Solution of Multi-dimensional Navier-Stokes Equation Through Gamar Transform Combined with Adomian Decomposition Method

ABDELILAH KAMAL.H.SEDEEG
Department of Mathematics, Faculty of Education, University of Holy Quran and Islamic Sciences, Omdurman P. O Box 14411, SUDAN.
Department of Mathematics, Faculty of Sciences and Arts-Almikwah, Al-Baha University, Al-Bahah P. O Box 1988, SAUDI ARABIA.
Department of Physics and Mathematics, College of Sciences and Technology, Merowe University of Technology- Abdulatif Alhamad, Merowe, SUDAN.

Abstract: - In this work, an attempt is made to combine Gamar transform and Adomian decomposition method (GTADM) in order to solve multi-dimensional Navier-Stokes Equation. Selected examples are discussed so as to prove the feasibility of this method. The efficacy of the current method in relation to finding exact and approximate solutions is strongly verified by the results of the study. The technique of numerical simulation is utilized to reach the exact and approximate solutions.

Key-Words: - Gamar transform; triple convolution theorem; partial derivatives; Mittin-Leffler functions; Adomian decomposition method ;Navier-Stokes Equation.

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1 Introduction

Fractional partial differential equations are essentially manifestation of classical partial differential equations. They have been developed and applied to a wide range of physical and engineering disciplines, including visco-elasticity, acoustics, electromagnetic and electro-chemistry. More recently, both double and triple Laplace decomposition methods were utilized to obtain solutions of fractional partial differential equations [1- 9]. Other methods have been equally successfully employed to solve linear and nonlinear problems natural Sciences [10,11].

Scholars exerted great efforts to obtain solutions to fractional partial differential equations. In principle, finding exact solutions to fractional partial differential equations entails much effort. Hence, scholars have focus on numerical methods, particularly the perturbation method. However, these methods suffer from some limitations. For instance, the fact that the approximate solution requires a series of small parameters is puzzling because the majority of nonlinear problems lack such parameters. While optimal choices of small parameters occasionally result in ideal solutions, in the majority of cases serious flaws in solutions ensue form unstable choices. The homotopy perturbation method was first

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developed in 1998[12-14] and was further studied by a host of authors in order to handle linear and nonlinear problems arising in scientific domains [15-20]. Recently, many researchers have attempted to find solutions of linear and nonlinear partial differential equations using a variety of methods in combination with all integral transform. Examples of these are Laplace decomposition method and homotopy perturbation transform method [21-27].

In a recent work, Kamal [28] suggested a novel general triple integral transform known as Gamar Transform, which is defined as follows:

$$\mathbb{T}_3[w(x,y,t),(r,s,v)] = \mathbb{G}[w(x,y,t),(r,s,v)] = \mathbb{T}_x\big[\mathbb{T}_y[\mathbb{T}_t[w(x,y,t);t \to v]y \to s]x \to r\big], r,s,v > 0,$$

$$= \mathcal{p}(r)\int_0^{\infty} e^{-\mathcal{W}(r)x}\left(\mathcal{q}(s)\int_0^{\infty} e^{-\psi(s)y}\left(\mathcal{u}(v)\int_0^{\infty} \right.\right.$$

$$= \mathcal{p}(r)\mathcal{q}(s)\mathcal{u}(v)\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\mathcal{W}(r)x-\psi(s)y-\varphi(v)t} w($$

$$= W(r,s,v). \quad (1)$$

provided that all integrals exists for some $\mathcal{W}(r), \psi(s)$ and $\varphi\,(v)$, where $\mathcal{W}(r), \psi(s)$ and $\varphi\,(v)$ are transform functions for x, y and t respectively. This transform can generate virtually all triple integral transform through changing the values of $\mathcal{W}(r), \psi(s)$, $\varphi\,(v)$, $\mathcal{W}(r), \psi(s)$ and $\varphi\,(v)$.For examples:

- If $\mathcal{p}(r) = \mathcal{q}(s) = \mathcal{u}(v) = 1$ and $\mathcal{W}(r) = r, \psi(s) = s, \varphi(v) = v$, then this new transform turns into the triple Laplace transform [38-41].

$$\mathcal{L}_x\mathcal{L}_y\mathcal{L}_t[w(x,y,t)] = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-rx-sy-vt}[w(x,y,t)]dxdy\,dt.$$

- If $\mathcal{p}(r) = \mathcal{W}(r) = \frac{1}{r}, \mathcal{q}(s) = \psi(s) = \frac{1}{s}$ and $\mathcal{u}(v) = \varphi(v) = \frac{1}{v}$, then this new transform turns into the triple Sumudu transform[42].

$$\mathbb{S}_x\mathbb{S}_y\mathbb{S}_t[w(x,y,t)] = \frac{1}{rsv}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{e^{-\frac{x}{r}-\frac{y}{s}-\frac{t}{v}}}[w(x,y,t)]dxdy\,dt.$$

- If $\mathcal{p}(r) = r, \mathcal{q}(s) = s, \mathcal{u}(v) = v$ and $\mathcal{W}(r) = \frac{1}{r}, \psi(s) = \frac{1}{s}, \varphi(v) = \frac{1}{v}$, then the new transform turns into the triple Elzaki transform[43].

$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_t[w(x,y,t)] = rsv\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-\frac{x}{r}-\frac{y}{s}-\frac{t}{v}}[w(x,y,t)]dxdy\,dt.$$

- If $\mathcal{p}(r) = \frac{1}{r}, \mathcal{q}(s) = \frac{1}{s}, \mathcal{u}(v) = \frac{1}{v}$ and $\mathcal{W}(r) = r, \psi(s) = s, \varphi(v) = v$, then this new transform turns into the triple Aboodh transform[44].

$$\mathbb{A}_x\mathbb{A}_y\mathbb{A}_t[w(x,y,t)] = \frac{1}{rsv}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-rx-sy-vt}[w(x,y,t)]dxdy\,dt.$$

- If $\mathcal{p}(r) = 1, \mathcal{q}(s) = \frac{1}{s}, \mathcal{u}(v) = \frac{1}{v}$ and $(r) = r, \psi(s) = s, \varphi(v) = \frac{1}{v}$, then this new transform turns into the triple Laplace-Sumudu-Aboodh transform[45].

