Exponential Ratio Class of Estimators of Finite Population Mean Using Deciles of an Auxiliary Variable

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Abstract - Practitioners of survey sampling have been working to reduce bias and increase efficiency in the estimation of finite population parameters. The present work aims at developing an exponential ratio class of finite population mean estimators with deciles of an auxiliary variable. Using the Taylor's series technique, the mean square error (MSE) of the developed estimators was obtained up to the first order of approximation. The study's suggested estimators outperform competing estimators, according to the findings of a numerical analysis that was done on them in comparison to the existing estimators that were taken into consideration.

Keywords: Population Mean, Deciles, Ratio Estimator, Mean Square Error, Estimators, Sampling.

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1. Introduction

Estimating population parameters, such as the population mean, variance, etc., is one of the main topics covered in sample surveys. The current sampling literature typically uses auxiliary information to increase the estimation procedure's efficiency. Numerous auxiliary information-based estimators, such as ratio, product, difference, exponential ratio, product type, and regression estimators, have been reported in the literature. The ratio estimator was created by Cochran (1940), who made the initial attempt to investigate the issue of population mean estimation when auxiliary variables are present. Following in his footsteps, numerous authors have suggested ratio estimators that make use of auxiliary data. These authors include Cochran (1977), Kadilar and Cingi (2004, 2006), Koyuncu and Kadilar (2009), Murthy (1967), Prasad (1989), Rao (1991), Singh and Chaundhary (1986), Singh (2003), Singh and Tailor (2003), Singh *et al.* (2004), Singh and Tailor (2005), Sisodia and Dwivedi (1981), Adejumobi *et al.* (2022), Upadhyaya and Singh (1999), Yan and Tian (2010), Bahl and Tuteja (1991), and numerous others.

The goal of the current study was to use deciles of an auxiliary variable to propose an exponential ratio class of estimators of finite population mean.

Consider a finite population which consist of N units $\varphi_i\left(1<i\le N\right)$ such that there is a complete information in the population. Now assume that a sample of size n drawn by using simple random sampling without replacement (SRSWOR) from a population of size N. Let y_i and x_i denote observations on the variable y and x respectively for the i^{th} unit $\left(i=1,2,....,N\right)$. According the above sampling scheme, let $\bar{y}=n^{-1}\sum_{i=1}^{n}y_i$ and $\bar{x}=n^{-1}\sum_{i=1}^{n}x_i$ be the sample mean of study variable $\boldsymbol{y}$ and auxiliary variable $\boldsymbol{x}$ and $\bar{Y}=N^{-1}\sum_{i=1}^{N}Y_i$ and $\bar{X}=N^{-1}\sum_{i=1}^{N}X_i$ be the population means.

Let S_y, S_x are population standard deviation, C_y, C_x are the coefficient of variation of study and auxiliary variable. $\beta_1=\frac{N\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^3}{\left(N-1\right)\left(N-2\right)\delta^3}$, $\beta_2=\frac{N\left(N+1\right)\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^4}{\left(N-1\right)\left(N-2\right)\left(N-3\right)\delta^4}-\frac{3\left(N-1\right)^2}{\left(N-2\right)\left(N-3\right)}$ are the coefficient of skewness and kurtosis, $\rho=\frac{s_{xy}}{s_x s_y}$ is the correlation coefficient between the auxiliary variable and study variable, $f=\frac{n}{N}$ is the sampling fraction, and $\gamma=\left(\frac{1}{n}-\frac{1}{N}\right)$ is the correction factor.

1.1 Existing Estimators

The usual unbiased sample mean estimator is given as

$$\bar{y}=n^{-1}\sum_{i=1}^{n}y_i \tag{1}$$

The variance of the estimator is given by

$$\operatorname{var}\left(\bar{y}\right)=\gamma\bar{Y}^2C_y^2 \tag{2}$$

The population mean of the study variable can be estimated using a traditional ratio estimator, as proposed by Cochran (1940), assuming a strong positive correlation between the study variable Y and the auxiliary variable X. The suggested estimator is provided by:

$$t_R=\bar{y}\frac{\bar{X}}{\bar{x}} \tag{3}$$

Given by is this estimator's MSE.

$$MSE\left(t_R\right)=\gamma\bar{Y}^2\left(C_y^2+C_x^2-2\rho_{yx}C_yC_x\right) \tag{4}$$

The exponential ratio and product type estimator for the $\bar{Y}$ were introduced by Bahl and Tuteja (1991) as follows:

$$\hat{\bar{Y}}_{BT1}=\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{5}$$

The mean square error of the estimator is given by

$$MSE\left(\hat{\bar{Y}}_{BT1}\right)=\gamma\bar{Y}^2\left[C_y^2+\frac{C_x^2}{4}-\rho C_yC_x\right] \tag{6}$$

$$\hat{\hat{Y}}_{BT2} = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \quad (7)$$

