likelihood (MLE) approach. Section 4, introduced Bayesian estimates under complete sample and censored sample . Section 5, Simulation study for the estimated parameter is provided. Finally, an application to a real data is obtained in Section 6. Despite the large number of generalized distributions proposed in the literature, there remains a need for models capable of simultaneously capturing heavy tails, skewness, and varying hazard rate behaviors under censored data conditions. The main contribution of this paper is the introduction of the Kumaraswamy Weibull Frechet (KWFR) distribution under Type II censoring, along with comprehensive estimation procedures based on both classical and Bayesian approaches. The proposed model provides enhanced flexibility compared to its sub-models and demonstrates strong performance in modeling complex lifetime data.

2 Kumaraswamy weibull Frechet Distribution and its Properties:

The Kumaraswamy Weibull Frechet Distribution (KWFR) is a new composite distribution that combines the Kumaraswamy, Weibull, and Frechet distributions. This new distribution is designed to model data exhibiting multiple behaviors from various families, particularly in contexts requiring the representation of extreme events, reliability analysis, and failure data. By combining the flexibility of the Kumaraswamy distribution (known for its versatility in modeling bounded data), the Weibull distribution (widely used for lifetime and reliability data), and the Frechet distribution (a member of the extreme value family), the Kumaraswamy Weibull Frechet Distribution offers a powerful tool for modeling complex real-world phenomena involving heavy-tailed data and extreme events [Kumaraswamy, 1980; Nadarajah & Kotz, 2004; Smith, 1989].

The cdf and pdf of the KWFR distribution can be obtained respectively by substituting equations (3) and (4) in equation (1) and (2) as follows:

$$G(x;\varphi) = 1 - \left\{1 - \left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a}\right\}^{b} \quad ; x > 0 \quad , a,b,\alpha,\theta,\gamma,\emptyset > 0 \tag{6}$$

$$g(x;\varphi) = g(x;a,b,\alpha,\theta,\gamma,\emptyset) = ab\theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-\emptyset-1} e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a-1}.$$

$$\left\{1 - \left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a}\right\}^{b-1} \quad ; x > 0 \quad , a,b,\alpha,\theta,\gamma,\emptyset > 0 \tag{7}$$

Where $a, b, \alpha, \theta, \gamma$ and $\emptyset$ are the parameters that control the shape and tail behaviors of the distribution. This distribution generalizes the existing models by combining the benefits of three important distributions, providing a more flexible and powerful tool for modeling real-life data that might exhibit varying patterns of behavior at the extremes. The Kumaraswamy Weibull Frechet Distribution finds applications in diverse fields such as environmental studies, actuarial science, and reliability engineering, where data often exhibits extreme behaviors and complex tail structures. This composite distribution offers a promising approach to handling data that would typically require separate modeling approaches under traditional methods [Lai & Balakrishnan, 2009].

The behaviors of the pdf and cdf of the KWFR distribution for different values of parameters are shown in Figure 1 respectively. It can be observed that the pdf can be decreasing, symmetric, and right-skewed shaped.

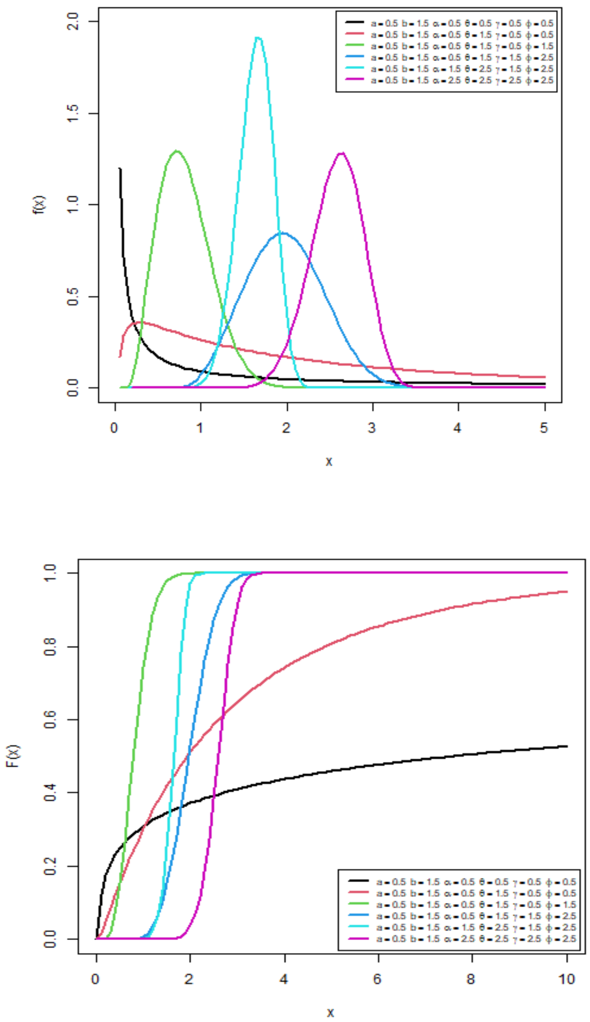

Figure (1): Plot of Pdf and cdf for KWFR distribution

The Survival Function $S(x;\varphi)$ is a key concept in reliability analysis and survival studies. It represents the probability that a random variable X exceeds a certain threshold x, i.e., the probability of survival beyond x. For KWFR distribution, the survival function is defined as the complement of the cumulative distribution function (CDF), which provides insights into the probability of an event not occurring up to a certain time or level. The Survival Function $S(x;\varphi)$ is expressed as:

$$S(x;\varphi) = 1 - G(x;\varphi)$$

$$S(x;\varphi) = \left\{1 - \left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-\theta}\right\}}\right]^{a}\right\}^{b} \quad ; x > 0 \quad , a, b, \alpha, \theta, \gamma, \emptyset > 0 \qquad (8)$$

The survival function in Eq. (8) represents the probability that a unit will survive beyond time x. This functional form is essential for deriving the likelihood function under censoring schemes, as it directly influences the contribution of the (n-r) censored items. The Survival Function for the Kumaraswamy Weibull Frechet Distribution provides critical information for analyzing lifetime data, failure times, and extreme events. It is particularly useful in contexts where the data exhibits heavy tails and extreme behavior, such as environmental risk analysis, reliability testing, and actuarial studies. By combining the flexibility of the Weibull, Frechet, and Kumaraswamy distributions, this survival function offers a more robust model for extreme value analysis and the modeling of extreme [Kumaraswamy, 1980; Nadarajah & Kotz, 2004; Smith, 1989].This function plays a crucial role in understanding the tail behavior of data and is widely used in survival analysis, reliability theory, and risk management. It allows practitioners to estimate the probability of survival beyond a certain threshold, providing insights into the expected lifetimes of systems or the likelihood of extreme occurrences in various fields.

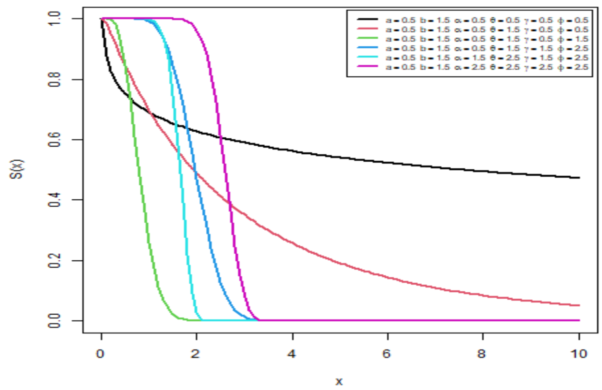

Figure (2): Plot of $S(x;\varphi)$ for KWFR distribution.

The Hazard Rate Function $h(x;\varphi)$ is an essential concept in survival analysis and reliability theory. It represents the instantaneous failure rate at time x, given that the event has survived up to that point. In other words, it provides the rate at which the event of interest (such as failure or death) is expected to occur at a specific time, conditioned on survival up to that time. The hazard rate function is closely related to the survival function $S(x;\varphi)$ and the probability density function (PDF) $g(x;\varphi)$, with the relationship given by:

$$h(x;\varphi) = \frac{g(x;\varphi)}{S(x;\varphi)}$$

$$h(x;\varphi) = ab\theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-\emptyset-1} e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a-1} .\left\{1-\left[1-e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a}\right\}^{-1} \quad (9)$$

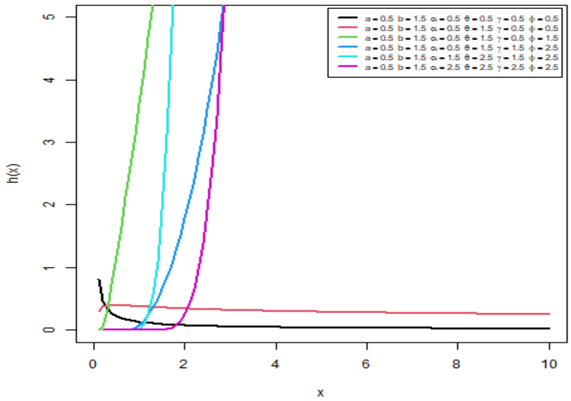

Figure (3): Plot of $h(x;\varphi)$ for KWFR distribution.

Fig. 1: highlights that the can exhibit diverse shapes, including decreasing, J- shape, and concave shapes. The graphical analysis of the PDF, CDF, and hazard rate functions demonstrates the flexibility of the KWFR distribution. The model is capable of capturing various shapes, including increasing, decreasing, and bathtub-shaped hazard rates, which are commonly observed in reliability and survival data. This flexibility makes the proposed distribution suitable for modeling a wide range of real-world phenomena, particularly those involving extreme events and heavy-tailed behavior

The quantile X_q of the KWFR distribution is obtained by inverting the cdf in equation (7) as follows:

$$g(X_q;\varphi) = q \quad ; 0 < q < 1$$

$$X_q = \gamma\left\{ln\left[1+\left\{(-\alpha^{-1})ln\left[1-\left\{1-(1-q)^{\frac{1}{b}}\right\}^{\frac{1}{a}}\right]\right\}^{\frac{-1}{\theta}}\right]\right\}^{\frac{-1}{\emptyset}} \quad (10)$$

This formula involves the parameters $a, b, \alpha, \theta, \gamma$ and $\emptyset$ that define the distribution. These parameters provide the flexibility to model a wide range of behaviors, including heavy-tailed distributions and extreme events.The quantile function is particularly useful in extreme value theory and risk management, as it allows one to compute thresholds corresponding to specific probabilities. For example, it can be used to estimate the value below which a certain percentage of data points are expected to fall, or conversely, to estimate the threshold above which extreme events are likely to occur. In reliability analysis, quantiles help assess the probability of system failure occurring within a given time frame or under specific conditions. Thus, the quantile function for the Kumaraswamy Weibull Frechet Distribution provides valuable insights into the tail behavior of the distribution, and it is widely used in applications involving risk assessment, extreme value analysis, and survival modeling [Kumaraswamy, 1980; Nadarajah & Kotz, 2004; Smith, 1989].