$$\mathcal{L}_x\mathbb{S}_y\mathbb{A}_t[w(x,y,t)] = \frac{1}{sv}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-rx-sy-\frac{1}{v}t}[w(x,y,t)]dxdy\,dt.$$

- If $\mathcal{p}(r) = 1, \mathcal{q}(s) = 1, \mathcal{u}(v) = 1$ and $(r) = \frac{1}{r}, \psi(s) = \frac{1}{s}, \varphi(v) = \frac{1}{v}$, then this new transform turns into the triple Kamal transform[46].

$$\mathbb{K}_x \mathbb{K}_y \mathbb{K}_t [w(x,y,t)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{1}{r}x - \frac{1}{s}y - \frac{1}{v}t} [w(x,y,t)] dxdy\, dt.$$

- If $\mathcal{p}(r) = 1, \mathcal{q}(s) = s, \mathcal{u}(v) = \frac{1}{v}$ and $\mathcal{W}(r) = r, \psi(s) = s, \varphi(v) = \frac{1}{v}$, then this new transform turns into the triple Laplace- ARA - Sumudu transform[47].

$$L_x \mathcal{G}_y \mathbb{S}_t [w(x,y,t)] = \frac{s}{v} \int_0^\infty \int_0^\infty \int_0^\infty e^{-r\,x - s\,y - \frac{1}{v}t} [w(x,y,t)] dxdy\, dt.$$

We note that the inverse Gamar transform is defined by

$$\mathbb{T}_3{}^{-1}[W(r,s,v)] = \mathbb{G}^{-1}[W(r,s,v)]$$
$$= \mathbb{T}_r{}^{-1}\left[\mathbb{T}_s{}^{-1}\left[\mathbb{T}_v{}^{-1}[W(r,s,v)]\right]\right] = w(x,y,t)$$
$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{\mathcal{p}(r)} e^{\mathcal{W}(r)x} dr \; \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{1}{\mathcal{q}(s)} e^{\psi(s)}$$
$\in \mathcal{R}$. (2)

where $\mathbb{G} = \mathbb{T}_x \mathbb{T}_y \mathbb{T}_t$ is the general triple transform with respect to x, y and t, and the inverse of general triple transform denoted by $\mathbb{G}^{-1} = \mathbb{T}_r{}^{-1} \mathbb{T}_s{}^{-1} \mathbb{T}_v{}^{-1}$ is with respect to r, s and v.

A well-known equation controlling the motion of viscous fluid flow designated Navier-Stokes Equation has been derived in the 19th century [29]. This equation is viewed as the equal to Newton's second law of motion as far as fluid substances are concerned and it is a fusion of the equations of momentum, continuity and energy. This equation covers many physical phenomena such as blood flow, liquid flow in tubes and air flow in the proximity of aircraft wings. The fractional modelling of Navier-Stokes Equation was first carried out by El-Shahed and Salem [30] who applied the classic Navier-Stokes Equation using Laplace and the finite Hankel and Fourier Sine transforms combining homotopy perturbation method and Laplace decomposition method. Kumar et al [31] have analytically solved a nonlinear fractional model of Navier-Stokes Equation. Furthermore utilizing the homotopy analysis method, Ragab et al and Ganji et al solved nonlinear time-fractional Navier-Stokes Equation [32,33].In contrast, Birajdar [34] and Momani and Odibat [35] have employed the Adomian decomposition method for numerical computation of time-fractional Navier-Stokes Equation. Kumar et al used both Adomian decomposition method and Laplace transform algorithm to find the analytical solution of time-fractional Navier-Stokes Equations [36]. Morever, Chaurasia and kumar solved the same problem by combining Laplace and Hankel finite transforms [37].In the current paper, we will study the system of multi-dimensional Navier-Stokes Equation of the following form:

$$\begin{aligned} D_t^\alpha w + w\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} - \rho_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) &= -\frac{1}{\rho}\frac{\partial r}{\partial x}, \\ D_t^\alpha m + w\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} - \rho_0\left(\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2}\right) &= -\frac{1}{\rho}\frac{\partial r}{\partial y}. \end{aligned} \quad (3)$$

Where $x, y, t > 0$ and $n - 1 < \alpha < n$.

Subject to the condition

$$\begin{aligned} w(x,y,0) &= k(x,y), \\ m(x,y,0) &= h(x,y). \end{aligned} \quad (4)$$

The objective of this paper is to present an approximate analytic of multi-dimensional solution of Navier-Stokes Equation using Gamar transform combined with Adomian decomposition method.

The remainder of the paper is structured as follows: In Section 2, basic concepts and properties of Gamar transformation are introduced. Some core definitions and notations on fractional calculus are outlined in Section 3. A succinct description of GTADM is presented in Section 4. In Section 5, the approximate analytical solutions of two elected examples of time-fractional order Navier-Stokes Equation are obtained. Section 6 concludes the study.

2 Fundamental Concepts of Gamar Transform

This section is concerned with the presentation of the Gamar transform in three-dimensional spaces. We out line basic properties regarding the existence conditions, linearity and the inverse of this transform. Moreover, some essential properties and results are used to compute the Gamar transform for elementary basic functions. We introduce the triple convolution theorem and the derivative properties of the new transform.

2.1 Some Properties and Theorems of Gamar Transform [28]

Property 2.1. (Linearity). If $\mathbb{G}[w(x,y,t)] = \Psi(r,s,v)$ and $\mathbb{G}[h(x,y,t)] = H(r,s,v)$, then for any constants A and B , we have

$$\mathbb{G}[A\, w(x,y,t) + B\, h(x,y,t)] = A\,\Psi(r,s,v) + B\, H(r,s,v). \quad (5)$$

Property 2.2. If $w(x,y,t) = f(x)h(y)z(t)$, $x > 0, y > 0$ and $t > 0$. Then

$$\mathbb{G}[w(x,y,t)] = \mathbb{T}_x[f(x)]\mathbb{T}_y[h(y)]\mathbb{T}_t[z(t)]. \quad (6)$$

where $\mathbb{T}_x$,$\mathbb{T}_y$ and $\mathbb{T}_t$ are general integral transform for $f(x)$, $h(y)$ and $z(t)$ respectively.