The mean square error of the estimator is given by:

$$MSE\left(\hat{\hat{Y}}_{BT2}\right) = \gamma\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4} + \rho C_y C_x\right] \quad (8)$$

Yan and Tian (2010) offered a ratio-type estimator for $\bar{Y}$ as:

$$\bar{Y}_{YT} = \frac{\left[\bar{y}+b\left(\bar{X}-\bar{x}\right)\right]}{\left(\bar{x}\beta_1+\beta_2\right)}\left(\bar{X}\beta_1+\beta_2\right) \quad (9)$$

The mean square error of the estimator $\bar{Y}_{YT}$ is given by:

$$MSE\left(\bar{Y}_{YT}\right) = \gamma\left(R_1^2 S_y^2 + S_y^2\left(1-\rho^2\right)\right) \quad (10)$$

Where, $R_1 = \frac{\bar{Y}\beta_1}{\bar{X}\beta_1+\beta_2}$

Subramani and Kumarapandiyan (2013) submitted an estimation strategy $\bar{Y}$ using deciles of an auxiliary variable as:

$$\bar{Y}_{SKi} = \bar{y}\left[\frac{\bar{X}+D_i}{\bar{x}+D_i}\right], \quad i=1,2,...,10 \quad (11)$$

The mean square error of the estimator $\bar{Y}_{SKi}$ is given as:

$$MSE\left(\bar{Y}_{SKi}\right) = \gamma\bar{Y}^2\left[C_y^2 + \varphi_i^2 C_x^2 - 2\varphi_i\rho C_y C_x\right] (12)$$

Where,

$$\varphi_1 = \frac{\bar{X}}{\bar{X}+D_1},\ \varphi_2 = \frac{\bar{X}}{\bar{X}+D_2},\ \varphi_3 = \frac{\bar{X}}{\bar{X}+D_3},\ \varphi_4 = \frac{\bar{X}}{\bar{X}+D_4},\ \varphi_5 = \frac{\bar{X}}{\bar{X}+D_5},$$
$$\varphi_6 = \frac{\bar{X}}{\bar{X}+D_6},\ \varphi_7 = \frac{\bar{X}}{\bar{X}+D_7},\ \varphi_8 = \frac{\bar{X}}{\bar{X}+D_8},\ \varphi_9 = \frac{\bar{X}}{\bar{X}+D_9},\ \varphi_{10} = \frac{\bar{X}}{\bar{X}+D_{10}}$$

Zaman (2020) an improved class of estimator for the estimation of population mean as:

$$\bar{y}_{zi} = \bar{y}\left(\frac{p}{P}\right)^{\alpha}\exp\left(\frac{(kP+l)-(kp+l)}{(kP+l)+(kp+l)}\right) \quad (13)$$

The MSE of the estimator is:

$$MSE\left(\bar{y}_{zi}\right) = \gamma\bar{Y}^2\begin{pmatrix} C_y^2 + \alpha^2 C_\phi^2 + \theta_i^2 C_\phi^2 - 2\alpha\theta_i C_\phi^2 + 2\alpha\rho_{y\phi}C_y C_\phi \\ -2\theta_i C_y C_\phi \end{pmatrix} \quad (14)$$

Where, $\alpha = \frac{\theta_i C_\phi - \rho_{y\phi}C_y}{C_\phi}$, $i = 0,1,2,...,9$.

$$MSE\left(\bar{y}_{zi}\right)_{\min} = \gamma\bar{Y}^2 C_y^2\left(1-\rho_{y\phi}^2\right) \quad (15)$$

Audu *et al.* (2022) modified class of estimators for the population mean of the study variable in the presence of auxiliary attribute as

$$t_{pi} = \left(\bar{y}+b_\phi\left(P-p\right)\right)\exp\left(\frac{(kP+l)-(kp+l)}{(kP+l)+(kp+l)}\right) \quad i=0,1,2,....,9 \quad (16)$$

$$t_{qi} = \frac{\left(\bar{y}+b_\phi\left(P-p\right)\right)(kP+l)}{\left(kp^*+l\right)}\exp\left(\frac{\left(kp^*+l\right)-(kP+l)}{\left(kp^*+l\right)+(kP+l)}\right) \quad i=0,1,2,..,9 \quad (17)$$

Where

$\bar{y} = \frac{1}{n}\sum_{i\subset S_2} y_i,\ p = \frac{1}{n}\sum_{i\subset S_2} a_i,\ p' = \frac{1}{n'}\sum_{i\subset S_1} y_i$,