The Moments are fundamental statistical measures that provide valuable information about the shape and characteristics of a probability distribution. Moments capture the central tendency, variability, skewness, and kurtosis of a distribution, offering a deeper understanding of the data's behavior. The rrr-th moment of a probability distribution is defined as the expected value of x , where xxx is the random variable and r is a non-negative integer [Feller, Morita, & Gillette, 1971].

$$\mu_r^{\backslash} = \int_0^{\infty} x^r\, g(x;\varphi)dx$$

$$\mu_r^{\backslash} = \int_0^{\infty} x^r\, ab\theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-\emptyset-1} e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}} .\left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a-1} .\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{ia}$$

$$\mu_r^{\prime} = \int_0^{\infty} x^r\, ab\theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-\emptyset-1}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^k (j+1)^k}{k!}(-1)^{i+j+k}\binom{b-1}{i}\binom{\alpha(i+1)-1}{j}\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}} - 1\right]^{-k\theta}$$

$$\mu_r^{\prime} = \gamma^r \Gamma\left(1-\frac{\gamma}{\emptyset}\right)\sum_{i,j,k,u=0}^{\infty}\left[\theta(k+1)+u\right]^{\frac{\gamma}{\emptyset}} ab\theta(-1)^{i+j+k}\binom{b-1}{i}\binom{\alpha(i+1)-1}{j}\frac{\Gamma[\theta(k+1)+n]\alpha^{k+1}(j+1)^k}{k!u!\Gamma[\theta(k+1)]} \quad (11)$$

Order statistics are critical tools in statistical analysis, representing the values obtained by arranging a random sample of size n in ascending order. The r-th order statistic, denoted as $X_r$corresponds to the r-th smallest value in the sample. These statistics are widely used in various fields such as reliability analysis, quality control, and extreme value theory, where the focus is often on the smallest or largest values in a dataset [David & Nagaraja, 2004). The probability density function (PDF) of the r-th order statistic for a random variable X with cumulative distribution function (CDF) $G(x;\varphi)$ and PDF $g(x;\varphi)$ is given by:

$$g_{r:n}(x;\varphi) = \frac{1}{\beta(r,n-r+1)}[G(x;\varphi)]^{r-1}[1 - G(x;\varphi)]^{n-r} g(x;\varphi)$$

We can use the binomial expansion of $[1 - G(x;\varphi)]^{n-r}$ given as following

$$[1 - G(x;\varphi)]^{n-r} = \sum_{i=0}^{n-r}(-1)^i\,[G(x;\varphi)]^i$$

$$g_{r:n}(x;\varphi) = \sum_{i=0}^{n-r}\frac{n!\,(-1)^r}{(r-1)!\,(n-r)!}g(x;\varphi)\,[G(x;\varphi)]^{i+r-1}$$

$$g_{r:n}(x;\varphi) = \frac{n!}{(r-1)!(n-r)!}ab\theta\alpha\emptyset\gamma^{\emptyset}x^{-\emptyset-1}\, e^{-\theta\left(\frac{\gamma}{x}\right)^{\emptyset}}\left[1 - e^{-\left(\frac{\gamma}{x}\right)^{\emptyset}}\right]^{-\emptyset-1} e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\left[1 - e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a-1} .\left\{1-\left[1- e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a}\right\}^{b+n-r-1} .\left\{1-\left[1- e^{\left\{-\alpha\left[e^{\left(\frac{\gamma}{x}\right)^{\emptyset}}-1\right]^{-\theta}\right\}}\right]^{a}\right\}^{b(r-1)} \quad (12)$$

Here, $\beta(r,n-r+1)$is the Beta function, which ensures the PDF integrates to 1. For the Kumaraswamy Weibull Frechet (KWFR) distribution, the order statistics provide a detailed characterization of the distribution's behavior within samples, capturing the variability and extreme properties of the data. Order statistics are particularly useful in deriving critical reliability metrics such as the minimum life and maximum life of systems, which are often modeled using advanced distributions like the KWFR[David & Nagaraja, 2004]. Furthermore, they play a fundamental role in non-parametric statistical methods, where sample quantiles and ranks are analyzed without assuming a specific distribution form [Arnold, 1992].The versatility of order statistics extends to hypothesis testing, parameter estimation, and modeling in applied fields like climatology, finance, and engineering. Their theoretical underpinnings are well-studied, making them indispensable tools in both classical and modern statistical frameworks.

3 Maximum likelihood Estimation of KWFR:

Maximum Likelihood Estimation (MLE) is a widely used method for parameter estimation in statistical models, particularly for distributions with complex structures. The method involves finding parameter values that maximize the likelihood function, representing the probability of observing the given data under the assumed model [Casella & Berger, 2002]. When dealing with censored samples, MLE is especially valuable, as it effectively utilizes incomplete data, which is common in survival analysis, reliability studies, and biomedical research. Censoring occurs when the exact value of a measurement is not fully observed but is known to fall within certain bounds. In the context of Type II censoring, only the smallest k observations out of n total observations are recorded, while the remaining $(n-k)$ values are censored. The likelihood function for a censored sample, under a probability density function $g(x;\varphi)$ and a cumulative distribution function $G(x;\varphi)$, is given by:

$$L(x;\varphi) = \left(ab\theta\alpha\emptyset\gamma^{\emptyset}\right)^n \prod_{i=1}^{n} x_i^{-\emptyset-1}e^{\left(\frac{\gamma}{x_i}\right)^{\emptyset}}[W(x_i)]^{-\emptyset-1}\, e^{\{-\alpha[W(x_i)]^{-\theta}\}}\left[1 - e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{a-1}\left\{1-\left[1- e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{a}\right\}^{b-1} \quad (13)$$

Where $W(x_i) = \left[e^{\left(\frac{\gamma}{x_i}\right)^{\emptyset}} - 1\right]$, $W(x_k) = \left[e^{\left(\frac{\gamma}{x_k}\right)^{\emptyset}} - 1\right]$

The natural logarithm of the likelihood function equation can be obtained as follows

$$Ln\,L(x;\varphi) = nln(a) + nln(b) + nln(\theta) + nln(\alpha) + n\emptyset\ln(\gamma) - (\emptyset+1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\left(\frac{\gamma}{x_i}\right)^{\emptyset} - (\emptyset+1)\sum_{i=1}^{n} ln[W(x_i)] + \sum_{i=1}^{k}\left\{-\alpha[W(x_i)]^{-\theta}\right\} + (a-1)\sum_{i=1}^{n} ln\left[1 - e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right] + (b-1)\sum_{i=1}^{n} ln\left\{1 - \left[1 - e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{a}\right\} \quad (14)$$

The estimates of all parameters are obtained by differentiating the log-likelihood function in (17) with respect to $a, b, \theta, \alpha, \emptyset$ *and* γ.

The first –order partial derivatives of $Ln\,L(x;\varphi)$ with respect to $a, b, \theta, \alpha, \emptyset$ *and* γ and equating them to zero are as follows:

$$\frac{\partial Log\,L}{\partial a} = \frac{k}{a} + \sum_{i=1}^{k} Log\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right) - (b-1)\sum_{i=1}^{k}\frac{\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a} Log\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)}{\left[1 - \left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a}\right]} - b(n-k)\frac{\left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a} Log\left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)}{\left[1 - \left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right]}$$

$$\frac{\partial Log\,L}{\partial a} = \frac{k}{b} + \sum_{i=1}^{k} Log\left[1 - \left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right] + (n-k)Log\left[1 - \left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right]$$

$$\frac{\partial Log\,L}{\partial \theta} = \frac{k}{\theta} - \sum_{i=1}^{k} Log\,w(x_i) + \alpha\sum_{i=1}^{k}\left(w(x_i)\right)^{-\theta} Log\,w(x_i) - \alpha(a-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha(w(x_i))^{-\theta}} Log\,w(x_i)}{\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)} + a\alpha(b-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha(w(x_i))^{-\theta}}\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a-1} Log\,w(x_i)}{\left[1 - \left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a}\right]} + a\alpha b(n-k)\frac{\left(w(x_k)\right)^{-\theta} e^{-\alpha(w(x_k))^{-\theta}}\left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a-1} Log\,w(x_k)}{\left[1 - \left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right]}$$

$$\frac{\partial Log\,L}{\partial \alpha} = \frac{k}{\alpha} - \sum_{i=1}^{k}\left(w(x_i)\right)^{-\theta} + (a-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha(w(x_i))^{-\theta}}}{\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)} - a(b-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha(w(x_i))^{-\theta}}\left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a-1}}{\left[1 - \left(1 - e^{-\alpha(w(x_i))^{-\theta}}\right)^{a}\right]} - ab(n-k)\frac{\left(w(x_k)\right)^{-\theta} e^{-\alpha(w(x_k))^{-\theta}}\left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a-1}}{\left[1 - \left(1 - e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right]} \quad (15)$$

Due to the complexity of the likelihood equations and the absence of closed-form solutions, numerical optimization techniques are required to obtain the parameter estimates. In this study, the Newton–Raphson method is employed. The estimation procedure can be summarized as follows:

Step 1: Select initial values for the parameter vector θ.

Step 2: Compute the score vector and the Hessian matrix of the log-likelihood function.

Step 3: Update the parameter estimates iteratively

Step 4: Repeat the iterative process until convergence is achieved, i.e.,

Step 5: Verify convergence by ensuring the Hessian matrix is negative definite.

All computations were implemented using R software.

For the Kumaraswamy Weibull Frechet (KWFR) distribution, MLE involves maximizing this likelihood function with respect to the distribution's parameters, making it a powerful tool for analyzing real-world data characterized by censoring. MLE under censored samples is extensively applied in reliability engineering, where lifetime data is often censored due to time or resource constraints [Butler et al., 2003].Additionally, it is used in survival analysis to estimate hazard and survival functions, critical for understanding the time-to-event data in medical studies [Kalbfleisch & Prentice, 2002].The robustness and efficiency of MLE, even under incomplete data, make it a cornerstone of modern statistical inference. Its implementation, however, often requires numerical optimization techniques due to the complexity of the likelihood functions involved, particularly for multi-parameter distributions like the KWFR.

3.1 Likelihood Function under Type II Censoring:

In the context of Type-II censoring, the likelihood function is constructed by combining the joint density of the first r observed failure times with the survival probability of the remaining (n-r) units. The refined expression for $S(x;\varphi)$ in Eq. (8) ensures that the likelihood surface is correctly defined for numerical optimization.

The likelihood function for a censored sample is given by substituting (6) and (7) into (5) as follows:

$$LogL(x;\varphi) = Log\frac{n!}{(n-k)!} + k\,Log\left(ab\theta\alpha\emptyset\gamma^{\emptyset}\right) - (\emptyset+1)\sum_{i=1}^{k} Log x_i + \sum_{i=1}^{k}\left(\frac{\gamma}{x}\right)^{\emptyset} - (\theta+1)\sum_{i=1}^{k} Log\, w(x_i) - \alpha\sum_{i=1}^{k}\left(w(x_i)\right)^{-\theta} + (a-1)\sum_{i=1}^{k} Log\left[1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right] + (b-1)\sum_{i=1}^{k} Log\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right] + b(n-k)Log\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right] \quad (16)$$

Where $w(x_i) = \left[e^{\left(\frac{\gamma}{x\_i}\right)^{\emptyset}} - 1\right]$ and $w(x_k) = \left[e^{\left(\frac{\gamma}{x\_k}\right)^{\emptyset}} - 1\right]$

Differentiating (18) partially with respect to $a, b, \alpha, \theta, \gamma\ and\ \emptyset$ the $LogL(x;\varphi)$ of (18), becomes

$$\frac{\partial Log\,L}{\partial a} = \frac{k}{a} + \sum_{i=1}^{k} Log\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right) - (b-1)\sum_{i=1}^{k}\frac{\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a} Log\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)}{\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right]} - b(n-k)\frac{\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a} Log\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)}{\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right]} \quad (17)$$

$$\frac{\partial Log\,L}{\partial b} = \frac{k}{b} + \sum_{i=1}^{k} Log\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right] + (n-k)Log\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right] \quad (18)$$

$$\frac{\partial Log\,L}{\partial \theta} = \frac{k}{\theta} - \sum_{i=1}^{k} Log\, w(x_i) + \alpha\sum_{i=1}^{k}\left(w(x_i)\right)^{-\theta} Log\, w(x_i) - \alpha(a-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha\left(w(x_i)\right)^{-\theta}} Log\, w(x_i)}{\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)} + a\alpha(b-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha\left(w(x_i)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a-1} Log\, w(x_i)}{\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right]} + a\alpha b(n-k)\frac{\left(w(x_k)\right)^{-\theta} e^{-\alpha\left(w(x_k)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a-1} Log\, w(x_k)}{\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right]} \quad (19)$$

$$\frac{\partial Log\,L}{\partial \alpha} = \frac{k}{\alpha} - \sum_{i=1}^{k}\left(w(x_i)\right)^{-\theta} + (a-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha\left(w(x_i)\right)^{-\theta}}}{\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)} - \boldsymbol{a(b-1)\sum_{i=1}^{k}\frac{\left(w(x_i)\right)^{-\theta} e^{-\alpha\left(w(x_i)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right]}} - \boldsymbol{ab(n-k)\frac{\left(w(x_k)\right)^{-\theta} e^{-\alpha\left(w(x_k)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a}\right]}} \quad (20)$$