Definition 2.1. If $w(x,y,t)$ defined on $[0,X] \times [0,Y] \times [0,T]$ satisfies the condition $|w(x,y,t)|$

$$\leq \mathbb{K}e^{\alpha x+\delta y+\lambda t},\ \exists \mathbb{K} > 0,\ \forall x > X,\ y > \mathrm{Y} \text{ and } t > T.$$

Then, $w(x,y,t)$ is called a function of exponential orders α ,δ and λ as $x,y,t \to \infty$.

Theorem 2.1. The existence condition of Gamar transform of the continuous function $w(x,y,t)$ defined on $[0,X] \times [0,Y] \times [0,T]$ is to be of exponential orders α ,δ and λ , for $\mathrm{Re}[\mathcal{W}(r)] > \alpha$, $\mathrm{Re}[\psi(s)] > \delta$ and $\mathrm{Re}[\varphi\,(v)] > \lambda$.

Theorem 2.2. Let $\mathbb{G}[w(x,y,t)] = \Psi(r,s,v)$. Then,

$$\mathbb{G}[w(x-\alpha,y-\delta,t-\lambda)H(x-\alpha,y-\delta,t-\lambda)] = e^{-\mathcal{W}(r)\alpha-\psi(s)\delta-\varphi(v)\lambda}\,\Psi(r,s,v). \quad (8)$$

where $H(x,y,t)$ denotes the unit step function defined by

$$H(x-\alpha,y-\delta,t-\lambda) = \begin{cases} 1, & x > \alpha, y > \delta, t > \lambda \\ 0, & otherwise. \end{cases}$$

Theorem 2.3. (Triple Convolution Theorem).

Let $\mathbb{G}[w(x,y,t)] = \Psi(r,s,v)$ and $\mathbb{G}[h(x,y,t)] = \mathrm{H}(r,s,v)$, then and $\mathbb{G}[\square(x,y,t)] = \mathrm{H}(r,s,v)$, then

$$\mathbb{G}[(w *** \square)(x,y,t)] = \frac{\Psi(r,s,v)H(r,s,v)}{p(r)q(s)u(v)}. (9)$$

2.2. Gamar Transform of Some Elementary Functions [28]

- $\mathbb{G}[x^n y^n t^n] = \frac{p(r)}{\mathcal{W}^{n+1}(r)} \frac{q(s)}{\psi^{n+1}(s)} \frac{u(v)}{\varphi^{n+1}(v)} \ (\Gamma(n+1))^3.$
- $\mathbb{G}[e^{ax+by+ct}] = \frac{p(r)}{(\mathcal{W}(r)-a)} \frac{q(s)}{(\psi(s)-b)} \frac{u(v)}{(\varphi(v)-c)}.$
- $\mathbb{G}[\cos(ax+by+ct)] = \frac{p(r)q(s)u(v)[\mathcal{W}(r)\psi(s)\varphi(v)-ab\varphi(v)-bc\mathcal{W}(r)-ac\psi(s)]}{(\mathcal{W}^2(r)+a^2)(\psi^2(s)+b^2)(\varphi^2(v)+c^2)}.$
- $\mathbb{G}[\sin(ax+by+ct)] = \frac{p(r)q(s)u(v)[b\mathcal{W}(r)\varphi(v)+a\psi(s)\varphi(v)+c\mathcal{W}(r)\psi(s)-abc]}{(\mathcal{W}^2(r)+a^2)(\psi^2(s)+b^2)(\varphi^2(v)+c^2)}$
- $\mathbb{G}[\cosh(ax+by+ct)] = \frac{p(r)q(s)u(v)[\mathcal{W}(r)\psi(s)\varphi(v)-ab\varphi(v)-bc\mathcal{W}(r)-ac\psi(s)]}{(\mathcal{W}^2(r)-a^2)(\psi^2(s)-b^2)(\varphi^2(v)-c^2)}.$
- $\mathbb{G}[\sinh(ax+by+ct)] = \frac{p(r)q(s)u(v)[b\mathcal{W}(r)\varphi(v)+a\psi(s)\varphi(v)+c\mathcal{W}(r)\psi(s)+abc].}{(\mathcal{W}^2(r)-a^2)(\psi^2(s)-b^2)(\varphi^2(v)-c^2)}$

2.3. Gamar Transform for Partial Differential Derivatives[28]

In this section, we present some theorems related to the new general triple integral transform of partial derivatives.

Theorem2.3. (*Derivative properties with respect to* x). Let $\Psi(r,s,v)$ is general triple transform of $w(x,y,t)$ and $G_D(0,s,v)$ is general double transform of $w(0,y,t)$, then

a) $\mathbb{G}\left[\frac{\partial w(x,y,t)}{\partial x}\right] = \mathcal{W}(r)\Psi(r,s,v) - p(r)G(0,s,v).$

b) $\mathbb{G}\left[\frac{\partial^2 w(x,y,t)}{\partial x^2}\right] = \mathcal{W}^2(r)\Psi(r,s,v) - p(r)\mathcal{W}(r)G(0,s,v) - p(r)\mathbb{T}_y\mathbb{T}_t\left[\frac{\partial w(0,y,t)}{\partial x}\right].$

c) $\mathbb{G}\left[\frac{\partial^n w(x,y,t)}{\partial x^n}\right] = \mathcal{W}^n(r)\Psi(r,s,v) - p(r)\sum_{i=0}^{n-1}\mathcal{W}^{n-1-i}(r)\mathbb{T}_y\mathbb{T}_t\left[\frac{\partial^i w(0,y,t)}{\partial x^i}\right].$

Theorem 2.4. (*Derivative properties with respect to* y). Let $\Psi(r,s,v)$ is Gamar transform of $w(x,y,t)$ and $G_D(r,0,v)$ is general double transform of $w(x,0,t)$, then

a) $\mathbb{G}\left[\frac{\partial w(x,y,t)}{\partial y}\right] = \psi(s)\Psi(r,s,v) - q(s)G_D(r,0,v).$

b) $\mathbb{G}\left[\frac{\partial^2 w(x,y,t)}{\partial y^2}\right] = \psi^2(s)\Psi(r,s,v) - q(s)\psi(s)G_D(r,0,v) - q(s)\mathbb{T}_x\mathbb{T}_t\left[\frac{\partial w(x,0,t)}{\partial y}\right].$

c) $\mathbb{G}\left[\frac{\partial^n w(x,y,t)}{\partial y^n}\right] = \psi^n(s)\Psi(r,s,v) - q(s)\sum_{i=0}^{n-1}\psi^{n-1-i}(s)\mathbb{T}_x\mathbb{T}_t\left[\frac{\partial^i w(x,0,t)}{\partial y^i}\right].$