$p^* = \frac{NP-np}{N-n}$

Bias and mean square error of the estimator t_{pi} and t_{qi} are given by

$$Bias\left(t_{pi}\right) = \gamma\left[\left(\frac{3w_2^2\bar{Y}}{8}+b_\phi w_2 P\right)C_\phi^2 - \bar{Y}w_2\rho_{y\phi}C_y C_\phi\right] \quad i=1,2,...,9 \quad (18)$$

$$Bias\left(t_{qi}\right) = -\gamma\bar{Y}\left(g^2 w_1 w_2 - \frac{g^2 w_2^2}{2}\right)C_\phi^2 \quad i = 0,1,2,...,9 \tag{19}$$

$$MSE\left(t_{pi}\right) = \gamma\bar{Y}^2\left(C_y^2 + \left(w_1 + \frac{b_\phi P}{\bar{Y}}\right)^2 C_\phi^2 - 2\left(w_1 + \frac{b_\phi P}{\bar{Y}}\right)\rho_{y\phi}C_y C_\phi\right) \; i = 1,2,..,9 \tag{20}$$

$$MSE\left(t_{qi}\right) = \gamma\left(\bar{Y}^2 C_y^2 + b_\phi^2 P^2 C_\phi^2 - 2\bar{Y}Pb_\phi\rho_{y\phi}C_y C_\phi\right) \quad i = 0,1,2,...,9 \tag{21}$$

Where, $w_1 = \dfrac{kP}{2\left(kP + l\right)}$, $w_2 = \dfrac{kP}{\left(kP + l\right)}$, $g = \dfrac{n}{N - n}$, $p^* = \dfrac{NP - np}{N - n}$, $b_\phi = \dfrac{\rho_{y\phi}\bar{Y}C_y}{C_\phi}$.

Sirait *et al.*, (2020) Modified ratio estimator of population mean using quartile and skewness coefficient. The estimators are as follows:

$$\hat{\bar{Y}}_{r1} = \frac{\bar{y}_n + b\left(\bar{X}_N - \bar{x}_n\right)}{\left(\bar{x}_n\beta_1 + \varphi_1\right)}\left(\bar{X}_N\beta_1 + \varphi_1\right) \tag{22}$$

$$\hat{\bar{Y}}_{r2} = \frac{\bar{y}_n + b\left(\bar{X}_N - \bar{x}_n\right)}{\left(\bar{x}_n\beta_1 + \varphi_2\right)}\left(\bar{X}_N\beta_1 + \varphi_2\right) \tag{23}$$

where, $\varphi_1 = \left(DM_N \times Q_{2N}\right)$ and $\varphi_2 = \left(DM_N \times qd_N\right)$

The biases and the MSEs of the estimators are given by

$$B\left(\hat{\bar{Y}}_{r1}\right) = \psi_{n,N}R_{r1}^2\frac{S_x^2}{\bar{Y}_N} \tag{24}$$

$$B\left(\hat{\bar{Y}}_{r2}\right) = \psi_{n,N}R_{r2}^2\frac{S_x^2}{\bar{Y}_N} \tag{25}$$

$$MSE\left(\hat{\bar{Y}}_{r1}\right) = \psi_{n,N}\left(\left(\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1 + \varphi_1\right)}\right)^2 s_x^2 + s_y^2\left(1 - \rho_{yx}^2\right)\right) \tag{26}$$

$$MSE\left(\hat{\bar{Y}}_{r2}\right) = \psi_{n,N}\left(\left(\frac{\bar{Y}_N\beta_1}{\left(X_N\beta_1 + \varphi_2\right)}\right)^2 s_x^2 + s_y^2\left(1 - \rho_{yx}^2\right)\right) \tag{27}$$

Where, $R_{r1} = \dfrac{\bar{Y}_N\beta_1}{\left(X_N\beta_1 + \varphi_1\right)}$, $R_{r2} = \dfrac{\bar{Y}_N\beta_1}{\left(X_N\beta_1 + \varphi_2\right)}$

Zakari *et al.*, (2020) offered a new alternative estimator by combining the ratio, product and exponential ratio type estimators using linear combination. The estimator is given as:

$$\hat{t}_{GR} = \bar{y}\left[k\frac{\bar{X}}{\bar{x}} + \left(1 - k\right)\frac{\bar{x}}{\bar{X}}\right]\exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right] \tag{28}$$

$$MSE\left(\bar{y}_{pr}\right) = \frac{1 - f}{n}\bar{Y}^2\left[C_y^2\left(1 - \rho^2\right)\right] \tag{29}$$

where, k = (0,1) is a suitably chosen constant to be determined.