$$\frac{\partial Log\,L}{\partial \gamma} = \frac{k\emptyset}{\gamma} - \sum_{i=1}^{k}\left(\frac{\emptyset}{x_i}\right)\left(\frac{\gamma}{x_i}\right)^{\emptyset-1} - (\theta+1)\sum_{i=1}^{k}\frac{\frac{w^{\backslash}(x_i)}{\partial\gamma}}{w(x_i)} + \theta\alpha\sum_{i=1}^{k}\frac{w^{\backslash}(x_i)\left(w(x_i)\right)^{-\theta-1}}{\partial\gamma} - \theta\alpha(a-1)\sum_{i=1}^{k}\frac{\frac{w^{\backslash}(x_i)}{\partial\gamma}\left(w(x_i)\right)^{-\theta-1} e^{-\alpha\left(w(x_i)\right)^{-\theta}}}{\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}} + a\theta\alpha(b-1)\sum_{i=1}^{k}\frac{\frac{w^{\backslash}(x_i)}{\partial\gamma}\left(w(x_i)\right)^{-\theta-1} e^{-\alpha\left(w(x_i)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right]} + a\theta\alpha b(n-k)\sum_{i=1}^{k}\frac{\frac{w^{\backslash}(x_k)}{\partial\gamma}\left(w(x_k)\right)^{-\theta-1} e^{-\alpha\left(w(x_k)\right)^{-\theta}}\left(1-e^{-\alpha\left(w(x_k)\right)^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha\left(w(x_i)\right)^{-\theta}}\right)^{a}\right]} \quad (21)$$

$$\frac{\partial Log\,L}{\partial \emptyset} = \frac{k}{\emptyset} + kLog\,\gamma - \sum_{i=1}^{k} Log\, w(x_i) + \sum_{i=1}^{k}\left(\frac{\gamma}{x_i}\right)^{\emptyset} Log\left(\frac{\gamma}{x_i}\right) - (\theta+1)\sum_{i=1}^{k}\frac{\frac{w^{\backslash}(x_i)}{\partial\gamma}}{w(x_i)} + \theta\alpha\sum_{i=1}^{k}\frac{w^{\backslash}(x_i)\left(w(x_i)\right)^{-\theta-1}}{\partial\emptyset} - \theta\alpha(a-$$

$$1)\sum_{i=1}^{k}\frac{\frac{w^{\diagdown}(x_i)}{\partial\emptyset}(w(x_i))^{-\theta-1}e^{-\alpha(w(x_i))^{-\theta}}e^{-\alpha(w(x_i))^{-\theta}}}{\left(1-e^{-\alpha(w(x_i))^{-\theta}}\right)}+a\theta\alpha(b-$$

$$1)\sum_{i=1}^{k}\frac{\frac{w^{\diagdown}(x_i)}{\partial\emptyset}(w(x_i))^{-\theta-1}e^{-\alpha(w(x_i))^{-\theta}}\left(1-e^{-\alpha(w(x_i))^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha(w(x_i))^{-\theta}}\right)^{a}\right]}+a\theta\alpha b(n-$$

$$k)\sum_{i=1}^{k}\frac{\frac{w^{\diagdown}(x_k)}{\partial\emptyset}(w(x_k))^{-\theta-1}e^{-\alpha(w(x_k))^{-\theta}}\left(1-e^{-\alpha(w(x_k))^{-\theta}}\right)^{a-1}}{\left[1-\left(1-e^{-\alpha(w(x_k))^{-\theta}}\right)^{a}\right]}$$

(22)

Since

$$\frac{w^{\diagdown}(x_i)}{\partial\gamma}=\left(\frac{\emptyset}{x_i}\right)\left(\frac{\gamma}{x_i}\right)^{\emptyset-1}e^{\left(\frac{\gamma}{x_i}\right)^{\emptyset}},\qquad \frac{w^{\diagdown}(x_k)}{\partial\gamma}=\left(\frac{\emptyset}{x_k}\right)\left(\frac{\gamma}{x_k}\right)^{\emptyset-1}e^{\left(\frac{\gamma}{x_k}\right)^{\emptyset}}$$

$$\frac{w^{\diagdown}(x_i)}{\partial\emptyset}=\left(\frac{\gamma}{x_i}\right)^{\emptyset}e^{\left(\frac{\gamma}{x_i}\right)^{\emptyset}}Log\left(\frac{\gamma}{x_i}\right),\qquad \frac{w^{\diagdown}(x_k)}{\partial\emptyset}=\left(\frac{\gamma}{x_k}\right)^{\emptyset}e^{\left(\frac{\gamma}{x_k}\right)^{\emptyset}}Log\left(\frac{\gamma}{x_k}\right)$$

Equating the Equations (19) – (24) to zero and solving them simultaneously will provide the MLE estimators of the KWFR parameters.

3.2 Confidence Interval Estimation:

For confidence interval estimation of the KWFR distribution the observed variance covariance matrix of the MLE is derived in by taking the negative expectation of the second derivatives of the natural logarithm of the likelihood function with respect to φ.

$$I(\varphi)^{-1}=-\begin{bmatrix}
E\left(\frac{\partial^2 \mathcal{L}}{\partial a^2}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial a\partial b}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial a\partial \alpha}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial a\partial \theta}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial a\partial \gamma}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial a\partial \emptyset}\right)\\
E\left(\frac{\partial^2 \mathcal{L}}{\partial b\partial a}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial b^2}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial b\partial \alpha}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial b\partial \theta}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial b\partial \gamma}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial b\partial \emptyset}\right)\\
E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha\partial a}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha\partial b}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha\partial \theta}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha\partial \gamma}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \alpha\partial \emptyset}\right)\\
E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta\partial a}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta\partial b}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta\partial \alpha}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta\partial \gamma}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \theta\partial \emptyset}\right)\\
E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma\partial a}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma\partial b}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma\partial \alpha}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma\partial \theta}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma^2}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \gamma\partial \emptyset}\right)\\
E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset\partial a}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset\partial b}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset\partial \alpha}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset\partial \theta}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset\partial \gamma}\right) & E\left(\frac{\partial^2 \mathcal{L}}{\partial \emptyset^2}\right)
\end{bmatrix}^{-1}$$

(23)

According to the estimate asymptotic multivariate normal $N_m(\varphi)(0,\ (\varphi)^{-1})$ distribution of φ can be used to construct approximate confidence interval for the parameters.

The asymptotic variance-covariance matrix of the MLEs is obtained by inverting the observed information matrix. This matrix is crucial for constructing asymptotic confidence intervals for the model parameters. Given the six-parameter structure of the KWFR distribution, the stability of the information matrix was monitored during numerical iterations to ensure it remained positive-definite, which is a key indicator of model identifiability and successful convergence of the Newton-Raphson algorithm. All second-order partial derivatives were computed and evaluated numerically using the R software environment to ensure high precision in estimating the standard errors.

4 Bayesian Estimation for the KWFR distribution

In Bayesian inference it is supposed that the unknown parameters are random variables with a joint prior PDF[Box & Tiao, 2011]. The prior PDF may be calculated using previous information and experience. When no prior information is available, non-informative priors can be used for the Bayesian inference[Debnath, Basu, & Technology, 2015]. In this section the Bayesian estimation for the parameters of the KWFR distribution under the assumption that the random variables $a,b,\theta,\alpha,\emptyset\ and\ \gamma$ have an independent gamma prior PDF as follows:

$$\pi_1(a)\propto a^{d_1-1}\exp(-\beta_1 a)\qquad a>0,\ d_1,\beta_1>0$$
$$\pi_2(b)\propto b^{d_2-1}\exp(-\beta_2 b)\qquad b>0,\ d_2,\beta_2>0$$
$$\pi_3(\theta)\propto \theta^{d_3-1}\exp(-\beta_3 \theta)\qquad \theta>0,\ d_3,\beta_3>0$$
$$\pi_4(\alpha)\propto \alpha^{d_4-1}\exp(-\beta_4 \alpha)\qquad \alpha>0,\ d_4,\beta_4>0$$
$$\pi_5(\emptyset)\propto \emptyset^{d_5-1}\exp(-\beta_5 \emptyset)\qquad \emptyset>0,\ d_5,\beta_5>0$$
$$\pi_6(\gamma)\propto \gamma^{d_6-1}\exp(-\beta_6 \gamma)\qquad \gamma>0,\ d_6,\beta_6>0$$

All the hyper-parameters $d_1,d_2,d_3,d_4,d_5,d_6,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6$ are chosen to be known and non-negative. The prior gamma has been used to obtain Bayesian estimate of $a,b,\theta,\alpha,\emptyset\ and\ \gamma$ using MCMC methods [Van Ravenzwaaij, Cassey, Brown, & review, 2018], it has been found that gamma prior gives quite good estimates compare to other priors. The joint prior for $a,b,\theta,\alpha,\emptyset\ and\ \gamma$ is given by

$$\pi(a,b,\theta,\alpha,\emptyset,\gamma)=\pi_1(a)\,\pi_2(b)\,\pi_3(\theta)\pi_4(\alpha)\pi_5(\emptyset)\pi_6(\gamma)$$

$$\pi(a,b,\theta,\alpha,\emptyset,\gamma)\propto a^{d_1-1}b^{d_2-1}\theta^{d_3-1}\alpha^{d_4-1}\emptyset^{d_5-1}\gamma^{d_6-1}\exp(-\beta_1 a-\beta_2 b-\beta_3\theta-\beta_4\alpha-\beta_5\emptyset-\beta_6\gamma)\quad(24)$$

The choice of gamma priors is motivated by their flexibility and conjugacy properties in modeling positive parameters. Gamma distributions allow for a wide range of shapes and can incorporate prior knowledge in a mathematically convenient form. To ensure the robustness of the Bayesian results, different prior specifications (informative and non-informative) were considered. The comparison indicates that informative priors lead to more stable estimates. The Metropolis–Hastings algorithm was implemented with carefully tuned proposal distributions to achieve acceptable acceptance rates. Convergence of the Markov chains was assessed through trace plots and monitoring the stability of posterior summaries across iterations.

4.1 Bayesian Estimation in Complete Sample

The joint posterior of complete data density of $a, b, \theta, \alpha, \emptyset$ and γ given the observed data $X = (x_1, x_2, \dots, x_n)$ can be obtained using equations (16) and (26) as follows:

$$\pi(\varphi|\boldsymbol{x}) \propto a^{n+d_1-1} b^{n+d_2-1} \theta^{n+d_3-1} \alpha^{n+d_4-1} \emptyset^{n+d_5-1} \gamma^{\emptyset n+d_6-1} \sum_{i=1}^{n} \boldsymbol{x_i^{-\emptyset-1}}\, \boldsymbol{e}^{\sum_{i=1}^{n}\left(\frac{\gamma}{x_i}\right)^{\emptyset}+(-\beta_1 a-\beta_2 b-\beta_3\theta-\beta_4\alpha-\beta_5\emptyset-\beta_6\gamma)} \sum_{i=1}^{n}[W(x_i)]^{-\emptyset-1}\, \boldsymbol{e}^{\sum_{i=1}^{n}\{-\alpha[W(x_i)]^{-\theta}\}} \sum_{i=1}^{n}\left[\boldsymbol{1}-\boldsymbol{e}^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{\alpha-1} \sum_{i=1}^{n}\left\{\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{a}\right\}^{b-1} \quad (25)$$

4.2 Bayesian Estimation in Type II Censored

Based on the likelihood function (18) and the joint prior density (26), the joint posterior of type II censored density function of $a, b, \theta, \alpha, \emptyset$ and γ can be obtained as follows:

$$\pi(\varphi|x) \propto a^{k+d_1-1} b^{k+d_2-1} \theta^{k+d_3-1} \alpha^{k+d_4-1} \emptyset^{k+d_5-1} \gamma^{\emptyset k+d_6-1} \sum_{i=1}^{k} x_i^{-\emptyset-1}\, e^{\sum_{i=1}^{k}\left(\frac{\gamma}{x_i}\right)^{\emptyset}+(-\beta_1 a-\beta_2 b-\beta_3\theta-\beta_4\alpha-\beta_5\emptyset-\beta_6\gamma)} \sum_{i=1}^{k}[W(x_i)]^{-\emptyset-1}\, e^{\sum_{i=1}^{k}\{-\alpha[W(x_i)]^{-\theta}\}} \sum_{i=1}^{k}\left[1-e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{\alpha-1} \sum_{i=1}^{k}\left\{1-\left[1-e^{\{-\alpha[W(x_i)]^{-\theta}\}}\right]^{a}\right\}^{b-1} \sum_{i=1}^{k}\left\{1-\left[1-e^{\{-\alpha[W(x_k)]^{-\theta}\}}\right]^{a}\right\}^{b(n-k)} \quad (26)$$

Under the squared error loss function it should be the estimation of all parameters are obtained by:

$$\tilde{h}(\Delta) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(\Delta)\pi(\Delta|\boldsymbol{x})\, da\, db\, d\theta\, d\alpha\, d\emptyset\, d\gamma \quad (27)$$

Where $\Delta = (a, b, \theta, \alpha, \emptyset, \gamma)$

Unfortunately, Equation. (29) cannot be computed for general $h(\Delta)$. The Markov Chain Monte Carlo (MCMC) approach and utilize the Metropolis-Hastings (MH) Algorithm to generate a posterior sample. Under LINEX loss function the Bayesian estimator of $\varphi(a, b, \theta, \alpha, \emptyset, \gamma)$ is the value $\hat{\varphi}(a, b, \theta, \alpha, \emptyset, \gamma)$ which minimizes $E_\varphi[\hat{\varphi}(a, b, \theta, \alpha, \emptyset, \gamma) - \varphi(a, b, \theta, \alpha, \emptyset, \gamma)]$, and is given by:

$$\hat{\varphi} = -\frac{1}{c} ln\left[E_\varphi(e^{-c\varphi})\right] = -\frac{1}{c} ln\left[\int_0^\infty \boldsymbol{\pi(\varphi|x)} e^{-c\varphi}\, d\varphi\right] \quad (28)$$

Where $E_\varphi(.)$ is the expectation with respect to $\boldsymbol{\pi(\varphi|x)}$. But the equation (30) has to be performed numerically using MCMC.