Theorem 2.5. (*Derivative properties with respect to* t). If $\Psi(r,s,v)$ is general triple transform of $w(x,y,t)$ and $G_D(r,s,0)$ is general double transform of $w(x,y,0)$, then

a) $\mathbb{G}\left[\frac{\partial w(x,y,t)}{\partial t}\right] = \varphi(v)\Psi(r,s,v) - u(v)G_D(r,s,0).$

b) $\mathbb{G}\left[\frac{\partial^2 w(x,y,t)}{\partial t^2}\right] = \varphi^2(v)\Psi(r,s,v) - u(v)\varphi(v)G_D(r,s,0) - u(v)\mathbb{T}_x\mathbb{T}_y\left[\frac{\partial w(x,y,0)}{\partial t}\right].$

c) $\mathbb{G}\left[\frac{\partial^n w(x,y,t)}{\partial t^n}\right] = \varphi^n(v)\Psi(r,s,v) - \mathcal{u}(v)\sum_{i=0}^{n-1}\varphi^{n-1-i}(v)\mathbb{T}_x\mathbb{T}_y\left[\frac{\partial^i w(x,y,0)}{\partial t^i}\right].$

Theorem 2.6. Let $\Psi(r,s,v)$ is general triple transform of $w(x,y,t)$, then

a) $\mathbb{G}[x^n w(x,y,t)] = (-1)^n \frac{\mathcal{p}(r)}{W'(r)}\frac{\partial^n}{\partial r^n}\left(\frac{\Psi(r,s,v)}{\mathcal{p}(r))}\right).$

b) $\mathbb{G}[y^n w(x,y,t)] = (-1)^n \frac{\mathcal{q}(s)}{\psi'(s)}\frac{\partial^n}{\partial s^n}\left(\frac{\Psi(r,s,v)}{\mathcal{q}(s)}\right).$

c) $\mathbb{G}[t^n w(x,y,t)] = (-1)^n \frac{\mathcal{u}(v)}{\varphi'(v)}\frac{\partial^n}{\partial v^n}\left(\frac{\Psi(r,s,v)}{\mathcal{u}(v)}\right).$

3. Basic Facts of the Fractional Calculus [38]

In this section, some definitions, and properties of the fractional calculus, which will be used in this work, are presented.

Definition 3.1. The left Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $w \in \Re^+$ is determined as

$$I_t^\alpha w(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^t (t-\tau)^{n-\alpha-1} w(\tau)d\tau,\ \ n-1<\alpha<n\,, x>0. \tag{10}$$

where the integral on the right is convergent point wise defined over $(0,\infty)$.

Definition 3.2. The Caputo time-fractional derivative operator order $\alpha > 0$ of a function $w(t)$ on $(0,\infty)$ is defined as

$$D_t^\alpha w(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{d^n}{d\tau^n} w(\tau)d\tau\ \ , n-1<\alpha<n\,. \tag{11}$$

where the integral on the right is convergent point wise defined over $(0,\infty)$.

Definition 3.3. The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\delta}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+\delta)}\ \ , t,\delta \in \mathbb{C}\ \ , \Re(\alpha) > 0. \tag{12}$$

General triple transform of some Mittag-leffler functions are given by

- $\mathbb{G}\left[xyt^{\alpha-1}E_{\alpha,\delta}(\rho t^\alpha)\right] = \frac{\mathcal{p}(r)}{W^2(r)}\frac{\mathcal{q}(s)}{\psi^2(s)}\frac{\mathcal{u}(v)\varphi^{\alpha-\delta}(v)}{(\varphi^\alpha(v)-\rho)}.$
- $\mathbb{G}\left[y^2 t^\alpha E_{1,\alpha+1}(t)\right] = 2!\frac{\mathcal{p}(r)}{W(r)}\frac{\mathcal{q}(s)}{\psi^3(s)}\frac{\mathcal{u}(v)}{\varphi(v)(\varphi(v)-1)}.$
- $\mathbb{G}\left[t^{2\alpha}E_{\alpha,2\alpha+1}(t)\right] = \frac{\mathcal{p}(r)}{W(r)}\frac{\mathcal{q}(s)}{\psi(s)}\frac{\mathcal{u}(v)}{\varphi^{2\alpha}(v)(\varphi(v)-1)}.$

Theorem 3.1: Let $\tau,\delta,\mu > 0$, $p-1<\tau\le p$, $m-1<\delta\le m$, $n-1<\mu\le n$ and $p,m,n\in\mathbb{N}$, so that $f\in C^l(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+), l=\max\{p,m,n\}$, $w^{(l)}\in L_1[(0,\beta)\times(0,\theta)\times(0,\vartheta)]$ for any positive β,θ and ϑ and let $|w(x,y,t)|\le\mathbb{K}e^{\tau x+\delta y+\mu t}$, $\exists\mathbb{K}>0$, $\forall x>\beta>0,\ y>\theta>0\ \ and\ \ t>\vartheta>0$ holds for constant $\mathbb{K},\tau,\delta,\mu>0$ .Then the general triple integral transforms of Caputo's fractional derivatives $D_x^\tau w(x,y,t), D_y^\delta\, w(x,y,t)\ \ and\ \ D_t^\mu\, w(x,y,t)$ are defined by

a) $\mathbb{G}[D_x^{\tau} w(x,y,t)]$

$= \mathcal{W}^{\tau}(r)\Psi(r,s,v)$

$- \wp(r)\sum_{i=0}^{n-1} \mathcal{W}^{\tau-1-i}(r)\mathbb{T}_y\mathbb{T}_t\left[\frac{\partial^i w(0,y,t)}{\partial x^i}\right].$

b) $\mathbb{G}\left[D_y^{\delta}\, w(x,y,t)\right]$

$= \psi^{\delta}(s)\Psi(r,s,v)$

$- q(s)\sum_{i=0}^{n-1} \psi^{\delta-1-i}(s)\mathbb{T}_x\mathbb{T}_t\left[\frac{\partial^i w(x,0,t)}{\partial y^i}\right].$

c) $\mathbb{G}\left[D_t^{\mu}\, w(x,y,t)\right]$

$= \varphi^{\mu}(v)\Psi(r,s,v)$

$- u(v)\sum_{i=0}^{n-1} \varphi^{\mu-1-i}(v)\mathbb{T}_x\mathbb{T}_y\left[\frac{\partial^i w(x,y,0)}{\partial t^i}\right].$