The Bias and MSE of the proposed estimator are given by:

$$Bias\left(\hat{t}_{GR}\right) = \bar{Y}\left[\frac{\left(16k - 1\right)}{8}\lambda C_x^2 + \frac{\left(1 - 4k\right)}{2}\lambda\rho_{(y,x)}C_y C_x\right] \tag{30}$$

$$MSE_{\min}\left(\hat{t}_{GR}\right) = \psi_{n,N}\bar{Y}^2 C_y^2\left(1 - \rho_{(y,x)}^2\right) \tag{31}$$

where, the optimum value of k is $k_{(opt)} = \dfrac{2\rho_{(y,x)}C_y + C_x}{4C_x}$.

Adichwal *et al*., (2022) put forth a class of difference-cum exponential ratio type estimators to estimate the parameter of the population.

$$t = \left[\hat{t}_{(a,b)} + K\left(\bar{X} - \bar{x}\right)\right]\exp\left[\frac{w_1\left(\bar{X} - \bar{x}\right)}{\bar{X} + \left(\alpha - 1\right)\bar{x}}\right]\exp\left[\frac{w_2\left(S_x^2 - s_x^2\right)}{S_x^2 + \left(\beta - 1\right)s_x^2}\right] \tag{32}$$

The mean square error t is given as:

$$MSE\left(t\right)_{\min} = MSE\left(\hat{t}_{(a,b)}\right) - \left(\frac{\hat{t}_{(a,b)}}{n}\frac{\left\{f_3\left(a,b\right)\right\}^2}{\delta_{04} - 1}\right) - \left(\frac{\hat{t}_{(a,b)}}{n}\right)\frac{\left\{\left(\delta_{04} - 1\right)f_2\left(a,b\right) - \delta_{03}f_3\left(a,b\right)\right\}^2}{\left(\delta_{04} - \delta_{03}^2 - 1\right)\left(\delta_{04} - 1\right)} \tag{33}$$

Where,

$$MSE\left(\hat{t}_{(a,b)}\right) = \left(\frac{\hat{t}_{(a,b)}}{n}\right)\left(a^2 C_Y^2 + ab\delta_{30}C_Y + \frac{b^2}{4}\left(\delta_{40} - 1\right)\right) \tag{34}$$

$$f_2(a,b)=\left\{a\rho_{yx}C_Y+\left(\frac{b}{2}\right)\delta_{21}\right\}; \qquad f_3(a,b)=\left\{a\delta_{12}C_Y+\left(\frac{b}{2}\right)(\delta_{22}-1)\right\}$$

2. Problem Formulation

After examining the estimators put forth by Subramani and Kumarapandiyan (2013), the following exponential ratio class of estimators was recommended for estimating population mean $\bar{Y}$:

$$\hat{\delta}_1=\frac{\lfloor \bar{y}_h+u_1(\bar{X}-\bar{x})+v_1\bar{y}\rfloor}{(\bar{x}+D_1)}(\bar{X}+D_1)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{35}$$

$$\hat{\delta}_2=\frac{\lfloor \bar{y}_h+u_2(\bar{X}-\bar{x})+v_2\bar{y}\rfloor}{(\bar{x}+D_2)}(\bar{X}+D_2)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{36}$$

$$\hat{\delta}_3=\frac{\lfloor \bar{y}_h+u_3(\bar{X}-\bar{x})+v_3\bar{y}\rfloor}{(\bar{x}+D_3)}(\bar{X}+D_3)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{37}$$

$$\hat{\delta}_4=\frac{\lfloor \bar{y}_h+u_4(\bar{X}-\bar{x})+v_4\bar{y}\rfloor}{(\bar{x}+D_4)}(\bar{X}+D_4)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{38}$$

$$\hat{\delta}_5=\frac{\lfloor \bar{y}_h+u_5(\bar{X}-\bar{x})+v_5\bar{y}\rfloor}{(\bar{x}+D_5)}(\bar{X}+D_5)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{39}$$

$$\hat{\delta}_6=\frac{\lfloor \bar{y}_h+u_6(\bar{X}-\bar{x})+v_6\bar{y}\rfloor}{(\bar{x}+D_6)}(\bar{X}+D_6)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{40}$$

$$\hat{\delta}_7=\frac{\lfloor \bar{y}_h+u_7(\bar{X}-\bar{x})+v_7\bar{y}\rfloor}{(\bar{x}+D_7)}(\bar{X}+D_7)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{41}$$

$$\hat{\delta}_8=\frac{\lfloor \bar{y}_h+u_8(\bar{X}-\bar{x})+v_8\bar{y}\rfloor}{(\bar{x}+D_8)}(\bar{X}+D_8)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{42}$$

$$\hat{\delta}_9=\frac{\lfloor \bar{y}_h+u_9(\bar{X}-\bar{x})+v_9\bar{y}\rfloor}{(\bar{x}+D_9)}(\bar{X}+D_9)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{43}$$