5 Simulation Study

In this section; Monte Carlo simulation is obtained for comparing between estimation methods based on complete data and Type-II Censoring Scheme. In all cases the RMSE and AIL using informative prior are less than RMSE and AIL using non informative prior. The Bayes estimate under LINEX loss function ($c = 0.5$) has the smallest estimated RMSE's compared with the Bayes estimates under squared error loss functions. The Bayes estimates under squared error loss functions has the smallest estimated RMSE's compared with the Bayes estimate under LINEX loss function ($c = -0.5$). As expected for most cases RMSE, estimate risk and the length of HDP decrease when n increase, also CP is greater than or equal 95%. The simulation results show that the RMSE and AIL using informative prior are less than RMSE and AIL using non informative prior. With increasing sample size and number of censors, the estimation of the parameters is closer to the real values, RMSE's and the length of the confidence intervals are reduced. Bayesian estimators of parameters under the LINEX loss function have the minimum RMSE ($c = 0.5$) compared with the Bayes estimates under squared error loss functions. The CP is greater than or equal 93%. The simulation results provide important insights into the performance of the estimators. As the sample size increases, the estimators become more accurate, as indicated by the reduction in RMSE and interval lengths. This behavior is consistent with the asymptotic properties of both maximum likelihood and Bayesian estimators. The effect of censoring is also evident, as higher censoring levels tend to increase the variability of the estimates. Nevertheless, the estimators remain relatively stable, indicating robustness of the proposed model under different censoring schemes. Furthermore, Bayesian estimators under the LINEX loss function generally show improved performance in terms of RMSE. The asymmetric nature of the LINEX loss function allows for better handling of estimation errors. The use of informative priors leads to more efficient estimates compared to non-informative priors, highlighting the importance of incorporating prior information when available.

5.1 Estimation under complete sample

Parameters of simulation

1. Number of replications = 1000

2. Sample sizes are: $n = 50, 100, 200$.
3. **Parameters of KWFR distribution are:**
   a. $a = 1.5, b = 2.5, \alpha = 1.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$
   b. $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 0.5, \phi = 0.5$
   c. $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$
   d. $a = 0.5, b = 1.5, \alpha = 1.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$
4. **Methods of point estimation are:**
   a. Non-Bayesian: MLE and MPS
   b. Bayesian estimation (BE): using Markov Chain Monte Carlo (MCMC)
      i. Number of MCMC: 10000
      ii. Type of prior: Informative (INF) and Non-Informative (Non-INF) prior
      iii. Type of loss function: squared error (SE) and Linear exponential (LN) loss functions
5. **Methods of interval estimation are:**
   a. Under Non-Bayesian: Asymptotic confidence interval (Asy-CI)
   b. Under Bayesian estimation: Highest posterior density credible interval (HPD-CI)

<u>Computations</u>:

1. For point estimation: Average (Avg.) and root mean square error (RMSE)
2. **For interval estimation:**
   a. Lower and upper limits of confidence interval
   b. Average interval length (AIL) and coverage probability (CP)

Table (1.1): Non-Bayesian estimates for MLE method of KWFR distribution at different sample sizes and $a = 1.5, b = 2.5, \alpha = 1.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$.

n	Parm.	MLE		**Asy-CI**			
		Avg.	RMSE	**Lower**	**Upper**	**AIL**	**CP (%)**
50	a	1.6856	0.9847	0.0000	3.5879	3.5879	93.3
	b	2.7922	1.5058	0.0000	5.6981	5.6981	95.6
	α	1.6181	0.8225	0.0167	3.2195	3.2028	97.0
	θ	0.5635	0.4366	0.0000	1.4133	1.4133	95.6
	γ	0.5887	0.5603	0.0000	1.6770	1.6770	95.6
	$\emptyset$	0.5506	0.1694	0.2325	0.8686	0.6362	96.3
100	a	1.5894	0.6878	0.2495	2.9293	2.6798	97.0
	b	2.4431	0.7882	0.8985	3.9876	3.0892	96.6
	α	1.6440	0.8318	0.0343	3.2537	3.2194	97.5
	θ	0.5118	0.2199	0.0803	0.9433	0.8630	97.0
	γ	0.5430	0.4321	0.0000	1.3878	1.3878	97.0
	$\emptyset$	0.5418	0.1145	0.3324	0.7512	0.4188	96.1
200	a	1.5491	0.6060	0.3631	2.7350	2.3719	97.0

	b	2.5492	0.7403	1.0987	3.9996	2.9008	96.3
	α	1.5034	0.4654	0.5896	2.4171	1.8275	96.3
	θ	0.4967	0.1167	0.2677	0.7258	0.4581	95.5
	γ	0.4973	0.2673	0.0000	1.0221	1.0221	96.3
	$\emptyset$	0.5312	0.0780	0.3907	0.6717	0.2809	97.0

Table (1.2): Non-Bayesian estimates for MLE method of KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$.

n	Parm.	MLE		**Asy-CI**			
		Avg.	RMSE	**Lower**	**Upper**	**AIL**	**CP (%)**
50	a	0.5022	0.2911	0.0000	1.0752	1.0752	97.3
	b	1.5506	0.7765	0.0251	3.0761	3.0510	97.3
	α	0.5790	0.3478	0.0000	1.2458	1.2458	95.6
	θ	1.6175	0.8979	0.0000	3.3701	3.3701	94.7
	γ	0.5873	0.5555	0.0000	1.6674	1.6674	95.6
	$\emptyset$	0.5856	0.2660	0.0899	1.0813	0.9914	96.5
100	a	0.5936	0.6046	0.0000	1.7684	1.7684	99.3
	b	1.5805	0.6925	0.2279	2.9332	2.7054	95.9
	α	0.5575	0.2926	0.0000	1.1217	1.1217	95.2
	θ	1.5916	0.6787	0.2691	2.9142	2.6451	94.5
	γ	0.6861	0.7747	0.0000	2.1651	2.1651	99.3
	$\emptyset$	0.5439	0.1958	0.1687	0.9191	0.7504	95.2
200	a	0.5082	0.2255	0.0651	0.9512	0.8861	97.0
	b	1.6870	0.7114	0.3378	3.0363	2.6985	95.3
	α	0.5834	0.3141	0.0000	1.1787	1.1787	96.4

	θ	1.5861	0.5397	0.5387	2.6336	2.0949	95.9
	γ	0.6139	0.4238	0.0000	1.4163	1.4163	95.3
	$\emptyset$	0.5187	0.1682	0.1901	0.8472	0.6571	95.9

Table (1.3): Non-Bayesian estimates for MLE method of KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$.

n	Parm.	MLE		**Asy-CI**			
		Avg.	RMSE	**Lower**	**Upper**	**AIL**	**CP (%)**
50	a	0.6424	0.4027	0.0000	1.3886	1.3886	97.9
	b	1.7250	0.7727	0.2607	3.1892	2.9286	97.9
	α	0.7440	0.5721	0.0000	1.7690	1.7690	95.8
	θ	1.3007	0.6484	0.0787	2.5227	2.4440	93.8
	γ	1.7018	0.4912	0.8148	2.5889	1.7741	91.7
	$\emptyset$	3.0460	1.1964	0.9375	5.1544	4.2170	95.8
100	a	0.5454	0.2990	0.0000	1.1281	1.1281	94.2
	b	1.6876	0.6139	0.5353	2.8400	2.3047	95.3
	α	0.6915	0.4090	0.0000	1.4039	1.4039	96.5
	θ	1.2786	0.4611	0.4813	2.0759	1.5946	96.5
	γ	1.6228	0.2827	1.1208	2.1248	1.0040	94.2
	$\emptyset$	3.0417	1.0390	1.2939	4.7895	3.4957	96.5
200	a	0.5431	0.1846	0.5339	0.3234	0.1898	0.8964
	b	1.6542	0.4440	1.6959	0.5928	0.8347	2.4737
	α	0.5778	0.2148	0.5362	0.3161	0.1838	0.9718
	θ	1.3252	0.3662	1.5994	0.4491	0.6919	1.9586
	γ	1.5698	0.1953	1.4975	0.3157	1.2107	1.9289

	$\emptyset$	2.8795	0.7885	2.4306	0.6784	1.5192	4.2398

Table (1.4): Non-Bayesian estimates MLE method of KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 1.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$.

n	Parm.	MLE		Asy-CI			
		Avg.	RMSE	**Lower**	**Upper**	**AIL**	**CP (%)**
50	a	0.5560	0.1922	0.1838	0.9283	0.7445	93.8
	b	1.5558	0.3261	0.9055	2.2061	1.3006	98.0
	α	1.6361	0.5304	0.5984	2.6739	2.0756	93.8
	θ	2.9297	1.0638	0.9598	4.8995	3.9398	98.0
	γ	1.4986	0.1201	1.2556	1.7417	0.4861	93.8
	$\emptyset$	2.5511	0.9616	0.6073	4.4950	3.8877	93.8
100	a	0.5451	0.2147	0.1269	0.9633	0.8364	93.5
	b	1.6264	0.6438	0.3687	2.8840	2.5153	96.8
	α	1.6782	0.8856	0.0000	3.4065	3.4065	93.5
	θ	2.8103	1.2536	0.3904	5.2303	4.8399	93.5
	γ	1.5009	0.1813	1.1397	1.8621	0.7224	96.8
	$\emptyset$	2.5578	0.9611	0.6463	4.4693	3.8229	93.5
200	a	0.5547	0.2743	0.0217	1.0877	1.0661	97.7
	b	1.5793	0.4460	0.7088	2.4497	1.7409	98.5
	α	1.5862	0.5304	0.5483	2.6241	2.0758	95.3
	θ	2.7967	1.1301	0.6342	4.9591	4.3250	95.3
	γ	1.4861	0.1160	1.2577	1.7144	0.4567	95.3
	$\emptyset$	2.6018	0.9064	0.8156	4.3879	3.5723	97.7

Table (2.1): Bayesian estimates for KWFR distribution at different sample sizes and $a = 1.5, b = 2.5, \alpha = 1.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$.

