# Computational Techniques for Accurate Solutions of Astrophysical Problems Using Transform-Based Collocation

A. O. ADEWUMI [a] , S. O. AKINDEINDE [b] , R. S. LEBELO [c]
[a]Department of Mathematics, Obafemi Awolowo University, 220005, Ile-Ife, NIGERIA.
[b] Department of Mathematics and Statistical Sciences,
Botswana International University of Science and Technology (BIUST),
Private Bag 016, Palapye, BOTSWANA.
[c] Faculty of Human Sciences, Vaal University of Technology, Vanderbijlpark, SOUTH AFRICA.

*Abstract:* This study applies three advanced techniques based on transforms to find approximate solutions to the Lane-Emden type equation, which is often encountered in mathematical physics and astrophysics. The proposed methods utilize new trial functions derived from expressing the second-order derivative of the variable function $y(x)$ using Bernoulli polynomials, and applying Laplace, Sumudu, and differential transforms. To assess the effectiveness of the proposed methods, the study establishes an error analysis and stability analysis, and provides numerical examples demonstrating their accuracy and efficiency. In addition, a comparison of the absolute errors is made among the three methods, namely, Laplace Transform Bernoulli Collocation Method (LTBCM), Sumudu Transform Bernoulli Collocation Method (STBCM), and Differential Transform Bernoulli Collocation Method (DTBCM), and with those obtained from prior literature. The results show that all three methods perform very well in terms of efficiency and accuracy, and can be considered as suitable techniques for solving the Lane-Emden type equation.

*Key-Words:* Fins problem, homotopy perturbation method, Laplace and differential transform methods; boundary value problems, polynomial projection

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## 1 Introduction

Many fields of physics and astrophysics, including stellar structure, thermodynamics, and fluid dynamics, use the Lame-Emden equation, credited to the studies of two astrophysicists, Jonathan Homer Lane and Robert Emden. These problems often appear as a singular initial/boundary valued second-order differential equation, which is a dimensionless structure of Poisson's equation for the gravitational potential of a self-gravitating spherically symmetric, polytropic fluid and the thermal behaviour of a spherical bunch of gas according to the laws of thermodynamics (see [1]). In general, the well-known Lane-Emden equation has the form

$$y''(x) + \frac{\alpha}{x} + f\big(x, y(x)\big) = g(x), x \in [0,1], \alpha > 0, \quad (1.1)$$

with initial conditions

$$y(0) = a \in \mathbb{R}, \qquad y'(0) = 0, \quad (1.2)$$

where $f(x, y)$ is a nonlinear continuous function in the variables $x$ and $y$, $g(x) \in C[0,1]$. Here, the prime denotes the usual differentiation operation with respect to variable $x$, and $\alpha$ is a constant. The function $f$ is assumed to be continuous, and the partial derivative of $f$ with respect to $x$ is assumed to exists and be continuous on the entire domain $[0,1]$ so that the existence and uniqueness of the solution are assured on the given interval.

Finding accurate solutions to (1.1)-(1.2) is essential for understanding the behaviour of these systems, and this has been an attractive research topic among mathematicians and physicists in the past decades. The exact analytical solutions to most problems in mathematical physics, engineering, astrophysics and many physical phenomena, which are normally modelled by differential equations except for a few, are difficult to obtain. Several numerical and analytical methods have been proposed and applied to solve Lane-Emden equation types (1.1) and (1.2). The analytical approaches that have been used in solving these equations are based on truncated series expansion and, as such, have been found to be desirable because they easily overcome the difficulty caused by the singularity at (see [2]). These approaches include the Adomian decomposition method [3], the differential transform method [5], the Laplace transform method [6], the homotopy analysis method [8], the variational iteration method [9], [10], [11], [12], the power series method [13], [14], etc. The collocation method is one of several numerical approaches developed to solve Lane-Emden equations, aiming to overcome the drawbacks and limitations of analytical methods.

The collocation method appears to be the simplest way to discretize functional equations. It requires that the residual equation be satisfied at the collocation points, thus yielding a number of collocation equations that are amenable to simple methods for solving linear systems. Methods such as Bessel collocation [15], Jacobi-Gauss collocation method [16], Legendre-Tau method [17], Sinc collocation [18], Chebyshev spectral method [19], and collocation method based on radial basis functions [20] have been used to solve Lane-Emden type equations in the literature. A number of other methods are also available, such as the successive linearization method [21], the optimal homotopy asymptotic method [23], the Lagrangian method [24], the Laguerre polynomial approach [25], the squared remainder minimization method [26], the artificial neural network [27], the Bernoulli wavelets functional matrix method [28], the Morlet wavelet neural network method [29], the gravitational decoupling method [30], and more. These methods exhibit versatility, enabling their application to a wide range of geometries and boundary conditions.

Recently, Adewumi et al. [31], [32], [33], and [34] developed numerical methods based on the Laplace transform to solve hyperbolic telegraph equations, boundary value problems of ordinary differential equations, and problems with semi-infinite domains. The main idea of the methods is based on the combination of the Laplace transform and collocation method with Taylor, Chebyshev and Laguerre polynomials as basis functions. Bernoulli polynomials defined on the interval are well-known to have many applications. Their excellent properties in function approximation make them widely used. They appear in the integral representation of differentiable periodic functions since they are employed to approximate such functions in polynomials. They are also used to represent the remainder term of the composite Euler-Maclaurin [35].

The objective of this paper is to develop efficient numerical techniques for approximating the solutions of linear and nonlinear Lane-Emden type equations (1.1)–(1.2). To achieve this, we propose coupling the collocation method with three integral transforms: Laplace, Sumudu, and differential transforms. The basis functions for the approximation are Bernoulli polynomials. Overall, this article aims to demonstrate the potential of transform-based collocation methods as a powerful computational tool for solving Lame-Emden-type problems.

The paper is structured as follows: Section 2 provides a brief overview of the fundamental properties of Laplace, Sumudu, differential transforms, and Bernoulli polynomials. Section 3 discusses the derivation of the methods, while Section 4 presents the error analysis and numerical stability of the proposed methods. We demonstrate the effectiveness of the proposed techniques through illustrative examples in Section 5. Finally, we present concluding remarks in Section 6.