$$\hat{\delta}_{10}=\frac{\lfloor \bar{y}_h+u_{10}(\bar{X}-\bar{x})+v_{10}\bar{y}\rfloor}{(\bar{x}+D_{10})}(\bar{X}+D_{10})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{44}$$

$$\hat{\delta}_{11}=\frac{\lfloor \bar{y}_h+u_{11}(\bar{X}-\bar{x})+v_{11}\bar{y}\rfloor}{(\bar{x}+D_{1^2})}(\bar{X}+D_{1^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{45}$$

$$\hat{\delta}_{12}=\frac{\lfloor \bar{y}_h+u_{12}(\bar{X}-\bar{x})+v_{12}\bar{y}\rfloor}{(\bar{x}+D_{2^2})}(\bar{X}+D_{2^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{46}$$

$$\hat{\delta}_{13}=\frac{\lfloor \bar{y}_h+u_{13}(\bar{X}-\bar{x})+v_{13}\bar{y}\rfloor}{(\bar{x}+D_{3^2})}(\bar{X}+D_{3^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{47}$$

$$\hat{\delta}_{14}=\frac{\lfloor \bar{y}_h+u_{14}(\bar{X}-\bar{x})+v_{14}\bar{y}\rfloor}{(\bar{x}+D_{4^2})}(\bar{X}+D_{4^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{48}$$

$$\hat{\delta}_{15}=\frac{\lfloor \bar{y}_h+u_{15}(\bar{X}-\bar{x})+v_{15}\bar{y}\rfloor}{(\bar{x}+D_{5^2})}(\bar{X}+D_{5^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{49}$$

$$\hat{\delta}_{16}=\frac{\lfloor \bar{y}_h+u_{16}(\bar{X}-\bar{x})+v_{16}\bar{y}\rfloor}{(\bar{x}+D_{6^2})}(\bar{X}+D_{6^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{50}$$

$$\hat{\delta}_{17}=\frac{\lfloor \bar{y}_h+u_{17}(\bar{X}-\bar{x})+v_{17}\bar{y}\rfloor}{(\bar{x}+D_{7^2})}(\bar{X}+D_{7^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{51}$$

$$\hat{\delta}_{18}=\frac{\lfloor \bar{y}_h+u_{18}(\bar{X}-\bar{x})+v_{18}\bar{y}\rfloor}{(\bar{x}+D_{8^2})}(\bar{X}+D_{8^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{52}$$

$$\hat{\delta}_{19}=\frac{\lfloor \bar{y}_h+u_{19}(\bar{X}-\bar{x})+v_{19}\bar{y}\rfloor}{(\bar{x}+D_{9^2})}(\bar{X}+D_{9^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{53}$$

$$\hat{\delta}_{20}=\frac{\lfloor \bar{y}_h+u_{20}(\bar{X}-\bar{x})+v_{20}\bar{y}\rfloor}{(\bar{x}+D_{10^2})}(\bar{X}+D_{10^2})\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{54}$$

Where,

$$\bar{y}_h=\frac{\bar{y}}{2}\left(\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right) \tag{55}$$

In general, the aforementioned estimators can be expressed as:

$$\hat{\delta}_i=\frac{\lfloor \bar{y}_h+u_i(\bar{X}-\bar{x})+v_i\bar{y}\rfloor}{(\bar{x}+\theta_i)}(\bar{X}+\theta_i)\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \quad i=1,2,\ldots,20. \tag{56}$$

The following error terms are defined as follows in order to determine the mean square error of the suggested estimators:

$\varepsilon_0 = \dfrac{\bar{y} - \bar{Y}}{\bar{Y}}, \quad \varepsilon_1 = \dfrac{\bar{x} - \bar{X}}{\bar{X}}$ such that $\bar{y} = \bar{Y}\left(1+\varepsilon_0\right), \quad \bar{x} = \bar{X}\left(1+\varepsilon_1\right)$

$$E\left(\varepsilon_0\right) = E\left(\varepsilon_1\right) = 0, \quad E\left(\varepsilon_0^2\right) = \gamma C_y^2,$$
$$E\left(\varepsilon_1^2\right) = \gamma C_x^2, \; E\left(\varepsilon_0 \varepsilon_1\right) = \gamma \rho C_y C_x$$

Expressing $\hat{\delta}_i$ in terms of error terms, we have

$$\hat{\delta}_i = \frac{\left[\bar{Y}\left(1+\frac{\varepsilon_1^2}{8}+\varepsilon_0\right)+u_i\bar{X}\varepsilon_1+v_i\bar{Y}\left(1+\varepsilon_0\right)\right]}{\left(\bar{X}\left(1+\varepsilon_1\right)+\theta_i\right)}\left(\bar{X}+\theta_i\right)\exp\left(\frac{\bar{X}\varepsilon_1}{2\bar{X}+\varepsilon_1}\right) \tag{57}$$