4. The Gamar Transform Adomian decomposition method

In this part of the paper, we give the fundamental idea of the Gamar Adomian decomposition method (GADM) for the two-dimensional time-fractional Navier–Stokes Equations. In order to show the fundamental plan of the general triple Adomian decomposition method, we consider the following system of two-dimensional time-fractional Navier–Stokes Equations:

$$D_t^{\alpha} w + w\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} - \rho_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = -\frac{1}{\rho}\frac{\partial r}{\partial x},$$
$$D_t^{\alpha} m + w\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} - \rho_0\left(\frac{\partial^2 m}{\partial x^2}+\frac{\partial^2 m}{\partial y^2}\right) = -\frac{1}{\rho}\frac{\partial r}{\partial y},$$
$$x,y,t>0, \qquad n-1<\alpha<n. \quad (13)$$

Subject to the conditions

$$w(x,y,0) = k(x,y),$$
$$m(x,y,0) = h(x,y). \quad (14)$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is fractional Caputo derivative , r is the pressure, in addition if r is known . Put $\mu = \frac{1}{\rho}\frac{\partial r}{\partial x}$ and $\tau = \frac{1}{\rho}\frac{\partial r}{\partial y}$.

Applying the Gamar transform for Eq. (13), we obtain

$$D_t^{\alpha} w + w\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} - \rho_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = -\mu,$$
$$D_t^{\alpha} m + w\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} - \rho_0\left(\frac{\partial^2 m}{\partial x^2}+\frac{\partial^2 m}{\partial y^2}\right) = -\tau,$$
$$x,y,t>0. \quad (15)$$

By linearity property and partial derivative properties of Gamar transform, we get

$$W(r,s,v) = \frac{u(v)}{\varphi(v)}K(r,s) + \frac{\rho_0}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_{xx}+w_{yy}\right]$$
$$-\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[ww_x + mw_y\right] - \frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\,\mu]. \quad (16)$$

$$M(r,s,v) = \frac{u(v)}{\varphi(v)}H(r,s) + \frac{\rho_0}{\varphi^{\alpha}(v)}\mathbb{G}\left[m_{xx}+m_{yy}\right]$$
$$-\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[wm_x + mm_y\right] - \frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\,\tau]. \quad (17)$$

Taking inverse Gamar transform to Eqs.(16) and (17),we get

$$w(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi(v)}K(r,s)\right] + \mathbb{G}^{-1}\left[\frac{\rho_0}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_{xx}+w_{yy}\right]\right]$$
$$-\mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[ww_x+mw_y\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\,\mu]\right]. \quad (18)$$

$$m(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi\,(v)}H(r,s)\right] + \mathbb{G}^{-1}\left[\frac{\rho_0}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[m_{xx}+m_{yy}\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[wm_x+mm_y\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}[\,\tau]\right]. \quad (19)$$

The Adomian decomposition method admits the decomposition of $w(x,y,t)$ and $m(x,y,t)$ into infinite series components as follows

$$w(x,y,t) = \sum_{n=0}^{\infty} w_n\,(x,y,t)\,,\qquad m(x,y,t) = \sum_{n=0}^{\infty} m_n\,(x,y,t). \quad (20)$$

and the nonlinear terms ww_x, mw_y , wm_x and mm_y be equated to an infinite series of polynomials as follows

$$ww_x = \sum_{n=1}^{\infty} A_n,\qquad mw_y = \sum_{n=1}^{\infty} B_n\;,\qquad wm_x = \sum_{n=1}^{\infty} C_n\;, m_y = \sum_{n=1}^{\infty} D_n\;. \quad (21)$$

where A_n , B_n , C_n and D_n are He's polynomial.

substituting Eqs.(20) and (21) into Eqs.(18) and (19),we have

$$\sum_{n=0}^{\infty} w_n\,(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi\,(v)}K(r,s)\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}[\mu]\right] + \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[\rho_0\left(\sum_{n=0}^{\infty} w_{n_{xx}} + \sum_{n=0}^{\infty} w_{n_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right]. \quad (22)$$

$$\sum_{n=0}^{\infty} m_n\,(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi\,(v)}H(r,s)\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}[\,\tau]\right] + \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[\rho_0\left(\sum_{n=0}^{\infty} m_{n_{xx}} + \sum_{n=0}^{\infty} m_{n_{yy}}\right) - \tau\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}\,(v)}\mathbb{G}\left[\sum_{n=0}^{\infty} C_n + \sum_{n=0}^{\infty} D_n\right]\right]. \quad (23)$$

The various components $w_n(x,y,t)$ and $m_n(x,y,t)$ of the solutions $w(x,y,t)$ and $m(x,y,t)$ respectively can be easily determined by using the following recursive relations

$$w_0(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi\,(v)}K(r,s)\right] - \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)},\qquad m_0(x,y,t) = \mathbb{G}^{-1}\left[\frac{u(v)}{\varphi\,(v)}H(r,s)\right] - \frac{\tau\, t^{\alpha}}{\Gamma(\alpha+1)}. \quad (24)$$

and,

$$w_{n+1}(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(w_{n_{xx}} + w_{n_{yy}}\right) - \mu\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[A_n + B_n]\right], n \geq 0. \quad (25)$$

$$m_{n+1}(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{n_{xx}} + m_{n_{yy}}\right) - \tau\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[C_n + D_n]\right], n \geq 0. \quad (26)$$

provided that the Gamar transform exist for Eq.(24),(25) and (26).