# 2 Basic Properties of Bernoulli Polynomials with Features of Laplace, Sumudu and Differential Transforms

## 2.1 Laplace transform

The Laplace transform is a strong mathematical technique that is widely utilized in engineering, physics, and other areas of applied mathematics. If $y(x)$ is a piecewise continuous function, the Laplace transform of $y(x)$ denoted by $\mathcal{L}\{y(x)\}$ is defined as

$$Y(s) := \mathcal{L}\{y(x)\} = \int_0^\infty e^{-sx}\, y(x)\mathrm{d}x, \tag{2.1}$$

and the inverse Laplace transform which recovers the original function, is given by

$$y(x) := \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{sx}\, Y(s)\mathrm{d}s, \quad \alpha > 0. \tag{2.2}$$

The real number $\alpha$ in (2.2) has to be such that all singularities of $Y(s)$ lies to the left of the path of integration. Furthermore, the Laplace transform of $n$th order derivative is given as

$$\mathcal{L}\{y^{(n)}(x)\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) \cdots s y^{(n-2)}(0) - y^{(n-1)}(0). \tag{2.3}$$

For more details on the basic theory and applications of Laplace transform, see [36].

## 2.2 Sumudu Transform

The Sumudu transform of a function f(x) is defined over a set of functions

$$A = \{f(x) : \exists M, \tau_1, \tau_2 > 0, |f(x)| < M e^{\frac{|x|}{\tau_j}}, \text{ if } x \in (-1)^j \times [0,\infty)\} \tag{2.4}$$

by the following formula

$$F(u) = \mathcal{S}(f(x); u) = \int_0^\infty f(ux)e^{-x}\mathrm{d}x, \quad u \in (-\tau_1, \tau_2).$$

For integer order, the Sumudu transform is given as

$$\mathcal{S}\left(\frac{df(x)}{dx}\right) = \frac{1}{u}\big(F(u) - f(0)\big),$$

and for the second derivative of the function $f(x)$, it is given by

$$\mathcal{S}\big(f''(x)\big) = \frac{F(u) - f(0)}{u^2} - \frac{f'(0)}{u}. \tag{2.8}$$

In general, the Sumudu transform of the $m$th-order derivative of a function f(x) is given by

$$\mathcal{S}\left(\frac{d^m f(x)}{dx^m}\right) = \frac{1}{u^m}\left(F(u) - \sum_{k=0}^{m-1} u^k \frac{d^k f(x)}{dx^k}\Big|_{x=0}\right).$$

We refer interested readers to [37] and the references therein for more details on the properties and applications of the Sumudu transform, where various examples and case studies demonstrate its efficacy in solving real-world problems.

### 2.3 Differential transform

The differential transform of the $k$-th differential function $y(x)$ at $x = 0$ is of the form

$$Y(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right)\Big|_{x=0}, \tag{2.9}$$

where $Y(k)$ is the transformed function and $u(x)$ is an auxiliary function related to y(x). The inverse differential transform allows one to recover the original function from its transformed form and is given by

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k. \tag{2.10}$$

In real applications, function $y(x)$ is often expressed as a finite series and (2.10) can be written as

$$y(x) = \sum_{k=0}^{N} Y(k)x^k. \tag{2.11}$$

From (2.9)- (2.11), we deduce the following properties of the differential transform which enable efficient manipulation and analysis of transformed functions. These properties include linearity, compatibility with derivatives, and the ability to handle products and powers of functions:

1. If $y(x) = g(x) \pm h(x)$, then $Y(k) = G(k) \pm H(k)$.
2. If $y(x) = cg(x)$, then $Y(k) = cG(k)$, where $c$ is a constant.
3. If $y(x) = \frac{d^n(x)}{dx^n}$, then $Y(k) = \frac{(k+n)!}{k!} G(k+n)$.
4. If $y(x) = g(x)h(x)$, then $Y(k) = \sum_{k_1=0}^{k} G(k_1) H(k-k_1)$.
5. If $y(x) = x^n$, then $Y(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$
6. If $y(x) = x^m f(x)$ with $m \in N$, then

$$Y(k) = \begin{cases} 0, & k = n, \\ F(k-m), & k \geq m. \end{cases}$$

### 2.4 Bernoulli Polynomials

Bernoulli polynomials are a family of orthogonal polynomials that play a fundamental role in number theory, combinatorics, and mathematical analysis. The classical Bernoulli polynomials, $B_n(x)$, are usually defined using the exponential generating function (see [38]):

$$\frac{\omega e^{x\omega}}{e^{w-1}} = \sum_{k=0}^{\infty} B_k(x) \frac{\omega^k}{k!}, \qquad (|\omega| < 2\pi).$$

This generating function provides a compact representation of Bernoulli polynomials and enables the derivation of various properties and identities associated with them. The primary property of the Bernoulli polynomials is given by the following familiar expansion:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n.$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \qquad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

and so on. Other interesting properties of Bernoulli polynomials include [39]

1. Differentiation: $B'_n(x) = nB_{n-1}(x), \qquad n = 1,2,\cdots$.
2. Integral means conditions: $\int_0^1 B_n(x)dx = 0, \qquad n = 1,2,\cdots$.
3. Differences: $B_n(x+1) - nB_n(x) = nx^{n-1}, \qquad n = 1,2,\cdots$.
4. A series representation in terms of monomials: $\sum_{k=0}^{n} \binom{n}{k} B_k(0) x^{n-k}$.

For more details on the properties of Bernoulli polynomials, readers are referred to [40] and the extensive literature available on this topic.

## 3. Description of the Methods

### 3.1 Laplace Transform Bernoulli Collocation Method

The second derivative, $y''(x)$ in (1.1) is sought in the truncated Bernoulli series form

$$y''(x) = \sum_{n=0}^{N} a_n B_n(x), \qquad (3.1)$$

where $a_n$, $n = 0,1,\cdots,N$ are the unknown Bernoulli coefficients, $N$ is chosen a positive integer, and $B_n(x)$, $n = 0,1,\cdots,N$ are the Bernoulli polynomials.