Simplify (57) we have

$$\hat{\delta}_i = \left[\bar{Y}\left(1+\frac{\varepsilon_1^2}{8}+\varepsilon_0\right)+u_i\bar{X}\varepsilon_1+v_i\bar{Y}\left(1+\varepsilon_0\right)\right]\left(1+\Delta_i\varepsilon_1\right)^{-1}\left(1-\frac{\varepsilon_1}{2}+\frac{3\varepsilon_1^2}{8}\right) \tag{58}$$

Simplifying (58) and subtracting $\bar{Y}$ from both sides, we obtain

$$\hat{\delta}_i - \bar{Y} = \bar{Y}\left[\begin{array}{l}\left(\varepsilon_0-\left(\Delta_i+\frac{1}{2}\right)\varepsilon_1-\left(\Delta_i+\frac{1}{2}\right)\varepsilon_0\varepsilon_1+\left(\Delta_i^2+\frac{\Delta_i}{2}+\frac{1}{2}\right)\varepsilon_1^2\right)-u_i\frac{\bar{X}}{\bar{Y}}\left(\varepsilon_1-\left(\Delta_i+\frac{1}{2}\right)\varepsilon_1^2\right)\\ +v_i\left(1+\varepsilon_0-\left(\Delta_i+\frac{1}{2}\right)\varepsilon_1-\left(\Delta_i+\frac{1}{2}\right)\varepsilon_0\varepsilon_1+\left(\frac{1}{8}+\frac{\Delta_i}{2}+\Delta_i^2\right)\varepsilon_1^2\right)\end{array}\right] \tag{59}$$

Squaring and taking expectation of (59) to obtain the mean square error of $\hat{\delta}_i$ as

$$MSE\left(\hat{\delta}_i\right) = \bar{Y}^2\left[A_i + B_i u_i^2 + C_i v_i^2 - 2D_i u_i + 2E_i v_i - 2u_i v_i F_i\right] \tag{60}$$

Where,

$$A_i = \gamma\left[C_y^2 + \left(\Delta_i+\frac{1}{2}\right)^2 C_x^2 - 2\left(\Delta_i+\frac{1}{2}\right)\rho C_y C_x\right], \quad B_i = \gamma\left(\frac{\bar{X}}{\bar{Y}}\right)^2 C_x^2$$
$$C_i = 1+\gamma\left(C_y^2 + \left(3\Delta_i^2+2\Delta_i+1\right)C_x^2 - 4\left(\Delta_i+\frac{1}{2}\right)\rho C_y C_x\right)$$
$$D_i = \gamma\left(\frac{\bar{X}}{\bar{Y}}\right)\left(\rho C_y C_x - \left(\Delta_i+\frac{1}{2}\right)C_x^2\right)$$
$$E_i = \gamma\left(C_y^2 - 3\left(\Delta_i+\frac{1}{2}\right)\rho C_y C_x + \left(2\Delta_i^2+\frac{3}{2}\Delta_i+\frac{3}{4}\right)C_x^2\right)$$
$$F_i = \gamma\left(\frac{\bar{X}}{\bar{Y}}\right)\left(\rho C_y C_x - 2\left(\Delta_i+\frac{1}{2}\right)C_x^2\right)$$

Differentiating MSE$\left(\hat{\delta}_i\right)$ with respect to u_i and equates to zero and solve for u_i and v_i, we obtain $u_i = \dfrac{E_i F_i - C_i D_i}{F_i^2 - B_i C_i}$ and $v_i = \dfrac{B_i E_i - D_i F_i}{F_i^2 - B_i C_i}$. Substituting u_i and v_i into MSE$\left(\hat{\delta}_i\right)$, we obtain the minimum mean square error (minMSE) as:

$$MSE\left(\hat{\delta}_i\right)_{\min} = \bar{Y}^2\left[A_i + \frac{\left(C_i D_i^2 + B_i E_i^2 - 2D_i E_i F_i\right)}{F_i^2 - B_i C_i}\right] \tag{61}$$

3. Problem Solution

To clarify how well the suggested estimators performed in comparison to the current estimators taken into consideration for the study, we employed the three (3) natural datasets in this section.