Note that, the first few terms of the Adomian polynomials A_n, B_n, C_n and D_n are given by

$$A_0 = w_0 w_{0_x} ,$$

$$A_1 = w_0 w_{1_x} + w_1 w_{0_x} , \quad (27)$$

$$A_2 = w_0 w_{2_x} + w_1 w_{1_x} + w_2 w_{0_x} ,$$

$$\vdots$$

$$B_0 = m_0 w_{0_y} ,$$

$$B_1 = m_0 w_{1_y} + m_1 w_{0_y} , \quad (28)$$

$$B_2 = m_0 w_{2_y} + m_1 w_{1_y} + m_2 w_{0_y} ,$$

$$\vdots$$

$$C_0 = w_0 m_{0_x} ,$$

$$C_1 = w_0 m_{1_x} + w_1 m_{0_x} , \quad (29)$$

$$C_2 = w_0 m_{2_x} + w_1 m_{1_x} + w_2 m_{0_x}$$

$$\vdots$$

$$D_0 = m_0 m_{0_y} ,$$

$$D_1 = m_0 m_{1_y} + m_1 m_{0_y} , \quad (30)$$

$$D_2 = m_0 m_{2_y} + m_1 m_{1_y} + m_2 m_{0_y}$$

$$\vdots$$

Thus, the solutions are

$$w(x,y,t) = \sum_{n=0}^{\infty} w_n(x,y,t) ,$$

$$m(x,y,t) = \sum_{n=0}^{\infty} m_n(x,y,t).$$

5. Applications

In this section of this paper, we discuss the achievement of our present methods and examine its accuracy by using the decomposition method with connection of the Gamar transform.

Example 5.1

Consider the time-fractional order two-dimensional Navier–Stokes Equation

$$\begin{aligned} \frac{\partial^{\alpha} w}{\partial t^{\alpha}} + w\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} - \rho_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -\mu, \\ \frac{\partial^{\alpha} m}{\partial t^{\alpha}} + w\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} - \rho_0\left(\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2}\right) = -\mu. \end{aligned} \quad (31)$$

where $x, y, t > 0$ and $n - 1 < \alpha < n$.

Subject to the conditions

$$\begin{aligned} w(x,y,0) &= -e^{\,x+y}, \\ m(x,y,0) &= e^{\,x+y}. \end{aligned} \quad (32)$$

Applying the Gamar transform for Eq. (31), we obtain

$$\begin{aligned} \mathbb{G}\left[D_t^\alpha w + ww_x + mw_y\right] &= \mathbb{G}\left[\rho_0\left(w_{xx} + w_{yy}\right) + \mu\right], \\ \mathbb{G}\left[D_t^\alpha m + wm_x + mm_y\right] &= \mathbb{G}\left[\rho_0\left(m_{xx} + m_{yy}\right) - \mu\right]. \end{aligned} \quad (33)$$

By linearity property and partial derivative properties of Gamar transform, we get

$$\begin{aligned} W(r,s,v) = &-\frac{u(v)}{\varphi(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)} \\ &+ \frac{1}{\varphi^\alpha(v)}\mathbb{G}[\mu] \\ &+ \frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{xx} + w_{yy}\right)\right] \\ &- \frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[ww_x + mw_y\right], \end{aligned} \quad (34)$$

$$\begin{aligned} M(r,s,v) = &\frac{u(v)}{\varphi(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)} \\ &- \frac{1}{\varphi^\alpha(v)}\mathbb{G}[\mu] \\ &+ \frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(m_{xx} + m_{yy}\right)\right] \\ &- \frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[wm_x + mm_y\right]. \end{aligned} \quad (35)$$

Taking inverse Gamar transform to Eqs.(34) and (35),we get

$$\begin{aligned} w(x,y,t) = &-e^{\,x+y} + \frac{\mu t^\alpha}{\Gamma(\alpha+1)} \\ &+ \mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{xx} + w_{yy}\right)\right]\right] \\ &- \mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[ww_x + mw_y\right]\right]. \end{aligned} \quad (36)$$

$$\begin{aligned} m(x,y,t) = &e^{\,x+y} - \frac{\mu t^\alpha}{\Gamma(\alpha+1)} \\ &+ \mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(m_{xx} + m_{yy}\right)\right]\right] \\ &- \mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[wm_x + mm_y\right]\right]. \end{aligned} \quad (37)$$

substituting Eqs.(20) and (21) into Eqs.(36) and (37) ,we have

$$\begin{aligned} w_0(x,y,t) &= -e^{\,x+y} + \frac{\mu t^\alpha}{\Gamma(\alpha+1)}, \\ m_0(x,y,t) &= e^{\,x+y} - \frac{\mu t^\alpha}{\Gamma(\alpha x+1)}. \end{aligned} \quad (38)$$

$$\begin{aligned} w_{n+1}(x,y,t) = &\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{n_{xx}} + w_{n_{yy}}\right)\right]\right] \\ &- \mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}[A_n + B_n]\right], \\ &n \geq 0. \end{aligned} \quad (39)$$

$$m_{n+1}(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{n_{xx}} + m_{n_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[C_n + D_n]\right], \quad n \geq 0. \qquad (40)$$

Putting $n = 0$ into Eq.(39) and Eq.(40) and using Eqs.(27-30), we get

$$w_1(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(w_{0_{xx}} + w_{0_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_0 w_{0_x} + m_0 w_{0_y}\right]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[-2\rho_0 e^{\,x+y}]\right]$$

$$= -2\rho_0\mathbb{G}^{-1}\left[\frac{u(v)}{\varphi^{\alpha+1}(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)}\right]$$

$$= -2\rho_0\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{\,x+y}.$$

in the same way, we get

$$m_1(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{0_{xx}} + m_{0_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_0 m_{0_x} + m_0 m_{0_y}\right]\right]$$

$$= -\mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[2\rho_0 e^{\,x+y}]\right]$$

$$= 2\rho_0\mathbb{G}^{-1}\left[\frac{u(v)}{\varphi^{\alpha+1}(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)}\right]$$

$$= 2\rho_0\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{\,x+y}.$$

Similarly if $n = 1$,

$$w_2(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(w_{1_{xx}} + w_{1_{yy}}\right)\right]\right]$$

$$= -\mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[(2\rho_0)^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{\,x+y}\right]\right]$$

$$= -(2\rho_0)^2\mathbb{G}^{-1}\left[\frac{u(v)}{\varphi^{2\alpha+1}(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)}\right]$$

$$= -(2\rho_0)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{\,x+y}.$$

and,

$$m_2(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{1_{xx}} + m_{1_{yy}}\right)\right]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[(2\rho_0)^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{\,x+y}\right]\right]$$

$$= (2\rho_0)^2\mathbb{G}^{-1}\left[\frac{u(v)}{\varphi^{2\alpha+1}(v)}\frac{p(r)}{(\mathcal{W}(r)-1)}\frac{q(s)}{(\psi(s)-1)}\right]$$

$$= (2\rho_0)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{\,x+y}.$$

In the same manner, we have

$$\left.\begin{aligned} w_n(x,y,t) &= -\frac{(2\rho_0)^n\, t^{n\alpha}}{\Gamma(n\alpha+1)}e^{\,x+y}, \\ m_n(x,y,t) &= \frac{(2\rho_0)^n\, t^{n\alpha}}{\Gamma(n\alpha+1)}e^{\,x+y}, \end{aligned}\right. \quad \forall n \geq 1.$$