n	Parm	Point Estimate						HPD-CI			
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)					
		AVG	RMSE	AVG	RMSE	AVG	RMSE	Lower	Upper	AIL	CP (%)
INF prior case											
50	a	1.7552	1.0654	1.6757	1.0195	1.8663	1.1234	0.1102	3.7203	3.6102	95.1
	b	2.3646	1.6429	2.2844	1.6261	2.4857	1.6604	0.0134	5.6718	5.6584	95.3
	α	1.8728	1.1293	1.7935	1.0592	1.9962	1.2024	0.1155	3.5007	3.3852	96.0
	θ	0.7478	0.6902	0.7251	0.6500	0.7993	0.7463	0.0648	2.4118	2.3471	95.2
	γ	0.8874	0.8930	0.8478	0.8477	0.9242	0.9401	0.0248	2.9307	2.9058	95.3
	ϕ	0.8721	0.8554	0.8115	0.7474	0.9298	0.9605	0.0815	2.9038	2.8223	96.6
100	a	1.7259	0.8097	1.6811	0.7469	1.8029	0.8670	0.3305	3.2756	2.9450	97.9
	b	2.1773	1.0215	2.0997	1.0174	2.2456	1.0399	0.3667	4.0518	3.6851	96.5
	α	1.8571	1.1163	1.7692	1.0674	1.9170	1.1771	0.1716	3.6458	3.4742	95.4
	θ	0.6710	0.4755	0.6313	0.4134	0.7063	0.5432	0.0262	1.6818	1.6556	95.1
	γ	0.5752	0.4474	0.5585	0.4224	0.6078	0.4742	0.0205	1.3337	1.3132	95.5
	ϕ	0.7903	0.6903	0.7385	0.5869	0.8477	0.8103	0.1905	2.3308	2.1403	96.8
200	a	1.6950	0.7932	1.6382	0.7573	1.7524	0.8365	0.2233	3.0705	2.8472	95.4
	b	2.3252	1.0292	2.2791	1.0026	2.4070	1.0291	0.2347	4.1866	3.9519	95.8
	α	1.6683	0.6733	1.6128	0.6417	1.7181	0.7187	0.6331	2.9367	2.3037	97.3
	θ	0.6133	0.4694	0.5974	0.3983	0.6383	0.5414	0.0084	1.6290	1.6206	95.2
	γ	0.6261	0.4506	0.6084	0.4199	0.6457	0.4808	0.0042	1.3747	1.3705	95.1
	ϕ	0.6841	0.5199	0.6540	0.4407	0.7215	0.6044	0.2982	1.9027	1.6046	96.2
Non-INF prior case											
50	a	1.9922	1.5671	1.9260	1.5112	2.0762	1.6408	0.0291	5.0230	4.9939	95.7
	b	2.9594	1.9872	2.8168	1.9342	3.1240	2.0600	0.1298	6.6400	6.5102	95.3
	α	1.9353	1.5458	1.8452	1.4704	2.0268	1.6259	0.0005	4.8179	4.8174	95.6
	θ	1.1019	1.0577	1.0399	0.9585	1.1663	1.1601	0.0252	3.1244	3.0992	96.0
	γ	0.9571	1.0697	0.9130	1.0116	1.0429	1.1567	0.0352	2.8829	2.8477	95.6
	ϕ	1.0428	1.1699	0.9779	1.0347	1.1534	1.3214	0.0382	2.5472	2.5089	95.1
100	a	1.8532	1.3254	1.7794	1.2477	1.9355	1.4051	0.0204	4.7839	4.7634	95.1
	b	2.7372	1.3537	2.6118	1.3011	2.8754	1.4205	0.1159	5.2248	5.1089	95.4
	α	1.9031	1.1843	1.8085	1.1328	1.9637	1.2522	0.0001	3.7004	3.7003	95.5
	θ	0.8887	0.8228	0.8185	0.6867	1.0003	0.9902	0.0471	2.4966	2.4495	95.8
	γ	0.8652	0.8467	0.8013	0.8038	0.8921	0.8829	0.0343	2.5088	2.4745	95.7
	ϕ	0.7742	0.6351	0.7413	0.5623	0.8196	0.7124	0.0048	1.8732	1.8684	95.1
200	a	1.7403	0.9864	1.6611	0.9492	1.8036	1.0387	0.1698	3.5792	3.4093	95.7
	b	2.5412	1.2255	2.4847	1.1919	2.6089	1.2612	0.4026	4.9230	4.5204	97.7
	α	1.5487	0.8965	1.4985	0.8534	1.6217	0.9382	0.0083	3.0968	3.0885	95.2
	θ	0.7876	0.6910	0.7406	0.6007	0.8456	0.7951	0.0225	2.0910	2.0685	95.3
	γ	0.6480	0.5104	0.6360	0.4757	0.6889	0.5580	0.0043	1.7012	1.6969	95.2
	ϕ	0.7771	0.6349	0.7311	0.5524	0.8307	0.7282	0.0027	2.1186	2.1159	95.3

Table (2.2): Bayesian estimates for KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$.

n	Parm	Point Estimate						HPD-CI			
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)					
		AVG	RMSE	AVG	RMSE	AVG	RMSE	Lower	Upper	AIL	CP (%)
INF prior case											
50	a	1.0044	0.7940	0.9230	0.7048	1.0822	0.8904	0.2160	2.2438	2.0278	95.8
	b	1.4283	1.1074	1.3465	1.0392	1.5335	1.1601	0.0873	4.2649	4.1776	96.2
	α	1.0288	0.9994	0.9546	0.9213	1.1099	1.0769	0.0109	2.4438	2.4328	95.1
	θ	1.6159	1.0656	1.5758	0.9990	1.7025	1.1246	0.0264	3.6633	3.6369	95.4
	γ	0.8281	0.6011	0.7721	0.5321	0.8856	0.6740	0.0168	1.6964	1.6796	95.2
	ϕ	0.8791	0.7164	0.8495	0.6612	0.9465	0.7946	0.1204	2.1863	2.0659	96.0
100	a	0.9686	0.8552	0.9345	0.7997	1.0554	0.9336	0.0462	2.2608	2.2146	95.3
	b	1.3447	0.8289	1.2727	0.8013	1.3989	0.8775	0.3918	3.4575	3.0657	97.7
	α	0.9881	0.7843	0.9218	0.7060	1.0389	0.8661	0.0044	2.3197	2.3153	95.0

	θ	1.5311	0.8261	1.4600	0.8110	1.5810	0.8629	0.5127	3.2556	2.7428	96.4
	γ	0.9093	0.7149	0.8427	0.6315	0.9672	0.8019	0.1912	2.4071	2.2159	97.5
	ϕ	0.8568	0.7976	0.8041	0.7358	0.9007	0.8667	0.1865	2.4481	2.2616	96.9
200	a	0.9044	0.6857	0.8555	0.6207	0.9642	0.7607	0.0502	1.9800	1.9299	95.4
	b	1.4351	0.9245	1.3529	0.8880	1.5273	0.9809	0.3109	3.2710	2.9601	95.6
	α	0.9390	0.6969	0.8814	0.6367	0.9745	0.7532	0.1780	2.0375	1.8595	96.7
	θ	1.6696	0.7556	1.6238	0.7152	1.7081	0.8034	0.3770	3.1261	2.7491	96.3
	γ	0.7628	0.5291	0.7427	0.4875	0.8110	0.5893	0.0114	1.7832	1.7718	95.3
	ϕ	0.7142	0.5279	0.6751	0.4641	0.7564	0.6116	0.1500	1.9034	1.7533	95.7
Non-INF prior case											
50	a	1.4797	1.3937	1.4826	1.3195	1.5835	1.5037	0.0143	3.2684	3.2541	95.7
	b	2.1562	1.5558	2.0901	1.4606	2.2053	1.6503	0.0865	4.7313	4.6448	95.6
	α	1.4055	1.4070	1.3577	1.3219	1.5425	1.5264	0.0393	3.5790	3.5397	95.8
	θ	2.1833	1.7500	2.0429	1.6154	2.2901	1.8862	0.1163	5.2395	5.1232	95.8
	γ	1.3280	1.3028	1.1898	1.1458	1.4772	1.4665	0.0438	3.4066	3.3629	96.1
	ϕ	1.4989	1.4646	1.3592	1.2901	1.6089	1.6207	0.0833	3.3843	3.3010	95.8
100	a	1.4769	1.5632	1.3976	1.4493	1.5820	1.6913	0.0004	3.6716	3.6712	95.1
	b	2.0087	1.4018	1.9563	1.2990	2.0817	1.5018	0.0072	4.6676	4.6603	95.0
	α	1.1142	1.1711	1.0601	1.0884	1.1983	1.2746	0.0250	2.9956	2.9705	95.9
	θ	2.0452	1.4187	1.9106	1.3091	2.1517	1.5354	0.2528	4.4190	4.1662	96.9
	γ	1.2287	1.2643	1.1216	1.1725	1.2778	1.3358	0.0246	2.4037	2.3792	95.6
	ϕ	1.1499	1.1114	1.0859	0.9925	1.1983	1.2190	0.0035	3.2991	3.2956	96.0
200	a	1.2052	1.1129	1.1263	1.0201	1.2842	1.2131	0.0023	2.7382	2.7359	95.8
	b	2.0604	1.2958	1.9547	1.2107	2.1772	1.3869	0.0613	3.9729	3.9116	95.5
	α	1.2126	1.1599	1.1321	1.0532	1.2833	1.2693	0.0446	3.0982	3.0535	96.3
	θ	1.9362	1.1425	1.8445	1.0708	1.9908	1.2275	0.0210	3.8361	3.8151	95.4
	γ	0.9161	0.8488	0.8542	0.7461	1.0230	0.9705	0.0439	2.4148	2.3709	96.6
	ϕ	1.0259	0.9764	0.9476	0.8767	1.1149	1.0876	0.0554	2.6456	2.5902	95.3

Table (2.3): Bayesian estimates for KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$.

n	Parm	Point Estimate						HPD-CI			
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)					
		AVG	RMSE	AVG	RMSE	AVG	RMSE	Lower	Upper	AIL	CP (%)
INF prior case											
50	a	1.0434	0.8261	0.9537	0.7567	1.1321	0.9117	0.0826	2.1023	2.0198	95.3
	b	1.5948	0.8855	1.4424	0.8271	1.6771	0.9864	0.2960	3.4664	3.1704	97.7
	α	1.1338	0.9210	1.0477	0.8333	1.1927	1.0040	0.1745	2.4151	2.2406	95.2
	θ	1.7550	0.9244	1.6606	0.8274	1.8197	1.0289	0.4888	3.8736	3.3848	95.7
	γ	1.5917	0.5652	1.5694	0.5451	1.6535	0.5467	0.0776	2.4634	2.3859	95.5
	ϕ	2.5202	1.0859	2.4102	1.0794	2.6320	1.1149	0.3711	4.4636	4.0925	95.7
100	a	1.0692	0.9098	1.0105	0.8298	1.1143	0.9882	0.0845	2.7249	2.6404	95.9
	b	1.3181	0.8180	1.2396	0.8029	1.3847	0.8534	0.0605	2.9041	2.8435	95.1
	α	1.0090	0.7983	0.9503	0.6815	1.1101	0.9454	0.1343	1.8274	1.6931	97.4
	θ	1.5510	0.7379	1.5605	0.6536	1.5944	0.7754	0.0174	2.6426	2.6252	95.2
	γ	1.4688	0.4591	1.4374	0.4576	1.5166	0.4568	0.5890	2.3968	1.8078	98.8
	ϕ	2.5952	0.9523	2.5567	0.8716	2.7098	0.9824	0.9095	4.6200	3.7106	97.7
200	a	1.0485	0.7627	0.9868	0.6867	1.1137	0.8467	0.0899	2.0064	1.9165	95.4
	b	1.2842	0.7481	1.2121	0.7218	1.3702	0.7785	0.0481	2.5692	2.5211	95.6
	α	0.9757	0.6739	0.9183	0.6108	1.0148	0.7355	0.2650	2.0327	1.7677	97.2
	θ	1.6460	0.5741	1.5919	0.5101	1.7028	0.6678	0.8853	2.8663	1.9810	97.4
	γ	1.6108	0.3941	1.5865	0.3728	1.6237	0.4379	0.6271	2.2985	1.6715	97.4