Table 1: Characteristics of the Population

Parameters	Population 1	Population 2	Population 3
N	40	34	34
n	20	20	20
$\bar{Y}$	5141.5363	856.4117	856.4117
$\bar{X}$	1221.6463	208.8823	199.4412
ρ	0.9244	0.4491	0.4453
S_y	256.1464	733.1407	733.1407
C_y	0.0557	0.8561	0.8561
S_x	102.5494	150.5054	150.2150
C_x	0.0839	0.7205	0.7531
β_2	-1.5154	0.0978	1.0445
β_1	0.3761	0.9782	1.1823
D_1	1111.8150	70.300	60.600
D_2	1119.4800	76.800	83.000
D_3	1139.200	108.200	102.700
D_4	1159.8400	129.400	111.200
D_5	1184.2250	150.000	142.500
D_6	1252.5500	227.200	210.200
D_7	1307.1000	250.400	264.500
D_8	1345.7200	335.600	304.400
D_9	1366.7880	436.400	373.200
D_{10}	1389.300	564.000	634.000

Table 2: MSEs and PREs of the proposed and existing estimators using dataset 1.

Estimators	MSE	PRE	Estimators	MSE	PRE
$\bar{y}$	2050.3888	100	$\bar{y}_R$	992.5383	206.58
			Bahl and Tuteja (1991)		
$\bar{Y}_{BT1}$	358.4367	572.04	$\bar{Y}_{BT2}$	6068.3946	33.79
			Yan and Tian (2010)		
$\bar{Y}_{YT1}$	4951.7534	41.41	$\bar{Y}_{YT2}$	4985.4911	41.13
			Subramani and Kumarapandiyan (2013)		
$\bar{Y}_{SK1}$	334.8577	612.23	$\bar{Y}_{SK6}$	363.7720	563.65
$\bar{Y}_{SK2}$	336.2980	609.69	$\bar{Y}_{SK7}$	376.1193	545.14
$\bar{Y}_{SK3}$	340.0837	602.91	$\bar{Y}_{SK8}$	385.1501	532.36
$\bar{Y}_{SK4}$	344.1636	595.76	$\bar{Y}_{SK9}$	390.1632	525.52
$\bar{Y}_{SK5}$	349.1280	587.28	$\bar{Y}_{SK10}$	395.5824	518.32
			Proposed Estimators		
$\hat{\delta}_1$	298.2978	687.36	$\hat{\delta}_{11}$	298.2195	687.54
$\hat{\delta}_2$	298.2978	687.36	$\hat{\delta}_{12}$	298.2195	687.54
$\hat{\delta}_3$	298.2976	687.36	$\hat{\delta}_{13}$	298.2195	687.54
$\hat{\delta}_4$	298.2474	687.36	$\hat{\delta}_{14}$	298.2195	687.54
$\hat{\delta}_5$	298.2971	687.36	$\hat{\delta}_{15}$	298.2195	687.54
$\hat{\delta}_6$	298.2964	687.37	$\hat{\delta}_{16}$	298.2195	687.54
$\hat{\delta}_7$	298.2958	687.37	$\hat{\delta}_{17}$	298.2195	687.54
$\hat{\delta}_8$	298.2953	687.37	$\hat{\delta}_{18}$	298.2195	687.54
$\hat{\delta}_9$	298.2951	687.37	$\hat{\delta}_{19}$	298.2195	687.54
$\hat{\delta}_{10}$	298.2948	687.37	$\hat{\delta}_{20}$	298.2195	687.54

Table 3: MSEs and PREs of the proposed and existing estimators using dataset 2.

Estimators	MSE	PRE	Estimators	MSE	PRE
$\bar{y}$	11067.0864	100	$\bar{y}_R$	10539.9739	105.00
			Bahl and Tuteja (1991)		
$\bar{Y}_{BT1}$	8843.8180	125.14	$\bar{Y}_{BT2}$	17209.7791	64.31
			Yan and Tian (2010)		
$\bar{Y}_{YT1}$	16600.5393	66.67	$\bar{Y}_{YT2}$	16655.9758	66.41
			Subramani and Kumarapandiyan (2013)		
$\bar{Y}_{SK1}$	9194.9620	120.36	$\bar{Y}_{SK6}$	8857.3224	124.95
$\bar{Y}_{SK2}$	9139.9570	121.08	$\bar{Y}_{SK7}$	8882.6263	124.59
$\bar{Y}_{SK3}$	8956.7638	123.56	$\bar{Y}_{SK8}$	9010.2560	122.83
$\bar{Y}_{SK4}$	8889.1069	124.50	$\bar{Y}_{SK9}$	9178.8233	120.57
$\bar{Y}_{SK5}$	8852.3417	125.02	$\bar{Y}_{SK10}$	9377.5847	118.02
			Proposed Estimators		
$\hat{\delta}_1$	8749.832	126.48	$\hat{\delta}_{11}$	8682.429	127.47
$\hat{\delta}_2$	8748.437	126.50	$\hat{\delta}_{12}$	8681.736	127.48
$\hat{\delta}_3$	8742.377	126.59	$\hat{\delta}_{13}$	8679.908	127.50
$\hat{\delta}_4$	8738.833	126.64	$\hat{\delta}_{14}$	8679.334	127.51
$\hat{\delta}_5$	8735.739	126.69	$\hat{\delta}_{15}$	8678.99	127.52
$\hat{\delta}_6$	8726.469	126.82	$\hat{\delta}_{16}$	8678.419	127.52
$\hat{\delta}_7$	8724.234	126.85	$\hat{\delta}_{17}$	8678.341	127.53
$\hat{\delta}_8$	8717.527	126.95	$\hat{\delta}_{18}$	8678.178	127.53
$\hat{\delta}_9$	8711.743	127.04	$\hat{\delta}_{19}$	8678.095	127.53
$\hat{\delta}_{10}$	8706.439	127.11	$\hat{\delta}_{20}$	8678.046	127.53