Applying Laplace transform to both sides of (3.1), we have

$$s^2Y(s) - sy(0) - y'(0) = \mathcal{L}\left(\sum_{n=0}^{N} a_n B_n(x)\right). \qquad (3.2)$$

Using the initial conditions (1.2) in (3.2) and simplifying gives

$$Y(s) = \frac{1}{s^2}\left(as + \mathcal{L}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right). \tag{3.3}$$

Taking the inverse Laplace transform of (3.3) yields

$$y(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(as + \mathcal{L}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right)\right]. \tag{3.4}$$

Now, substituting (3.1) and (3.4) into (1.1), we obtain

$$\sum_{n=0}^{N} a_n\, B_n(x) + \frac{\alpha}{x} + f\left(x, \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(as + \mathcal{L}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right)\right]\right) = g(x). \tag{3.5}$$

The residual function $R(x)$ corresponding to (3.5) is then collocated at the points $x_i$ to give

$$R_N(x_i) = \sum_{n=0}^{N} a_n\, B_n(x_i) + \frac{\alpha}{x_i} + f(x_i, \mathcal{L}^{-1}[\frac{1}{s^2}(as + \mathcal{L}(\sum_{n=0}^{N} a_n\, B_n(x_i)))]) - g(x_i), \tag{3.6}$$

where

$$x_i = a + \frac{(b-a)}{N+2} i, \qquad i = 1,2, \cdots N+1. \tag{3.7}$$

Equation (3.6) generates $N+1$ system of algebraic equations in $N+1$ unknowns. The unknown coefficients are computed using the Gaussian elimination method for the linear case, while Newton's method is used for nonlinear cases. The required approximate solution is then obtained by substituting the values of unknown constants into (3.4).

## 3.2 Sumudu Transform Bernoulli Collocation Method

Here, we also approximate the second derivative of the function $y(x)$ by a truncated Bernoulli polynomial series as in (3.1) and then apply Sumudu transform on both sides of the equation to give

$$\frac{1}{u^2}[F(u) - f(0) - uf'(0)] = \mathcal{S}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right). \tag{3.8}$$

Again, using the initial conditions (1.2) in (3.8) and simplifying, the resulting equation yields

$$F(u) = a + u^2 \mathcal{S}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right). \tag{3.9}$$

Thus, the inverse Sumudu transform is then taking on (3.9) to give

$$y(x) = \mathcal{S}^{-1}\left(a + u^2 \mathcal{S}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right). \tag{3.10}$$

Substituting (3.1) and (3.10) into (1.1), we obtain

$$\sum_{n=0}^{N} a_n\, B_n(x) + \frac{\alpha}{x} + f\left(x, \mathcal{S}^{-1}\left[a + u^2 \mathcal{S}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right]\right) = g(x). \tag{3.11}$$

Again, collocating the residual function $R(x)$ of (3.11) at the points $x_i$ gives

$$R_N(x_i) = \sum_{n=0}^{N} a_n\, B_n(x_i) + \frac{\alpha}{{}_i x} + f\left(x_i, \mathcal{S}^{-1}\left[a + u^2 \mathcal{S}\left(\sum_{n=0}^{N} a_n\, B_n(x_i)\right)\right]\right) - g(x_i), \tag{3.12}$$

where $x_i$ is as defined in (3.7). Apparently, (3.12) constitutes $N+1$ system of algebraic equations in $N+1$ unknowns which can be solved depending on whether the system of equations is linear or nonlinear. We obtain the approximate solution by substituting the values of unknown coefficients into (3.10).

## 3.3 Differential Transform Bernoulli Collocation Method

As before, the second derivative, $y''(x)$, in (1.1) is sought in the truncated Bernoulli series form (3.1). The differential transform is applied on the equation to obtain

$$(k+1)(k+2)Y(k+2) = \mathcal{DT}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right). \tag{3.13}$$

By rearranging (3.13), we have

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[\mathcal{DT}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right]. \tag{3.14}$$

Thus, the new trial function takes the form

$$y(x) = Y(0) + Y(1)x + \sum_{k=0}^{N}\left(\frac{1}{(k+1)(k+2)}\left[\mathcal{DT}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right]x^{k+2}\right). \tag{3.15}$$

Then substitute (3.1) and (3.15) into (1.1), we have

$$\sum_{n=0}^{N} a_n\, B_n(x) + \frac{\alpha}{x} + f\left(x, a + \sum_{k=0}^{N}\left(\frac{1}{(k+1)(k+2)}\left[\mathcal{DT}\left(\sum_{n=0}^{N} a_n\, B_n(x)\right)\right]x^{k+2}\right)\right) = g(x), \tag{3.16}$$

where $Y(0) = a$ and $Y(1) = 0$ from initial conditions (1.2).
The residual function $R(x)$ of (3.16) is also collocated at the points $x_i$ to give

$$R_N(x_i) = \sum_{n=0}^{N} a_n\, B_n(x_i) + f\left(x_i, a + \sum_{k=0}^{N}\left(\frac{1}{(k+1)(k+2)}\left[\mathcal{DT}\left(\sum_{n=0}^{N} a_n\, B_n(x_i)\right)\right]x_i^{k+2}\right)\right) + \frac{\alpha}{x_i} - g(x_i), \tag{3.17}$$

where $x_i$ is as defined in (3.7). Equation (3.17) produces $N+1$ system of equations in $N+1$ unknowns which can be solved by Gaussian elimination or Newton's method for linear and nonlinear cases, respectively. The required approximate solution is then obtained by substituting the values of unknown constants into (3.15).

***Remark 3.1***. *To the best of author's knowledge, the combination of differential and Sumudu transforms with collocation method for the numerical solutions of Lane-Emden type equation* (1.1)-(1.2) *has not been reported in the published literature.*

# 4 Error Analysis and Accuracy of the Solution

Following [41], we illustrate the convergence of the methods by assuming that the unknown and the known solutions are in the space of $C^m[0,1]$ with bounded derivatives. Thus, we state the following theorems:

**Theorem 4.1**. *([42]) Suppose that $g(x) \in C^m[0,1]$ and is approximated by Bernoulli polynomials. Assume that $P_N[g](x)$ is the approximate polynomial of $g(x)$ in terms of Bernoulli polynomials and $R_N[g](x)$ is the remainder term. Then, the associated formulae are stated as follows:*

$$\begin{aligned} g(x) &= P_N[g](x) + R_N[g](x), \qquad x \in [0,1], \\ P_N[g](x) &= \int_0^1 g(x)\mathrm{d}x + \sum_{j=1}^{N} \frac{B_j(x)}{j!}\left(g^{(j-1)}(1) - g^{(j-1)}(0)\right), \\ R_N[g](x) &= -\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)\mathrm{d}t, \end{aligned}$$

where $B_N^*(x) = B_N(x - [x])$ and $[x]$ denotes the largest integer not greater than $x$.

***Lemma 4.2***. *([42],[43]) Suppose $g(x) \in C^\infty[0,1]$ (with bounded derivatives) and $g_N(x)$ is approximated using Bernoulli polynomials. Then the error bound would be obtained as follows:*

$$||E(g_N(x))||_\infty \le C\hat{G}(2\pi)^{-N}, \qquad x \in [0,1], \tag{4.1}$$

*where $\hat{G}$ denotes a bound for all the derivatives of function $g(x)$ (i.e., $||g^i(x)||_\infty \le \hat{G}$, for $i = 0,1,\cdots$), and $C$ is a positive constant.*

Furthermore, we state the following main theorem and show that if both $y(x)$ and $g(x)$ are approximated by the Bernoulli polynomials in (1.1), then the error of the approximation of $y(x)$ depends directly on the approximation of $g(x)$. Thus, using a large value of $N$ will produce high-order approximation of the required solutions.