Therefore, the solution of Eq.(31) is given by

$$w(x,y,t) = w_0 + w_1 + w_2 + \cdots + w_n + \cdots,$$

$$m(x,y,t) = m_0 + m_1 + m_2 + \cdots + m_n + \cdots.$$

$$w(x,y,t) = \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)} - e^{\,x+y}\sum_{n=0}^{\infty}(2\rho_0)^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= -e^{\,x+y}E_{\alpha,1}(2\rho_0 t^{\alpha}) + \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)},$$

$$m(x,y,t) - \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)} + e^{\,x+y} \sum_{n=0}^{\infty} (2\rho_0)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$
$$= e^{\,x+y} E_{\alpha,1}(2\rho_0 t^{\alpha}) - \frac{\mu t^{\alpha}}{\Gamma(\alpha+1)}.$$

By taking $\alpha = 1$ and $\mu = 0$, then the exact solution of the classical Navier–Stokes Equation for the velocity is

$$w(x,y,t) = -e^{x+y+2\rho_0 t},$$

$$m(x,y,t) = e^{x+y+2\rho_0 t}.$$

The following figures, Fig.1 illustrates the 3D graph of exact solution of Example 5.1, for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.4$.

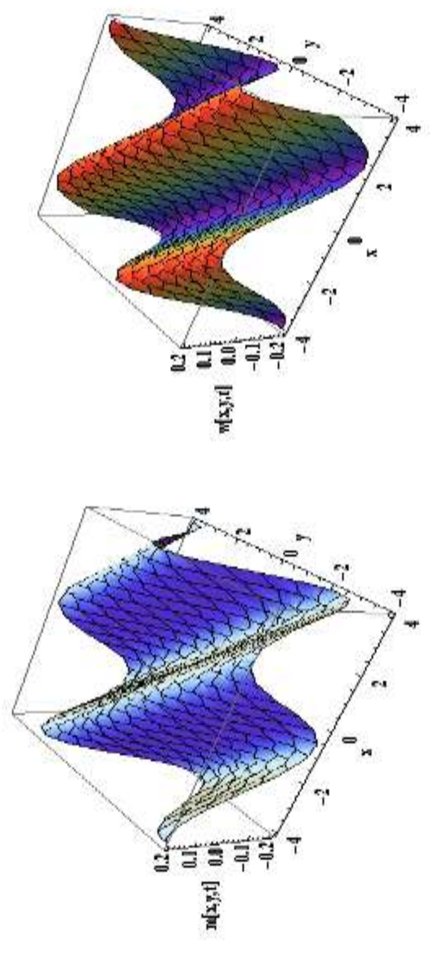

Fig.1. The 3D graph of exact solution of Eq.(31) for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.4$.

The following figures, Fig.2 illustrates the 3D graph of exact solution of Example 5.1, for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.6$.

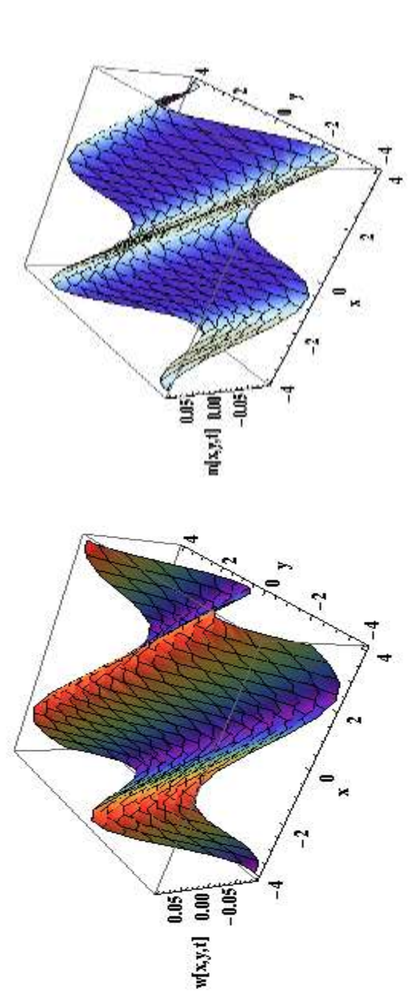

Fig.2. The 3D graph of exact solution of Eq.(31) for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.6$.

Example 5.2

Consider the time-fractional order two-dimensional Navier–Stokes Equation

$$\begin{aligned} &\frac{\partial^{\alpha} w}{\partial t^{\alpha}} + w\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} - \rho_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \mu, \\ &\frac{\partial^{\alpha} m}{\partial t^{\alpha}} + w\frac{\partial m}{\partial x} + m\frac{\partial m}{\partial y} - \rho_0\left(\frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2}\right) = -\mu. \end{aligned} \tag{41}$$

where $x, y, t > 0$ and $n - 1 < \alpha < n$.