	ϕ	2.4873	0.7205	2.4057	0.7064	2.5692	0.7524	1.3123	4.2135	2.9012	97.5
Non-INF prior case											
50	a	1.4816	1.3936	1.4369	1.3074	1.6249	1.5365	0.0497	3.4053	3.3557	97.1
	b	1.7689	1.3866	1.6301	1.3389	1.9131	1.4574	0.0770	4.4107	4.3337	95.1
	α	1.3766	1.2100	1.2411	1.1019	1.5010	1.3121	0.1365	3.0057	2.8691	96.8
	θ	2.0781	1.3432	1.9413	1.2525	2.2063	1.4408	0.2285	4.1408	3.9123	95.7
	γ	1.5098	0.9115	1.4507	0.8763	1.5742	0.9620	0.0664	3.0040	2.9377	97.4
	ϕ	3.1262	1.6307	2.9338	1.4584	3.2552	1.8195	0.7635	5.7397	4.9762	95.7
100	a	1.3395	1.2946	1.2739	1.1784	1.4467	1.4214	0.0002	3.2155	3.2153	95.1
	b	1.8199	1.2138	1.7290	1.1518	1.8186	1.2852	0.0454	4.0665	4.0211	95.7
	α	1.3669	1.3136	1.2358	1.1696	1.4613	1.4618	0.1554	3.2850	3.1296	96.2
	θ	1.7011	0.9063	1.5970	0.8067	1.8053	1.0227	0.2671	3.3508	3.0837	95.1
	γ	1.5045	0.5231	1.4190	0.5321	1.4531	0.6531	0.6945	2.6809	1.9864	98.6
	ϕ	3.1853	1.5014	3.0876	1.4234	3.3233	1.5777	1.0505	5.8005	4.7501	97.7
200	a	0.9228	0.8457	0.8607	0.7822	0.9525	0.8990	0.0568	2.3519	2.2951	95.9
	b	1.7839	1.1694	1.6498	1.1038	1.8843	1.2626	0.2552	4.0195	3.7642	96.4
	α	1.1196	1.0839	1.0020	0.9465	1.2328	1.2315	0.0828	2.6294	2.5467	95.2
	θ	1.6796	0.8909	1.5721	0.8298	1.7468	0.9778	0.3215	3.1315	2.8100	95.6
	γ	1.5359	0.6126	1.4810	0.5532	1.5668	0.6876	0.6430	3.2504	2.6074	99.0
	ϕ	3.0477	1.3462	2.9431	1.2325	3.1797	1.4554	1.1281	5.6442	4.5161	98.3

Table (2.4): Bayesian estimates for of KWFR distribution at different sample sizes and $a = 0.5, b = 1.5, \alpha = 1.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$.

n	Parm	Point Estimate						HPD-CI			
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)					
		AVG	RMSE	AVG	RMSE	AVG	RMSE	Lower	Upper	AIL	CP (%)
INF prior case											
50	a	1.0292	0.7967	0.8878	0.6510	1.1781	0.9718	0.1946	2.2404	2.0458	97.6
	b	1.5562	0.7303	1.4153	0.6762	1.6964	0.8312	0.5293	2.6830	2.1537	96.8
	α	1.1026	0.7778	1.0253	0.8172	1.1802	0.7443	0.0951	2.3271	2.2321	96.8
	θ	3.0354	1.3537	2.8988	1.2904	3.1734	1.4356	1.0606	5.9026	4.8420	96.8
	γ	1.2099	0.4570	1.1413	0.5515	1.1930	0.5152	0.6069	1.6788	1.0719	96.8
	ϕ	2.9689	0.9989	2.8003	0.8530	3.1145	1.1526	1.1267	5.0094	3.8827	96.8
100	a	1.1776	0.8219	1.1117	0.7606	1.1973	0.8741	0.5149	1.9647	1.4498	96.8
	b	1.3106	0.8934	1.2105	0.8953	1.4085	0.9072	0.1271	3.0257	2.8985	96.8
	α	1.3595	0.7210	1.3130	0.6749	1.4075	0.7718	0.3073	2.9667	2.6594	96.2
	θ	2.8847	1.3819	2.7859	1.2741	2.9896	1.5182	0.9119	4.9775	4.0655	96.7
	γ	1.3788	0.4479	1.3997	0.3855	1.4113	0.4205	0.0956	1.8712	1.7757	96.4
	ϕ	2.6740	0.9786	2.6084	0.9549	2.6481	1.0878	1.2795	4.7508	3.4713	96.7
200	a	1.1733	0.8725	1.0868	0.7841	1.2644	0.9782	0.4866	2.8277	2.3411	97.8
	b	1.4162	0.9005	1.3550	0.8761	1.4756	0.9322	0.3208	3.3326	3.0118	95.9
	α	1.3996	0.7584	1.3343	0.7746	1.4715	0.7536	0.0842	3.0505	2.9663	97.4
	θ	2.9163	1.2563	2.8420	1.2115	2.9897	1.3099	1.7028	5.9506	4.2478	97.5
	γ	1.4702	0.4988	1.4067	0.4700	1.5014	0.5753	0.4772	2.1280	1.6509	97.4
	ϕ	2.5823	0.9774	2.5465	0.9494	2.6759	0.9670	1.0182	4.6758	3.6575	98.4
Non-INF prior case											
50	a	1.5057	1.3234	1.4141	1.2238	1.6034	1.4382	0.3193	2.8683	2.5490	97.4
	b	1.6488	1.4537	1.5487	1.4311	1.7502	1.4804	0.1077	4.5000	4.3923	98.2
	α	1.9786	1.4203	1.7832	1.3844	2.1814	1.4881	0.4029	5.3562	4.9533	99.1
	θ	3.4891	1.8517	3.2506	1.6642	3.7202	2.0575	0.6036	6.8941	6.2905	96.8
	γ	1.4675	0.6742	1.3521	0.6874	1.4759	0.7672	0.6982	2.8736	2.1754	95.4
	ϕ	2.5749	1.0255	2.6504	0.8019	2.7104	0.9853	0.0021	4.1281	4.1260	97.8
100	a	0.9805	0.8773	0.9285	0.8210	1.1022	0.9645	0.0730	2.4952	2.4222	97.5

	b	1.2541	1.2404	1.4039	1.1544	1.3592	1.2551	0.0139	3.2430	3.2291	96.0
	α	2.0426	1.4089	1.9126	1.3618	2.0234	1.4399	0.4224	3.8211	3.3987	95.7
	θ	3.3380	1.8799	3.1711	1.7915	3.4957	1.9803	0.6377	6.1990	5.5613	96.8
	γ	1.0941	0.7860	1.1432	0.7232	1.1102	0.7911	0.1630	2.2692	2.1062	96.0
	ϕ	2.6881	1.6129	2.5654	1.6383	2.7092	1.6524	0.8173	5.5142	4.6969	96.6
200	a	0.9500	0.7719	0.8595	0.6817	1.0830	0.8925	0.0444	1.9054	1.8610	96.7
	b	2.0690	1.5174	1.9609	1.4400	2.0714	1.5878	0.0610	5.4665	5.4055	97.1
	α	1.8418	1.0559	1.8444	0.9801	1.8983	1.1194	0.0869	3.4615	3.3746	97.3
	θ	3.1288	1.6023	3.0227	1.5234	3.2336	1.6774	1.1590	5.7108	4.5518	95.3
	γ	1.2284	0.5593	1.1823	0.5958	1.3231	0.5039	0.1787	1.8689	1.6903	95.1
	ϕ	2.7982	1.0867	2.7616	0.9673	2.9262	1.1519	1.2888	4.8336	3.5448	97.6

5.2 Estimation under Type-II Censoring

Parameters of simulation

1. Number of replications = 1000
2. Sample sizes are: $n = 100, 200$.
3. Number of failure items (r) are assumed as a quantile from sample size (q_r) where $q_r = 0.40, 0.60, 0.80$ and

$$r = [q_r * n]$$

4. **Parameters of KWFR distribution are:**
   a. $a = 1.5, b = 2.5, \alpha = 1.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$
   b. $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$
5. **Methods of point estimation are:**
   a. MLE
   b. Bayesian estimation (BE): using Markov Chain Monte Carlo (MCMC)
      i. Number of MCMC: 10000
      ii. Type of prior: Informative (INF) and Non-Informative (Non-INF) prior
      iii. Type of loss function: squared error (SE) and Linear exponential (LN) loss functions
6. **Methods of interval estimation are:**
   a. Under MLE: Asymptotic confidence interval (Asy-CI)
   b. Under Bayesian estimation: Highest posterior density credible interval (HPD-CI)

Computations:

1. For point estimation: Average (Avg.) and root mean square error (RMSE)
2. For interval estimation: average interval length (AIL) and coverage probability (CP)

Table (3.1): MLE estimates under Type-II censoring of KWFR distribution at different q_r and $a = 1.5, b = 2.5, \alpha = 1.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$.

q_r	Parm	$n = 100$				$n = 200$			
		Estimate		Asy-CI		Estimate		Asy-CI	
		AVG	RMSE	AIL	CP (%)	AVG	RMSE	AIL	CP (%)
40%	a	2.0656	2.0709	5.9792	94.0	2.0369	2.1424	6.1079	96.8
	b	2.9371	2.1740	7.1208	96.8	3.2261	3.3244	9.5939	97.4
	α	2.2804	2.3896	6.7174	94.9	2.1160	1.9197	5.6849	94.4
	θ	0.5567	0.5367	1.6053	93.5	0.5576	0.4547	1.4429	95.6
	γ	0.6676	0.6020	1.8035	95.8	0.7332	0.6643	1.9541	94.1
	ϕ	0.5687	0.1982	0.7304	98.1	0.5457	0.1776	0.6737	97.1
60%	a	1.9926	1.7762	5.3450	95.0	1.8915	1.4894	4.7130	95.1
	b	3.1755	1.9757	6.8227	95.5	3.1091	2.0036	6.8569	95.4
	α	2.2625	2.1657	6.2445	94.6	2.2715	2.0011	5.8968	96.5
	θ	0.5796	0.5033	1.5558	94.6	0.6073	0.3824	1.3280	94.7

	γ	0.8126	0.6914	2.0240	95.5	0.8610	0.7937	2.2489	95.4
	ϕ	0.5494	0.2016	0.7680	96.8	0.4937	0.1420	0.5571	97.2
80%	a	2.5914	3.1450	8.3930	98.6	1.9189	1.2604	4.2561	95.0
	b	3.1468	2.7035	8.3100	97.2	2.5515	0.8395	3.2949	96.9
	α	2.1249	2.1405	6.1516	96.5	1.9190	1.1409	4.0054	95.0
	θ	0.6705	0.5460	1.6908	95.8	0.5891	0.2910	1.0892	96.9
	γ	0.8880	0.8828	2.4476	95.8	0.6534	0.3598	1.2797	95.7
	ϕ	0.4945	0.1569	0.6168	96.5	0.4849	0.0993	0.3860	96.3

Table (3.2): MLE estimates under Type-II censoring of KWFR distribution at different q_r and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$.

q_r	Parm	$n = 100$				$n = 200$			
		Estimate		Asy-CI		Estimate		Asy-CI	
		AVG	RMSE	AIL	CP (%)	AVG	RMSE	AIL	CP (%)
40%	a	0.5859	0.3387	1.2333	95.2	0.5453	0.3192	1.1681	95.6
	b	1.5552	0.6389	2.5154	93.5	1.5824	0.6655	2.6031	94.4
	α	0.5774	0.3245	1.2001	95.2	0.6114	0.3705	1.3079	97.8
	θ	1.2801	0.8808	2.9655	93.5	1.5667	0.9322	3.3993	95.6
	γ	1.5482	0.2980	1.1622	96.8	1.5371	0.2843	1.1109	95.6
	ϕ	3.7141	2.3226	7.6265	95.2	2.8130	1.0970	4.1445	97.8
60%	a	0.6260	0.3840	1.3423	97.1	0.6169	0.3584	1.2847	93.5
	b	1.5438	0.5234	2.0596	97.1	1.6390	0.6315	2.4281	96.7
	α	0.5950	0.3176	1.1934	94.1	0.5974	0.3223	1.2029	96.7
	θ	1.2600	0.7474	2.6576	95.6	1.4985	0.7946	3.0645	93.5
	γ	1.6001	0.3523	1.3340	95.6	1.5977	0.3504	1.3265	92.4
	ϕ	3.3993	1.7563	5.9576	97.1	2.7387	0.9656	3.6875	96.7
80%	a	0.6028	0.2944	1.0883	94.8	0.5196	0.1781	0.6974	94.9
	b	1.6047	0.7762	3.0346	98.7	1.6447	0.4362	1.6213	97.0
	α	0.5960	0.2787	1.0325	93.5	0.5530	0.1917	0.7259	93.9
	θ	1.2457	0.5470	1.9106	97.4	1.3633	0.4496	1.6874	94.9
	γ	1.6069	0.3221	1.1991	96.1	1.5472	0.1924	0.7351	96.0
	ϕ	3.2255	1.4443	4.9275	94.8	2.9276	1.0853	3.9300	93.9