Table 4: MSEs and PREs of the proposed and existing estimators using dataset 3.

Estimators	MSE	PRE	Estimators	MSE	PRE
$\bar{y}$	11067.0864	100	$\bar{y}_R$	10960.8417	100.97
			Bahl and Tuteja (1991)		
$\bar{Y}_{BT1}$	8872.9002	124.73	$\bar{Y}_{BT2}$	17543.4002	63.08
			Yan and Tian (2010)		
$\bar{Y}_{YT1}$	17336.9770	63.84	$\bar{Y}_{YT2}$	17362.2582	63.74
			Subramani and Kumarapandiyan (2013)		
$\bar{Y}_{SK1}$	9454.2668	117.06	$\bar{Y}_{SK6}$	8874.7609	124.70
$\bar{Y}_{SK2}$	9214.1709	120.11	$\bar{Y}_{SK7}$	8921.3976	124.05
$\bar{Y}_{SK3}$	9074.5845	121.96	$\bar{Y}_{SK8}$	8975.8044	123.30
$\bar{Y}_{SK4}$	9029.7423	122.56	$\bar{Y}_{SK9}$	9085.0541	121.82
$\bar{Y}_{SK5}$	8922.5155	124.04	$\bar{Y}_{SK10}$	9481.5539	116.72
			Proposed Estimators		
$\hat{\delta}_1$	8790.067	125.90	$\hat{\delta}_{11}$	8715.451	126.98
$\hat{\delta}_2$	8754.633	125.98	$\hat{\delta}_{12}$	8712.613	127.02
$\hat{\delta}_3$	8780.55	126.04	$\hat{\delta}_{13}$	8711.473	127.04
$\hat{\delta}_4$	8778.725	126.07	$\hat{\delta}_{14}$	8711.114	127.05
$\hat{\delta}_5$	8773.144	126.15	$\hat{\delta}_{15}$	8710.359	127.06
$\hat{\delta}_6$	8763.65	126.23	$\hat{\delta}_{16}$	8709.716	127.07
$\hat{\delta}_7$	8757.839	126.37	$\hat{\delta}_{17}$	8709.513	127.07
$\hat{\delta}_8$	8754.296	126.42	$\hat{\delta}_{18}$	8709.427	127.07
$\hat{\delta}_9$	8749.258	126.49	$\hat{\delta}_{19}$	8709.339	127.07
$\hat{\delta}_{10}$	8737.319	126.66	$\hat{\delta}_{20}$	8709.224	127.07

Using three natural datasets, Tables 2, 3, and 4 present the Mean Square Errors (MSEs) and Percentage Relative Efficiencies (PREs) of the suggested and other related estimators that are currently in use and taken into consideration in the study. The findings obtained from every category demonstrated that the suggested estimators, for every population, have higher PREs and lower MSEs than competing, currently available estimators. These demonstrate the superior efficiency of the suggested estimators over their equivalents and increase the likelihood of generating estimates that are more accurate in approximating the true mean values for any population of interest.

4. Conclusion

The present study introduced an exponential ratio class of finite population mean estimators that utilize deciles of an auxiliary variable. The properties of this class, including mean square error (MSE), are derived. The proposed estimators outperformed other existing estimators taken into consideration in the study, as evidenced by the empirical results, which showed that they had higher percentage relative efficiencies (PREs) and minimum mean square errors (MSEs). This means that compared to the current estimators taken into consideration in this study, the suggested estimators are more efficient. Because of this, it is strongly advised that the suggested estimators be used in real-world applications such as commerce, agriculture, health, and so forth, as they have the potential to generate significantly better estimates.

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Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author(s) used [MATHTYPE/QUILBOT] in order to write the equations correctly and also rephrasing of words. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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