**Theorem 4.3**. *Assume that* $F(x,y(x)) = x^{-\alpha}\int_0^x t^\alpha f(t,y(t))\mathrm{d}t$ *and* $G(x) = a + L_\alpha(g(x))$, *where* $L_\alpha(\cdot) = \int_0^x x^{-\alpha}\int_0^x t^\alpha(\cdot)\mathrm{d}t\mathrm{d}x$ *is a linear integral operator. If we approximate* $y(x)$ *and* $G(x)$ *by* $y_N(x)$ *and* $G_N(x)$, *respectively, using Bernoulli polynomials, then*

$$||y(x) - y_N(x)||_\infty \leq \frac{1}{1-L_F}||G(x) - G_N(x)||_\infty, \tag{4.2}$$

*where* $L_F$ *is the Lipschitz constant of the function* $F(x,y(x))$ *with respect to its second variable* $y(x)$ *and also* $L_F \ll 1$.

*Proof.* For the proof of this theorem, see [41].

# 5 Numerical Stability of the Methods

The stability of any numerical approach is related to the errors incurred at every stage of computation. Therefore, if errors introduced at some stage in the calculations, such as from erroneous initial conditions or local truncation or round-off errors, propagate without bound throughout subsequent calculations, the solution becomes unstable. Thus, a method is stable if small changes in the initial data produce correspondingly small changes in the final results. That is, the difference between the theoretical and numerical solutions remains bounded at a given time t, as time and space steps tend to zero or the time step remains fixed at every level and $t \to \infty$.

To demonstrate the numerical stability of the transform-based collocation methods, we introduce some random noises to the initial data. We present the plots of the approximate solutions with and without noise. Following [44], a random noise is added to the initial data $y(0)$ and $y'(0)$ in (1.2). The noise functions $y^\delta(0)$ and $y'^\delta(0)$ are obtained by adding $\delta$, the random noise, to $y(0)$ and $y'(0)$ respectively such that

$$y^\delta(0) = y(0) + \delta \tag{5.1}$$

and

$$y'^\delta(0) = y'(0) + \delta, \tag{5.2}$$

where the quantity, $\delta$, is usually very small (of some $p\%$ of the maximum absolute errors) and may be considered bounded. Thus, the corresponding residual functions for LTBCM, STBCM and DTBCM with noise, $\delta$, for the Lane-Emden equation (1.1) with initial conditions (1.2) are given by

$$R_N^\delta(x_i) = \sum_{n=0}^N a_n B_n(x_i) + \frac{\alpha}{x_i} + f\left(x_i, \mathcal{L}^{-1}\left[\frac{1}{s^2}\left((a+\delta)s + \delta + \mathcal{L}\left(\sum_{n=0}^N a_n B_n(x_i)\right)\right)\right]\right) - g(x_i), \tag{5.3}$$

$$R_N^\delta(x_i) = \sum_{n=0}^N a_n B_n(x_i) + \frac{\alpha}{x_i} + f\left(x_i, \mathcal{S}^{-1}\left[(a+\delta) + \delta u + u^2\mathcal{S}\left(\sum_{n=0}^N a_n B_n(x_i)\right)\right]\right) - g(x_i), \tag{5.4}$$

and

$$R_N^\delta(x_i) = \sum_{n=0}^N a_n B_n(x_i) + \frac{\alpha}{x_i} + f(x_i, (a+\delta) + \delta x + \sum_{k=0}^N \left(\frac{1}{(k+1)(k+2)}\left[\mathcal{DT}\left(\sum_{n=0}^N a_n B_n(x_i)\right)\right] x_i^{k+2}\right)) - g(x_i). \tag{5.5}$$

respectively.

# 6 Numerical Examples

In this section, we consider some examples for numerical illustrations of the methods and compare the absolute errors obtained with other methods in the literature. To establish the stability of the methods, we present the plots of approximate solutions with and without noise terms in the initial data. The noises $\delta_i, (i = 1,2,3)$ are taken to be $10^{-2}$, $10^{-3}$, and $10^{-4}$ respectively.

**Example 6.1**. We first consider the equation

$$y''(x) + \frac{2}{x}y'(x) + 1 = 0, \tag{6.1}$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{6.2}$$

The exact solution is $y(x) = 1 - \frac{1}{6}x^2$ [41].

**Method 1:** Applying LTBCM developed in Section 3 for $N = 0$, we have

$$y''(x) = a_0 B_0(x). \tag{6.3}$$

Taking the Laplace transform of both sides of (6.3), we have

$$s^2 Y(s) - sy(0) - y'(0) = \mathcal{L}\big(a_0 B_0(x)\big). \tag{6.4}$$

Using the initial conditions (6.2) in (6.4) and then rearrange to get

$$Y(s) = \frac{1}{s} + \frac{1}{s^3}a_0. \tag{6.5}$$

On taking the inverse Laplace transform of (6.5), we have

$$y(x) = 1 + \frac{1}{2}a_0 x^2. \tag{6.6}$$

The residual function is obtained and collocated at the point $x = \frac{1}{2}$ to yield

$$3a_0 + 1 = 0. \tag{6.7}$$

or $a_0 = -\frac{1}{3}$. Substituting this value into (6.6) yields

$$y(x) = 1 - \frac{1}{6}x^2, \tag{6.8}$$

which is the same as the exact solution.

**Method 2:** Similarly, we approximate the second derivative $y''(x)$ as in (6.3) and on taking the Sumudu transform on both sides of the equation, we have

$$\frac{1}{u^2}[Y(u) - y(0) - uy'(0)] = \mathcal{S}\big(a_0 B_0(x)\big).$$

Substituting the initial conditions (6.2) into (6.9) and simplifying, we have

$$Y(u) = 1 + u^2 a_0. \tag{6.10}$$

Then taking the inverse Sumudu transform of (6.10), we obtain

$$y(x) = 1 + \frac{1}{2}a_0 x^2. \tag{6.11}$$

On substituting (6.11) into (6.1), we obtain a residual function which is also collocated at the point $x = \frac{1}{2}$ to give $3a_0 + 1 = 0$ whose solution is trivially $a_0 = -\frac{1}{3}$. Plugging this into (6.11) to yield $y(x) = 1 - \frac{1}{6}x^2$, that is, the exact solution.