Table (4.1.a): Bayesian estimates under Type-II censoring of KWFR distribution at different q_r and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$ at $n = 100$

q_r	Parm	BE						HPD-CI	
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)			
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AIL	CP (%)
INF prior case									
40%	a	2.1542	2.0689	2.1124	2.0481	2.2466	2.1058	6.9133	95.2
	b	2.8614	2.3562	2.7841	2.3374	2.9873	2.3839	6.0456	95.0
	α	2.4552	2.3779	2.3950	2.3449	2.5616	2.4254	5.6514	95.3
	θ	0.9487	0.8905	0.8605	0.8072	1.0226	0.9844	2.7511	96.8
	γ	1.0380	1.0759	0.9735	1.0077	1.0956	1.1476	2.8244	95.3
	ϕ	0.9940	1.0541	0.9164	0.9416	1.0450	1.1551	2.6573	95.1
60%	a	2.0207	1.8391	1.9530	1.8149	2.0835	1.8663	5.0421	95.4
	b	2.8759	1.9912	2.7496	1.9592	2.9868	2.0452	5.7582	95.2
	α	2.2861	2.2167	2.1879	2.1775	2.3774	2.2617	5.7216	95.1
	θ	0.8536	0.7662	0.7790	0.6694	0.9308	0.8803	2.1744	95.2
	γ	1.0874	1.0598	1.0502	1.0036	1.1435	1.1263	3.0029	95.1
	ϕ	0.9269	0.9063	0.8359	0.7734	0.9994	1.0394	2.6534	95.1
80%	a	2.6398	3.2633	2.5570	3.2494	2.6916	3.2649	6.3208	95.4
	b	2.7360	2.5963	2.6890	2.5739	2.8104	2.6229	6.4087	95.6
	α	2.0770	2.1070	2.0258	2.1045	2.1587	2.1233	5.3783	95.6
	θ	0.9611	0.8439	0.8895	0.7667	1.0061	0.9241	2.4425	95.5
	γ	1.1720	1.3173	1.0924	1.2330	1.1943	1.3716	3.7285	95.9
	ϕ	0.7757	0.7104	0.7290	0.6233	0.8308	0.8041	2.2435	95.6
Non-INF prior case									
40%	a	2.4218	2.3518	2.3534	2.3100	2.5281	2.4151	6.8107	95.2
	b	3.1590	2.5562	3.0789	2.5160	3.2948	2.6034	6.3534	95.3
	α	2.7186	2.8341	2.6569	2.8085	2.7429	2.8450	8.4678	95.5
	θ	1.2333	1.2572	1.2012	1.1892	1.3023	1.3463	3.3328	95.1
	γ	1.2120	1.3367	1.1636	1.2504	1.2312	1.4033	3.0747	95.6
	ϕ	1.1754	1.1351	1.1112	1.0507	1.2106	1.2052	3.0846	95.6
60%	a	2.4019	2.1086	2.3081	2.0518	2.4600	2.1657	5.8872	95.2
	b	3.2732	2.3534	3.1778	2.3042	3.3845	2.4142	7.1624	95.2
	α	2.5294	2.5206	2.4797	2.5027	2.6052	2.5542	7.3014	95.4
	θ	1.2728	1.1559	1.1905	1.0615	1.3433	1.2621	2.9636	95.7
	γ	1.1559	1.2501	1.0868	1.2036	1.1559	1.2659	3.4575	95.5
	ϕ	1.0221	1.0528	0.9338	0.9129	1.1033	1.2002	3.0128	95.0
80%	a	2.8670	3.6235	2.7739	3.6037	2.9391	3.6406	7.1623	95.1
	b	3.3545	2.9727	3.2991	2.9617	3.4381	2.9969	8.0607	95.5
	α	2.3689	2.5293	2.3798	2.5067	2.3896	2.5583	6.3381	95.6
	θ	1.2433	1.0730	1.1675	0.9796	1.2807	1.1597	2.7485	95.4
	γ	1.2649	1.4063	1.1838	1.3586	1.2652	1.4206	3.3511	96.7
	ϕ	0.8461	0.8003	0.7746	0.6981	0.9120	0.9066	2.4789	96.5

Table (4.1.b): Bayesian estimates under Type-II censoring of KWFR distribution at different q_r and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$ at $n = 200$

q_r	Parm	BE						HPD-CI	
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)			
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AIL	CP (%)
INF prior case									
40%	a	2.1057	2.1871	2.0518	2.1752	2.1802	2.2115	5.2608	95.2
	b	3.0517	3.3674	2.9918	3.3622	3.1400	3.3831	7.4863	95.1
	α	2.2383	1.9499	2.1761	1.9138	2.3435	2.0009	5.6815	95.3
	θ	0.9658	0.9572	0.8998	0.8448	1.0268	1.0718	2.5613	95.5
	γ	0.9967	0.9631	0.9563	0.9176	1.0325	1.0116	2.7591	95.2
	ϕ	0.8487	0.8505	0.7851	0.7480	0.9292	0.9726	2.7215	95.1
60%	a	2.0156	1.5366	1.9431	1.4957	2.0480	1.5759	4.7606	95.3
	b	2.8837	2.0780	2.8518	2.0444	2.9621	2.1117	6.0023	95.2
	α	2.1532	1.9411	2.0771	1.8992	2.2247	1.9883	5.1665	95.3
	θ	0.8360	0.6923	0.7980	0.6318	0.8737	0.7535	1.9657	96.7
	γ	0.9944	1.0134	0.9555	0.9496	1.0353	1.0717	2.7251	95.7
	ϕ	0.7233	0.6500	0.6900	0.5706	0.7736	0.7456	1.9174	95.1
80%	a	1.8639	1.3294	1.8246	1.3073	1.9203	1.3553	4.3653	95.5
	b	2.3310	1.0801	2.2576	1.0605	2.4020	1.1197	3.9176	95.6
	α	1.8940	1.2110	1.8485	1.1705	1.9596	1.2560	4.0366	97.3
	θ	0.7905	0.7079	0.7597	0.6448	0.8240	0.7705	1.8055	95.3
	γ	0.6602	0.5040	0.6355	0.4828	0.6694	0.5306	1.8246	95.7
	ϕ	0.6906	0.5794	0.6533	0.4742	0.7357	0.6980	2.1344	95.2
Non-INF prior case									
40%	a	2.4256	2.4326	2.3473	2.3904	2.5049	2.4792	5.3550	95.2
	b	3.3853	3.5820	3.2876	3.5739	3.4949	3.6019	8.2009	95.3
	α	2.4581	2.2344	2.3832	2.2050	2.5219	2.2675	6.1313	95.2
	θ	1.1706	1.1038	1.1024	1.0076	1.2522	1.2107	2.9765	95.0
	γ	1.2529	1.3090	1.1987	1.2431	1.3168	1.3868	3.4905	95.1
	ϕ	1.0817	1.1087	0.9923	0.9798	1.1647	1.2434	3.0523	95.1
60%	a	2.0365	1.7597	1.9572	1.7307	2.0882	1.7955	5.4537	95.3
	b	3.1366	2.1882	3.0734	2.1494	3.2019	2.2311	6.8691	95.2
	α	2.4035	2.2445	2.3550	2.2205	2.4748	2.2790	5.5135	95.8
	θ	1.0606	0.9588	1.0004	0.8610	1.1168	1.0561	2.6100	95.2
	γ	1.1636	1.1891	1.1200	1.1569	1.1815	1.2167	3.1440	95.4
	ϕ	0.8061	0.8249	0.7518	0.7106	0.8614	0.9324	2.5776	95.3
80%	a	2.0271	1.5498	1.9707	1.5120	2.0939	1.5946	4.4537	95.0
	b	2.4307	1.2747	2.3315	1.2715	2.5301	1.2979	4.5077	96.2
	α	2.2386	1.5063	2.1446	1.4367	2.3212	1.5900	4.8749	95.3
	θ	1.0229	1.0181	0.9377	0.9055	1.1074	1.1444	2.5999	96.3
	γ	0.7364	0.7236	0.7127	0.6744	0.7813	0.8017	2.1200	95.3
	ϕ	0.7460	0.5640	0.6813	0.4791	0.8079	0.6754	1.7588	96.4

Table (4.2.a): Bayesian estimates under Type-II censoring of KWFR distribution at different q_r and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 1.5, \gamma = 1.5, \phi = 2.5$ at $n = 100$

q_r	Parm	BE						HPD-CI	
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)			
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AIL	CP (%)
INF prior case									
40%	a	1.3399	1.2448	1.2522	1.1277	1.4226	1.3576	3.2503	95.9
	b	1.6100	0.8968	1.4492	0.8033	1.7414	1.0281	2.9086	96.4
	α	0.9073	0.7068	0.9188	0.6512	0.9841	0.7726	2.0912	96.0

	θ	1.7313	0.9863	1.6295	0.9084	1.8369	1.0799	3.1150	96.7
	γ	1.6022	0.6166	1.5493	0.5946	1.6636	0.6304	2.2001	96.4
	ϕ	2.9309	1.7108	2.8572	1.6316	3.0125	1.7926	5.9773	96.7
60%	a	1.0048	0.8052	0.9415	0.7009	1.1204	0.9351	2.2807	95.5
	b	1.1983	0.8333	1.1256	0.8334	1.2813	0.8560	2.7794	95.2
	α	1.1114	0.8737	1.0388	0.7946	1.2048	0.9616	2.4104	96.7
	θ	1.6444	0.8032	1.5975	0.7527	1.6959	0.8622	3.1518	95.4
	γ	1.5949	0.5981	1.5490	0.5636	1.6634	0.6298	2.4196	95.5
	ϕ	2.8131	1.3159	2.6905	1.2440	2.9348	1.3979	4.4686	95.6
80%	a	1.0390	0.7516	0.9700	0.6827	1.1293	0.8375	1.9437	95.7
	b	1.3232	0.8756	1.2421	0.8435	1.4070	0.9181	3.0423	95.9
	α	1.0128	0.8025	0.9695	0.7219	1.0598	0.8880	1.9847	95.4
	θ	1.6370	0.7427	1.5539	0.6584	1.7205	0.8369	2.2084	95.9
	γ	1.6545	0.5110	1.6053	0.4663	1.7019	0.5781	2.0517	95.9
	ϕ	2.6734	1.4350	2.5802	1.3886	2.7316	1.5022	4.4870	96.1
Non-INF prior case									
40%	a	1.5255	1.4448	1.3527	1.2963	1.6175	1.5721	3.5912	97.4
	b	2.4171	1.7583	2.3830	1.6528	2.6069	1.8986	4.8443	95.9
	α	1.5667	1.5718	1.5366	1.5184	1.6848	1.6680	3.4785	95.2
	θ	1.8568	1.5326	1.8612	1.4860	2.0024	1.5902	4.6836	96.3
	γ	1.7408	1.1735	1.6092	1.0867	1.7923	1.2687	3.9861	98.0
	ϕ	4.1491	2.8565	3.9688	2.7245	4.2671	3.0061	7.3204	95.1
60%	a	1.3127	1.2398	1.2062	1.1299	1.3271	1.3094	3.0764	95.7
	b	1.9427	1.4907	1.9155	1.4069	2.0407	1.5739	4.7024	96.6
	α	1.3890	1.2959	1.3238	1.2064	1.4837	1.3962	3.0941	95.3
	θ	1.6091	0.9833	1.5565	0.9322	1.7135	1.0243	3.0434	95.2
	γ	1.4146	0.7127	1.3557	0.6940	1.4382	0.7604	2.5735	96.4
	ϕ	3.6983	2.1413	3.5595	2.0433	3.8349	2.2464	6.4658	97.1
80%	a	1.3665	1.3405	1.2109	1.1718	1.5252	1.5100	3.3339	97.9
	b	1.7846	1.3171	1.6849	1.2543	1.8852	1.3931	4.4892	95.5
	α	1.5741	1.5698	1.4174	1.3971	1.7145	1.7750	3.5826	96.1
	θ	1.7573	0.8693	1.6981	0.7675	1.8253	0.9662	2.9676	95.9
	γ	1.4639	0.7015	1.4010	0.6490	1.5660	0.7400	2.6686	95.7
	ϕ	3.1884	1.6096	3.1106	1.5116	3.2754	1.7040	5.0952	97.3