**Method 3:** Here, we also approximate the second derivative $y''(x)$ as in (6.3) and on taking the differential transform on both sides of the equation, we have

$$(k+1)(k+2)Y(k+2) = \mathcal{DT}\big(a_0 B_0(x)\big). \tag{6.12}$$

Rearranging (6.12) and substituting the resulting equation into (3.15) yields

$$y(x) = Y(0) + Y(1)x + \frac{1}{2}a_0x^2. \qquad (6.13)$$

Applying the initial conditions given by (6.2), we have

$$y(x) = 1 + \frac{1}{2}a_0x^2. \qquad (6.14)$$

Thus, the following equation is obtained after collocating the residual function of (6.1) at the point $x = \frac{1}{2}$ to get $a_0 = -\frac{1}{3}$ and this is substituted into (6.14) to yield $y(x) = 1 - \frac{1}{6}x^2$, which is the exact solution.

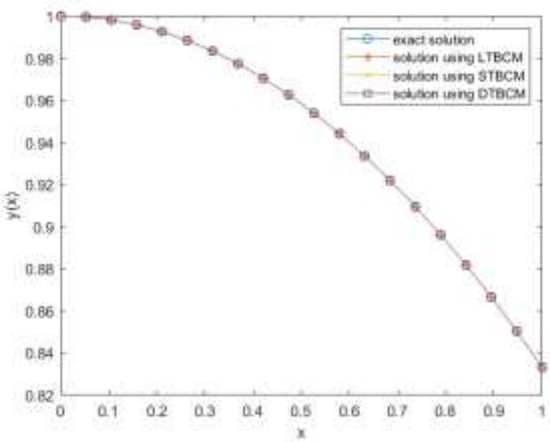


**Figure 1**: Comparison of solution of Example 6.1 by the proposed methods and the exact solution

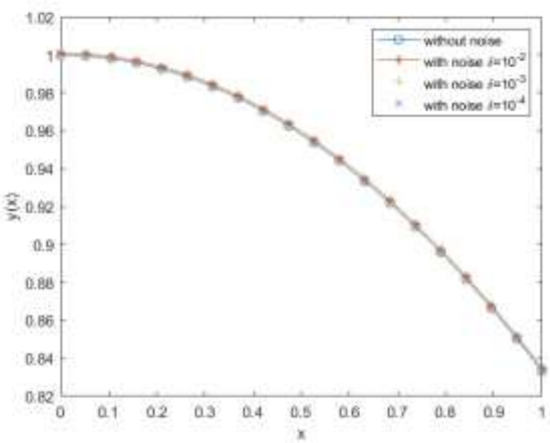


**Figure 2:** Behaviour of the solutions of Example 6.1 by our proposed methods with and without noise

Figures 1 and 2 show the plots of the behaviours of the solutions with and without noise terms in the initial data using LTBCM, STBCM and DTBCM for case $N = 0$. From the two figures, it is observed that the variations in the solutions with noise and without noise are negligible, and hence, the methods are numerically stable.

**Example 6.2**. We also consider the equation

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 0, \qquad (6.15)$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \qquad (6.16)$$

The exact solution is $y(x) = \frac{\sin(x)}{x}$.
The errors listed in Table 1 are obtained by applying the three methods to solve this problem when $N = 10$. Figure 3 depicts the plot of the exact solution and the approximate solutions obtained by LTBCM, STBCM and DTBCM, while Figure 4 shows the behaviour of approximate solutions without and with noises, $\delta = 10^{-1}, 10^{-2}, 10^{-3}$ when $N = 8$. It shows that the methods are numerically stable since the variations in the approximate solutions are insignificant.

***Table 1****: Comparison of absolute errors for Example 6.2 when $N = 10$*

| $x$ | Exact solution | LTBCM | STBCM | DTBCM |
|---|---|---|---|---|
| 0.1 | 0.998334166468281 | $1.350e\ 17$ | $1.350e\ 17$ | $1.350e-17$ |
| 0.2 | 0.993346653975306 | $1.571e-17$ | $1.571e-17$ | $1.571e-17$ |
| 0.3 | 0.985067355537798 | $1.644e-17$ | $1.644e-17$ | $1.644e-17$ |
| 0.4 | 0.973545855771626 | $1.663e-17$ | $1.663e-17$ | $1.663e-17$ |
| 0.5 | 0.958851077208406 | $1.644e-17$ | $1.644e-17$ | $1.644e-17$ |

| $x$ | Exact solution | LTBCM | STBCM | DTBCM |
|---|---|---|---|---|
| 0.6 | 0.94107078899172 | $1.627e-17$ | $1.627e-17$ | $1.627e-17$ |
| 0.7 | 0.92031098176813 | $1.613e-17$ | $1.613e-17$ | $1.613e-17$ |
| 0.8 | 0.89669511362440 | $1.564e-17$ | $1.564e-17$ | $1.564e$ 17 |
| 0.9 | 0.87036323291942 | $1.555e-17$ | $1.555e-17$ | $1.555e-17$ |
| 1.0 | 0.84147098480789 | 5.000 e-20 | $5.000e-20$ | $5.000e-20$ |

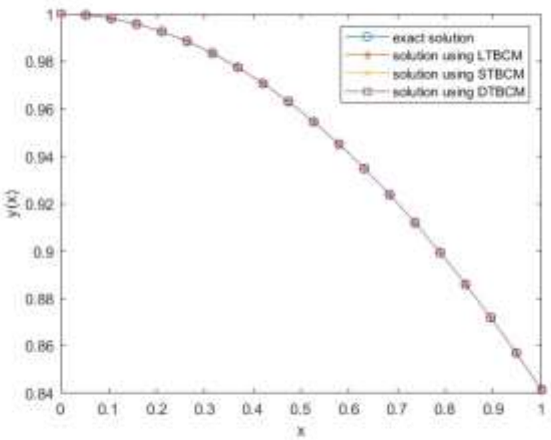


**Figure 3:** Comparison of solution of Example 6.2 by the proposed methods and the exact solution

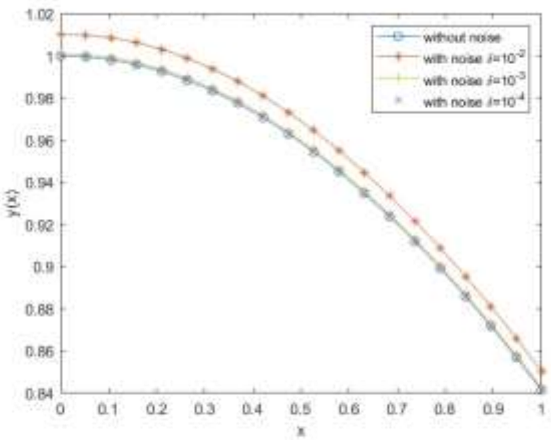


**Figure 4:** Behaviour of the solutions of Example 6.2 by our proposed methods with and without noise.

**Example 6.3** We consider a case where $f(x,y) = e^{y(x)}$, $g(x) = 0$ which gives the isothermal gas sphere equation

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \qquad (6.17)$$

with initial conditions

$$y(0) = 0, \qquad y'(0) = 0. \qquad (6.18)$$

The nonlinear function, $f(x,y) = e^{y(x)}$, is approximated by using five terms of its Maclaurin expansion, that is, $f(x,y) = e^{y(x)} \approx 1 + y + y^2/2 + y^3/6 + y^4/24$. The methods are applied to solve this problem when $N = 8$ and $N = 10$. Table 2 shows the comparison of the absolute errors with other methods in the literature. From the table, it is observed that the errors are in good agreement with [41] but more accurate than those obtained in [47]. Also Figures 5 and 6 show the comparison of approximate solutions using the methods and the behaviour of approximate solutions without and with noises, respectively. Since the variations in the approximate solutions are negligible, it can therefore be concluded that the methods are numerically stable.

***Table 2****: Comparison of absolute errors for Example 6.3 when $N = 10$*

| $x$ | Exact solution | LTBCM | STBCM | DTBCM | Error in [41] | Error in [47] |
|---|---|---|---|---|---|---|
| 0.0 | 0.00000000 | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $3.01e-31$ | $9.24e-18$ |
| 0.1 | $-0.0016658$ | $6.21e-11$ | $6.21e-11$ | $6.21e-11$ | $6.21e-11$ | $5.28e-10$ |
| 0.2 | $-0.0066533$ | $9.22e-13$ | $9.22e-13$ | $9.22e-13$ | $9.22e-13$ | $3.37e-08$ |
| 0.5 | $-0.041153$ | $5.95e-10$ | $5.95e-10$ | $5.95e-10$ | $5.95e-10$ | $8.12e-06$ |

| $x$ | Exact solution | LTB CM | STB CM | DTB CM | Error in [41] | Error in [47] |
|---|---|---|---|---|---|---|
| 1.0 | $-0.158827$ | $3.97e-07$ | $3.97e-07$ | $3.97e-07$ | $3.97-07$ | $4.93-04$ |

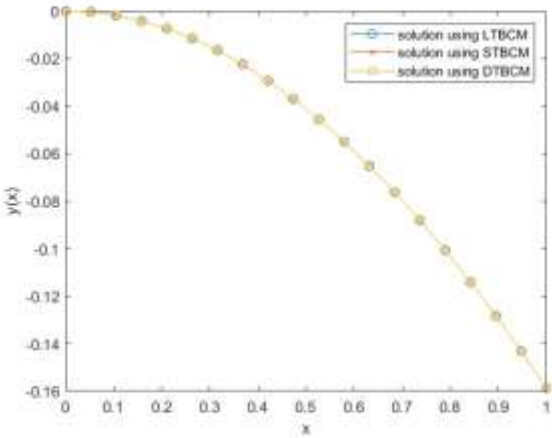


**Figure 5**: Comparison of solution of Example 6.3 by the proposed methods and the exact solution

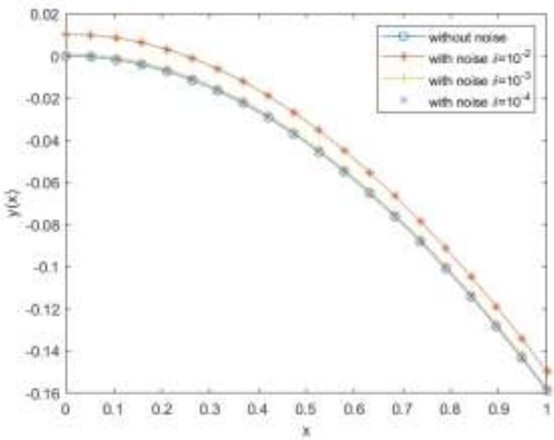


**Figure 6:** Behaviour of the solutions of Example 6.3 by our proposed methods with and without noise

**Example 6. 4**. We also consider case $f(x,y) = \sin(y)$, $g(x) = 0$ and this gives the equation

$$y''(x) + \frac{2}{x}y'(x) + \sin(y) = 0, \quad (6.19)$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \quad (6.20)$$

In a similar manner, we apply the three methods to solve this problem where the nonlinear function, $f(x,y) = \sin(y)$ is approximated by using three terms of its Maclaurin expansion series, that is, $f(x,y) = \sin(y) \approx y - y^3/6 + y^5/120$. The point-wise absolute errors of the methods for case $N = 10$ together with the absolute errors of [41] and [45] are presented in Table 3. It is observed that the errors are in good agreement with [41] but superior to those obtained in [45]. Figures 7 and 8 depict the plots of numerical results of Lane-Emden equation for $N = 8$ and behaviour of the solution with and without noise terms in the initial data, respectively. In Figure 8 it is observed that the variations in the approximate solutions are very negligible and hence the methods are numerically stable.

***Table 3***: *Comparison of absolute errors for Example 6.4 when* $N = 10$

| $x$ | Exact solution | LTB CM | STB CM | DTB CM | Error in [41] | Error in [45] |
|---|---|---|---|---|---|---|
| 0.0 | 1.0000000 | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $1.11e-16$ | $0.00e+00$ |
| 0.1 | 0.9985979 | $3.25e-07$ | $3.25e-07$ | $3.25e-07$ | $3.25e-07$ | $7.21e-06$ |
| 0.2 | 0.9943962 | $1.28e-06$ | $1.28e-06$ | $1.28e-06$ | $1.28e-06$ | $1.00e-05$ |
| 0.5 | 0.9651777 | $7.53e-06$ | $7.53e-06$ | $7.53e-06$ | $7.53e-06$ | $1.04e-05$ |
| 1.0 | 0.8636807 | $2.35e-05$ | $2.35e-05$ | $2.35e-05$ | $2.35e-05$ | $7.03e-06$ |