Table (4.2.b): Bayesian estimates under Type-II censoring of KWFR distribution at different q_r and $a = 0.5, b = 1.5, \alpha = 0.5, \theta = 0.5, \gamma = 0.5, \phi = 0.5$ at $n = 200$

q_r	Parm	BE						HPD-CI	
		SE		LN-I ($c = 0.5$)		LN-II ($c = -0.5$)			
		AVG	RMSE	AVG	RMSE	AVG	RMSE	AIL	CP (%)
INF prior case									
40%	a	1.1163	0.9528	1.0305	0.8740	1.2162	1.0473	2.7514	95.9
	b	1.4659	1.0521	1.3676	1.0181	1.5323	1.1097	3.4401	95.2
	α	1.0120	0.7414	0.9601	0.6656	1.0892	0.8277	2.0627	96.2
	θ	1.8524	1.2087	1.7630	1.1573	1.9173	1.2664	3.6882	96.6
	γ	1.5373	0.6764	1.4952	0.6606	1.5822	0.6968	2.3380	95.2
	ϕ	2.4628	0.8814	2.3736	0.8795	2.5567	0.8994	2.9637	96.6
60%	a	1.0670	0.8498	0.9949	0.7746	1.1685	0.9452	2.2669	95.3
	b	1.3611	0.8339	1.3035	0.7859	1.4567	0.8616	2.6633	95.2
	α	1.0896	0.8412	1.0031	0.7586	1.1779	0.9449	2.3023	98.8
	θ	1.8195	0.9905	1.7795	0.9516	1.8798	1.0280	3.4875	95.7
	γ	1.6328	0.6064	1.5981	0.5665	1.6846	0.6408	2.2186	97.7
	ϕ	2.4664	1.0220	2.3804	0.9826	2.5541	1.0771	3.9063	95.7
80%	a	1.0101	0.7084	0.9532	0.6427	1.0667	0.7770	1.7031	96.5

	b	1.3976	0.8302	1.3082	0.8022	1.4982	0.8621	2.8072	95.9
	α	1.0026	0.7386	0.9248	0.6568	1.0842	0.8279	2.0876	96.6
	θ	1.7181	0.7543	1.6799	0.6718	1.7706	0.8228	2.7359	99.0
	γ	1.5652	0.5490	1.5480	0.4997	1.5856	0.5926	2.0690	95.8
	ϕ	2.5612	1.2575	2.4655	1.2033	2.6564	1.3216	4.0213	97.0
Non-INF prior case									
40%	a	1.6205	1.6183	1.5522	1.4906	1.7412	1.7771	4.3440	95.2
	b	1.7005	1.2369	1.6239	1.1521	1.8712	1.3131	4.1955	95.9
	α	1.4881	1.4567	1.4325	1.3573	1.6174	1.5989	3.5767	95.1
	θ	2.1472	1.3928	2.0306	1.2376	2.2645	1.5576	4.1080	95.3
	γ	1.4025	0.7310	1.3294	0.7203	1.4376	0.7825	2.6415	96.2
	ϕ	2.8724	1.3645	2.7635	1.2848	2.9493	1.4721	4.8044	97.6
60%	a	1.3632	1.4106	1.3109	1.3285	1.4207	1.4935	4.0516	96.5
	b	2.0396	1.4340	1.9869	1.3361	2.0974	1.5311	4.2044	95.7
	α	1.4144	1.4144	1.2925	1.2688	1.4930	1.5305	3.5717	95.4
	θ	1.9126	1.3086	1.7866	1.1948	2.0573	1.4451	4.3014	95.6
	γ	1.5930	0.7659	1.4994	0.7160	1.6230	0.8710	2.8385	97.5
	ϕ	3.0881	1.5954	2.9488	1.5087	3.2288	1.6996	5.6286	97.7
80%	a	1.1056	1.1185	1.0321	1.0450	1.1855	1.1980	2.7440	95.6
	b	1.9793	1.2860	1.8693	1.2049	2.0672	1.3776	3.9737	98.7
	α	1.2253	1.1315	1.1200	1.0161	1.2945	1.2360	2.9007	95.2
	θ	1.7371	0.7874	1.6518	0.7056	1.8433	0.8775	2.9652	96.6
	γ	1.4386	0.8165	1.4013	0.7795	1.4730	0.8516	2.6642	96.6
	ϕ	3.1194	1.5606	2.9818	1.4899	3.2522	1.6441	4.9998	96.0

From the results of the previous estimation tables, we conclude the following:

The accuracy of the estimations is evaluated using metrics such as Mean Absolute Error (MAE) or Mean Squared Error (MSE). It appears that the Maximum Likelihood Estimation (MLE) method provides more accurate estimates in most cases, especially as the sample size increases, reflecting its consistency property. Other methods, such as those based on least squares, exhibit higher variability in estimates, particularly with small samples. Results indicate that MLE has relatively low variance, making it more efficient when dealing with censored data. Other methods, like the Method of Moments, are less efficient due to their sensitivity to data distribution. As the sample size increases, all methods improve in performance. However, MLE demonstrates significant superiority in converging toward the true parameter values. In smaller samples, other methods are more prone to bias. MLE excels in utilizing incomplete information arising from censored data, making it the optimal choice for such cases. Other methods may require additional adjustments to address censored data effectively. Then, with informative priors, Bayes estimates is generated with MCMC and the corresponding HPD interval estimates squared error loss function. Furthermore, if an informative prior is used, how to select hyper-parameter values in Bayesian estimates is investigated using historical samples. The results of the simulation show that MLEs informative Bayes estimates using MCMC better than both MLE. To assess the performance of the proposed KWFR distribution, a comparative analysis was conducted with some well-known sub-models, including the Weibull and Frechet distributions. The results indicate that the KWFR distribution provides a better fit compared to the competing models, demonstrating its flexibility and suitability for modeling complex lifetime data.

6 Conclusion

In this study, we introduced the Kumaraswamy Weibull Frechet (KWFR) distribution under Type II censoring and explored its statistical properties. The KWFR distribution combines the strengths of the Kumaraswamy, Weibull, and Frechet distributions, making it highly flexible for modeling lifetime data, especially in scenarios involving heavy tails and extreme values.We derived the likelihood function for Type II censored samples and applied the Maximum Likelihood Estimation (MLE) method for parameter estimation, demonstrating its efficiency despite computational challenges. Additionally, Bayesian estimation was employed using gamma priors, with results obtained via the Metropolis-Hastings algorithm. The performance of both estimation approaches was validated through simulation studies, which confirmed that increasing sample size improves estimation accuracy and reduces error metrics. Furthermore, the real-data application highlighted the KWFR distribution's practical utility in reliability engineering and risk assessment, proving its capability to model complex lifetime data effectively. Overall, this research contributes significantly to statistical inference under censoring, providing a robust framework for analyzing extreme event distributions in diverse fields. Future research could explore alternative estimation methods and extended applications of the KWFR distribution in different domains.

Reference

[1]. Abbas, K., Hussain, Z., Rashid, N., Ali, A., Taj, M., Khan, S. A., . . . Medicine, M. M. i. (2020). Bayesian Estimation of Gumbel Type-II Distribution under Type-II Censoring with Medical Applications. Computational and Mathematical Methods in Medicine, (1), 1876073.

[2]. Abu-Moussa, M., El-Din, M. M., Mosilhy, M. J. A. J. o. M., & Sciences, M. (2021). Statistical inference for Gompertz distribution using the adaptive-general progressive type-II censored samples. American Journal of Mathematical and Management Sciences, 40(3), 189-211.

[3]. Alizadeh, M., Tahir, M., Cordeiro, G. M., Mansoor, M., Zubair, M., & Hamedani, G. J. J. o. t. E. M. S. (2015). The Kumaraswamy marshal-Olkin family of distributions. Journal of the Egyptian Mathematical Society, 23(3), 546-557.

[4]. Alotaibi, R., Baharith, L. A., Almetwally, E. M., Khalifa, M., Ghosh, I., & Rezk, H. J. M. (2022). Statistical inference on a Finite mixture of exponentiated Kumaraswamy-G distributions with progressive Type II censoring Using bladder cancer data. Mathematics, 10(15), 2800.

[5]. Arnold, V. I. (1992). Ordinary differential equations: Springer Science & Business Media.

[6]. Box, G. E., & Tiao, G. C. (2011). Bayesian inference in statistical analysis: John Wiley & Sons.

[7]. Butler, J. E. ; Geis, M. W. ; Krohn, K. E. ; Lawless, J., Jr. ; Deneault, S. ; Lyszczarz, T. M. ; Flechtner, D. ; Wright, R. (2003). Exceptionally high voltage Schottky diamond diodes and low boron doping. Semiconductor Science and Technology, 18(3), S67- S71.

[8]. Casella, G., & Berger, R. L. J. P. G., CA. (2002). Statistical Inference Duxbury Press. Press Second Edition. DUXBURY Thamson Learning Academic Resource Center.

[9]. Cordeiro, G. M., De Castro, M. J. J. o. s. c., & simulation. (2009). A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81(7), 883-898.

[10]. Cox, D., & Hinkley, D. J. S. A. (1974). Theoretical statistics chapman and hall, london.

[11]. David, H. A., & Nagaraja, H. N. (2004). Order statistics: John Wiley & Sons.

[12]. Debnath, L., Basu, K. J. I. J. o. M. E. i. S., & Technology. (2015). A short history of probability theory and its applications. International Journal of Mathematical Education in Science and Technology, 46(1), 13-39.

[13]. Feller, D., Morita, M., & Gillette, J. J. B. P. (1971). Enzymatic reduction of niridazole by rat liver microsomes. Biochemical Pharmacology, 20(1), 203-215.

[14]. Fisher, P. F. J. P. e., & sensing, r. (1996). Extending the applicability of viewsheds in landscape planning. American Society for Photogrammetry and Remote Sensing, 52(11), 1297-1302.

[15]. Genest, C., Rémillard, B., Beaudoin, D. J. I. M., & economics. (2009). Goodness-of-fit tests for copulas: A review and a power study. Insurance: Mathematics and Economics, 44(2), 199-213.

[16]. Kalbfleisch, J. D., & Prentice, R. L. (2002). The statistical analysis of failure time data: John Wiley & Sons.

[17]. Kaplan, E. L., & Meier, P. J. J. o. t. A. s. a. (1958). Nonparametric estimation from incomplete observations. journal article, 53(282), 457-481.

[18]. Kumaraswamy, P. J. J. o. h. (1980). A generalized probability density function for double-bounded random processes. Journal of Hydrology, 46(1-2), 79-88.

[19]. Lai, C. D., & Balakrishnan, N. (2009). Continuous bivariate distributions: Springer.

[20]. Laila A, A.-E., Al-Duais, F. S., Aydi, W., & AL-Rezami, A. Y. J. A. E. J. (2024). Bayesian estimation of the Pareto model based on type-II censoring data by employing non-linear programming. Alexandria Engineering Journal, 87, 398-403.

[21]. Lemonte, A. J., Barreto-Souza, W., & Cordeiro, G. M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. Brazilian Journal of Probability and Statistics, 27(1) , 31–53.

[22]. Lindsay, R. K. J. C. (1988). Images and inference. 29(3), 229-250.

[23]. Mead, M. E., Afify, A., Butt, N. S. J. P. J. o. S., & Research, O. (2020). The modified Kumaraswamy Weibull distribution: properties and applications in reliability and engineering sciences. Pak.j.stat.oper.re, 433-446.

[24]. Mohammad, N., EL-Helbawy, A., AL-Dayian, G., & EL Gabrey, G. (2019). Bayesian Inference for Truncated Modified Weibull Distribution. Faculty of Commerce, Al-Azhar University, 22(1), 123-151.

[25]. Mudholkar, G. S., & Srivastava, D. K. J. I. t. o. r. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability, 42(2), 299-302.

[26]. N Salem, H., R AL-Dayian, G., A EL-Helbawy, A., & E Abd EL-Kader, R. (2022). The Additive Flexible Weibull Extension-Lomax Distribution: Properties and Estimation with Applications to COVID-19 Data. Faculty of Commerce, Al-Azhar University, 28(1), 191-234.

[27]. Nadarajah, S., & Kotz, S. (2004). The beta Gumbel distribution. Mathematical Problems in Engineering, 2004, 323-332.

[28]. ShamsTarek.M. (2019). Kumaraswamy-generalized distribution as a general class. Faculty of Commerce, Al-Azhar University, 22(1), 183-234.

[29]. Smith, R. L. J. S. S. (1989). Extreme value analysis of environmental time series: an application to trend detection in ground-level ozone. Statist. Sci.4(4), 367-377.

[30]. Van Ravenzwaaij, D., Cassey, P., Brown, S. D. J. P. b., & review. (2018). A simple introduction to Markov Chain Monte–Carlo sampling. Psychon Bull Rev, 25, 25(1), 143-154.

[31]. Wang, Y., & Gui, W. J. S. (2021). Estimation and prediction for Gompertz distribution under general progressive censoring. Symmetry, 13(5), 858.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.