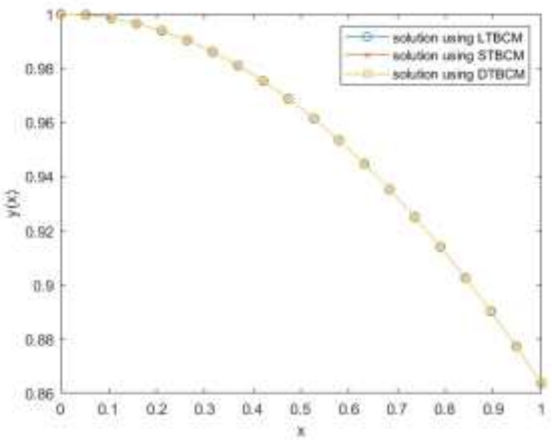


**Figure 7:** Comparison of solution of Example 6.4 by the proposed methods and the exact solution

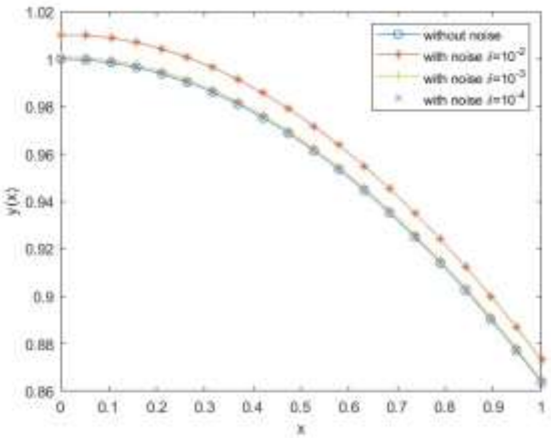


**Figure 8**: Behaviour of the solutions of Example 6.4 by our proposed methods with and without noise

**Example 6. 5**. We also consider case $f(x,y) = y^3(x)$, $g(x) = 0$ and this gives the equation

$$y''(x) + \frac{2}{x}y'(x) + y^3(x) = 0, \qquad (6.21)$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \qquad (6.22)$$

In [46], the exact solution of this problem is reported. We apply the methods in Section 3 to solve this problem. In Table 4, an interesting comparison is made with the errors of [41] and [48].

For small value of Bernoulli polynomials, that is, $N = 10$, we obtain absolute errors which are in good agreement with those in [48] where the authors had used $N = 25$ . This is an indication that our methods are superior to that of [48]. Also Figure 9 shows the plot of approximate solutions and Figure 10 depicts the behaviour of solutions with and without noise terms in the initial data. It is then observed that the variations in the approximate solutions are very negligible which means that the methods are numerically stable.

***Table 4**: Comparison of absolute errors for Example 6.5 when $N = 10$*

| $x$ | Exact solution | LTBCM | STBCM | DTBCM | Error in [41] | Error in [48] N=25 |
|---|---|---|---|---|---|---|
| 0.0 | 1.0000000 | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ |
| 0.1 | 0.9983358 | $2.95e-08$ | $2.95e-08$ | $2.95e-08$ | $2.95e-08$ | $2.95e-08$ |
| 0.5 | 0.9598391 | $3.00e-08$ | $3.00e-08$ | $3.00e-08$ | $3.00e-08$ | $3.00e-08$ |
| 1.0 | 0.8550576 | $3.14e-08$ | $3.14e-08$ | $3.14e-08$ | $3.14e-08$ | $3.14e-08$ |

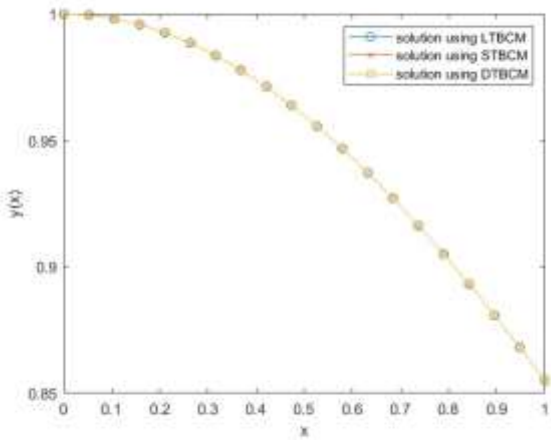

**Figure 9**: Comparison of solution of Example 6.5 by the proposed methods and the exact solution

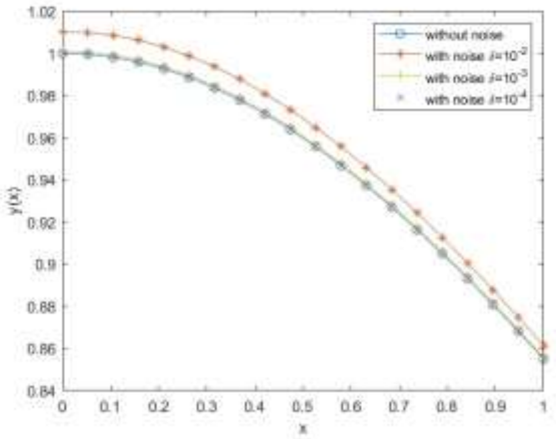


**Figure 10**: Behaviour of the solutions of Example 6.5 by our proposed methods with and without noise

**Example 6.6**. We consider case $f(x,y) = y^4(x)$, $g(x) = 0$ which gives the equation

$$y''(x) + \frac{2}{x}y'(x) + y^4(x) = 0, \tag{6.23}$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{6.24}$$

The exact solution of this problem is also reported in [46]. The point-wise errors are presented in Table 5 when $N = 10$ and we compare the errors obtained using the three methods with those of [41] and [48]. Figures 11 and 12 show the approximate solutions obtained by the methods and the behaviour of the solutions with and without noise terms in the initial data, respectively for case $N = 8$. From Figure 12 it is observed that the variations in approximate solutions are very negligible and hence, the methods are numerically stable.

***Table 5**: Comparison of absolute errors for Example 6.6 when $N = 10$*

| $x$ | Exact solution | LTBCM | STBCM | DTBCM | Error in [41] | Error in [48] N=25 |
|---|---|---|---|---|---|---|
| 0.0 | 1.0000000 | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ |
| 0.1 | 0.9983367 | $4.05e-08$ | $4.05e-08$ | $4.05e-08$ | $4.05e-08$ | $4.04e-08$ |
| 0.2 | 0.9933862 | $1.35e-08$ | $1.35e-08$ | $1.35e-08$ | $1.35e-08$ | $1.35e-08$ |
| 0.5 | 0.9603109 | $2.33e-09$ | $2.33e-09$ | $2.33e-09$ | $2.33e-09$ | $2.34e-09$ |
| 1.0 | 0.8608138 | $1.22e-08$ | $1.22e-08$ | $1.22e-08$ | $1.22e-08$ | $1.22e-08$ |

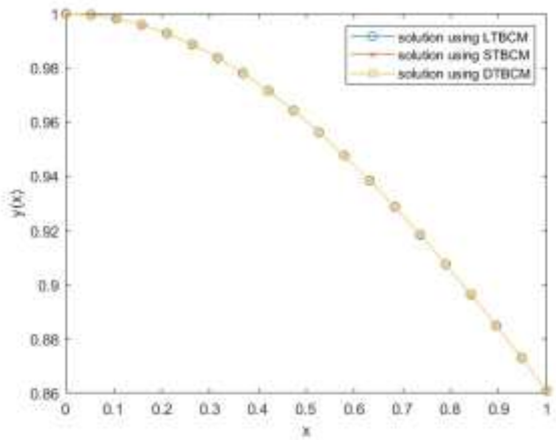


**Figure 11**: Comparison of solution of Example 6.6 by the proposed methods and the exact solution

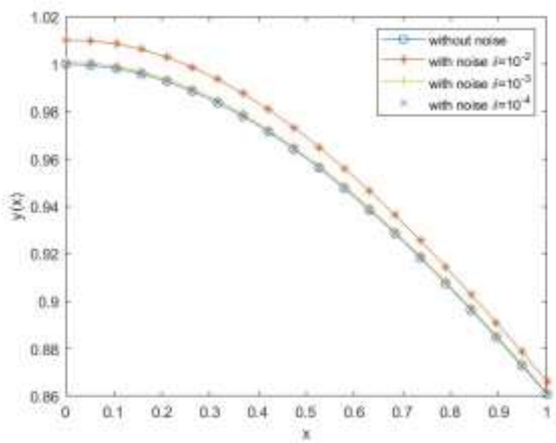

**Figure 12**: Behaviour of the solutions of Example 6.6 by our proposed methods with and without noise

**Example 6.7.** Finally, we consider the white-dwarf equation

$$y''(x) + \frac{2}{x}y'(x) + (y^2(x) - C)^{3/2} = 0, \qquad (6.25)$$

with initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \qquad (6.26)$$

This example is solved by the three methods with $N = 8, C = 0.4$ and $C = 0.2, N = 10$. In Table 6 we present a comparison of the absolute errors obtained by the methods for $C = 0.2$ when $N = 10$ with those obtained in [10] and [26]. From the table, it is observed that the errors obtained using our methods are in close agreement with those in [10] but more accurate than those in [26]. Furthermore, Figure 13 shows the plot of approximate solutions by the methods while Figure 14 depicts the plot of behaviour of solutions with and without noise terms in the initial data for the case $C = 0.4, N = 8$. It is observed that the methods are numerically stable since the variations in the approximate solutions with and without noises are negligible.

***Table 6**: Comparison of absolute errors for Example 6.7 when $N = 10$*

| $x$ | RK4 | LTBCM | STBCM | DTBCM | Error in [10] | Error in [26] |
|---|---|---|---|---|---|---|
| 0.0 | 1.0000000 | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ | $0.00e+00$ |
| 0.5 | 0.9711547 | $1.60e-09$ | $1.60e-09$ | $1.60e-09$ | $8.58e-09$ | $1.61e-05$ |
| 1.0 | 0.8949287 | $1.48e-08$ | $1.48e-08$ | $1.48e-08$ | $7.45e-08$ | $6.38e-05$ |

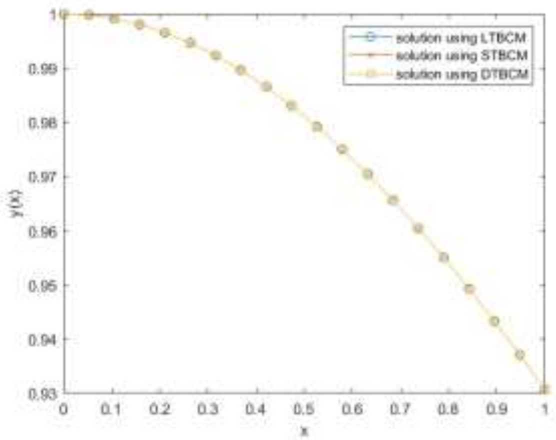


**Figure 13**: Comparison of solution of Example 6.7 by the proposed methods and the exact solution

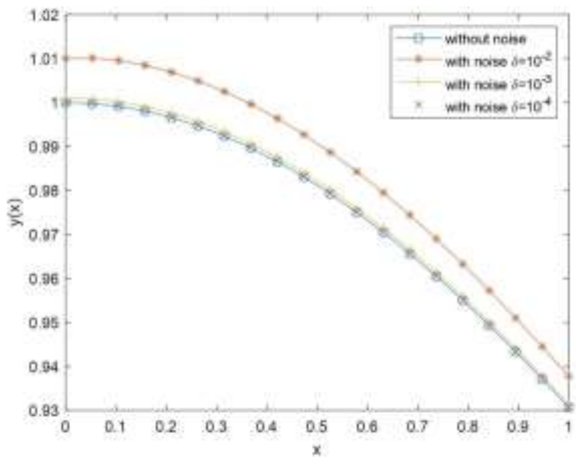


**Figure 14**: Behaviour of the solutions of Example 6.7 by our proposed methods with and without noise

## 7 Conclusion

This paper developed and applied transform-based collocation methods to solve Lane-Emden type equations that usually arise in mathematical physics and astrophysics. Successfully applied to both linear and nonlinear Lane-Emden equations within the interval [0, 1], these methods demonstrate exceptional accuracy and efficiency. Through comprehensive comparisons of absolute errors with existing literature (Tables 1-6), it is evident that our methods not only agreed with results from established approaches but also exhibit superiority in certain cases. The findings underscore the robustness of these methods, further supported by their demonstrated numerical stability, even amidst high levels of noise in initial data (as evidenced by Figures 2-14).

For future research, exploring the extension of these methods to solve Lane-Emden equations in higher dimensions or incorporating adaptive strategies to enhance efficiency and accuracy could offer promising avenues. Additionally, investigating the application of the proposed methods to other classes of differential equations or their adaptation for solving problems in different scientific disciplines could further expand their utility and impact.

**Declaration of Generative AI and AI-assisted Technologies in the Writing Process**

During the preparation of this work the authors used Grammarly for grammar checking. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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