Subject to the conditions

$$\begin{aligned} w(x,y,0) &= -\sin(x+y), \\ m(x,y,0) &= \sin(x+y). \end{aligned} \tag{42}$$

Applying the Gamar transform for Eq. (41), we obtain

$$\begin{aligned} \mathbb{G}\left[D_t^\alpha w + ww_x + mw_y\right] &= \mathbb{G}\left[\rho_0\left(w_{xx}+w_{yy}\right) + \mu\right], \\ \mathbb{G}\left[D_t^\alpha m + wm_x + mm_y\right] &= \mathbb{G}\left[\rho_0\left(m_{xx}+m_{yy}\right) - \mu\right]. \end{aligned} \tag{43}$$

By linearity property and partial derivative properties of Gamar transform, we get

$$\begin{aligned} W(r,s,v) = &-\frac{u(v)}{\varphi(v)}\frac{p(r)q(s)[\mathcal{W}(r)+\psi(s)]}{(\mathcal{W}^2(r)+1)(\psi^2(s)+1)} \\ &+\frac{1}{\varphi^\alpha(v)}\mathbb{G}[\mu] \\ &+\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{xx}+w_{yy}\right)\right] \\ &-\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[ww_x+mw_y\right]. \end{aligned} \tag{44}$$

$$\begin{aligned} M(r,s,v) = &\frac{u(v)}{\varphi(v)}\frac{p(r)q(s)[\mathcal{W}(r)+\psi(s)]}{(\mathcal{W}^2(r)+1)(\psi^2(s)+1)} \\ &-\frac{1}{\varphi^\alpha(v)}\mathbb{G}[\mu] \\ &+\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(m_{xx}+m_{yy}\right)\right] \\ &-\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[wm_x+mm_y\right]. \end{aligned} \tag{45}$$

Taking inverse Gamar transform to Eqs.(44) and (45),we get

$$\begin{aligned} w(x,y,t) = &-\sin(x+y)+\frac{\mu t^\alpha}{\Gamma(\alpha+1)} \\ &+\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{xx}+w_{yy}\right)\right]\right] \\ &-\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[ww_x + mw_y\right]\right]. \end{aligned} \tag{46}$$

$$\begin{aligned} m(x,y,t) = &\sin(x+y)-\frac{\mu t^\alpha}{\Gamma(\alpha+1)} \\ &+\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(m_{xx} + m_{yy}\right)\right]\right] \\ &-\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[wm_x + mm_y\right]\right]. \end{aligned} \tag{47}$$

substituting Eq.(20) and Eq.(21) into Eq.(46) and Eq.(47) , we have

$$\begin{aligned} w_0(x,y,t) &= -\sin(x+y)+\frac{\mu t^\alpha}{\Gamma(\alpha+1)}, \\ m_0(x,y,t) &= \sin(x+y)-\frac{\mu t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{48}$$

and,

$$\begin{aligned} w_{n+1}(x,y,t) = &\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(w_{n_{xx}} + w_{n_{yy}}\right)\right]\right] \\ &-\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[A_n + B_n\right]\right], \qquad n \geq 0. \end{aligned} \tag{49}$$

$$\begin{aligned} m_{n+1}(x,y,t) = &\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[\rho_0\left(+m_{n_{yy}}\right)\right]\right] \\ &-\mathbb{G}^{-1}\left[\frac{1}{\varphi^\alpha(v)}\mathbb{G}\left[C_n + D_n\right]\right], \qquad n \geq 0. \end{aligned} \tag{50}$$

Putting $n = 0$ into Eq.(49) and Eq.(50) and using Eqs.(27-30), we get

$$w_1(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(w_{0_{xx}} + w_{0_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_0 w_{0_x} + m_0 w_{0_y}\right]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[2\rho_0 \sin(x+y)]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\sin(x+y)\cos(x+y) - \sin(x+y)\cos(x+y)]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[2\rho_0 \sin(x+y)]\right] = 2\rho_0\mathbb{G}^{-1}\left[\frac{u(v)p(r)q(s)[\mathcal{W}(r)+\psi(s)]}{\varphi^{\alpha+1}(v)(\mathcal{W}^2(r)+1)(\psi^2(s)+1)}\right]$$

$$= 2\rho_0\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x+y).$$

in the same way, we get

$$m_1(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{0_{xx}} + m_{0_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[w_0 m_{0_x} + m_0 m_{0_y}\right]\right]$$

$$= -\mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[2\rho_0 \sin(x+y)]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\sin(x+y)\cos(x+y) - \sin(x+y)\cos(x+y)]\right]$$

$$= -2\rho_0\mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[\sin(x+y)]\right] = -2\rho_0\mathbb{G}^{-1}\left[\frac{u(v)p(r)q(s)[\mathcal{W}(r)+\psi(s)]}{\varphi^{\alpha+1}(v)(\mathcal{W}^2(r)+1)(\psi^2(s)+1)}\right]$$

$$= -2\rho_0\frac{t^{\alpha}}{\Gamma(\alpha+1)}\sin(x+y).$$

Similarly if $n = 1$,

$$w_2(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(w_{1_{xx}} + w_{1_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[A_1 + B_1]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\frac{-(2\rho_0)^2 t^{\alpha}}{\Gamma(\alpha+1)}\sin(x+y)\right]\right] = -(2\rho_0)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x+y).$$

and,

$$m_2(x,y,t) = \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\rho_0\left(m_{1_{xx}} + m_{1_{yy}}\right)\right]\right] - \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}[C_1 + D_1]\right]$$

$$= \mathbb{G}^{-1}\left[\frac{1}{\varphi^{\alpha}(v)}\mathbb{G}\left[\frac{(2\rho_0)^2 t^{\alpha}}{\Gamma(\alpha+1)}\sin(x+y)\right]\right] = (2\rho_0)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x+y).$$

In the same manner, we have

$$\begin{aligned} w_n(x,y,t) &= \frac{(-2\rho_0)^n\, t^{n\alpha}}{\Gamma(n\alpha+1)}\sin(x+y), \\ m_n(x,y,t) &= \frac{(-2\rho_0)^n\, t^{n\alpha}}{\Gamma(n\alpha+1)}\sin(x+y). \end{aligned} \quad \forall n \geq 1.$$

Therefore, the solution of Eq.(41) is given by

$$w(x,y,t) = w_0 + w_1 + w_2 + \cdots + w_n + \cdots,$$

$$m(x,y,t) = m_0 + m_1 + m_2 + \cdots + m_n + \cdots.$$

$$w(x,y,t) =$$

$$-\sin(x+y)\sum_{n=0}^{\infty}(-2\rho_0)^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}+\frac{\mu t^{\alpha}}{\Gamma(\alpha+1)}$$

$$m(x,y,t) =$$

$$\sin(x+y)\sum_{n=0}^{\infty}(-2\rho_0)^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}-\frac{\mu t^{\alpha}}{\Gamma(\alpha+1)}$$

By taking $\alpha = 1$ and $\mu = 0$, then the exact solution of the classical Navier–Stokes Equation for the velocity is

$$w(x,y,t) = -\sin(x+y)\, e^{-2\rho_0 t},$$

$$m(x,y,t) = \sin(x+y)\, e^{-2\rho_0 t}.$$

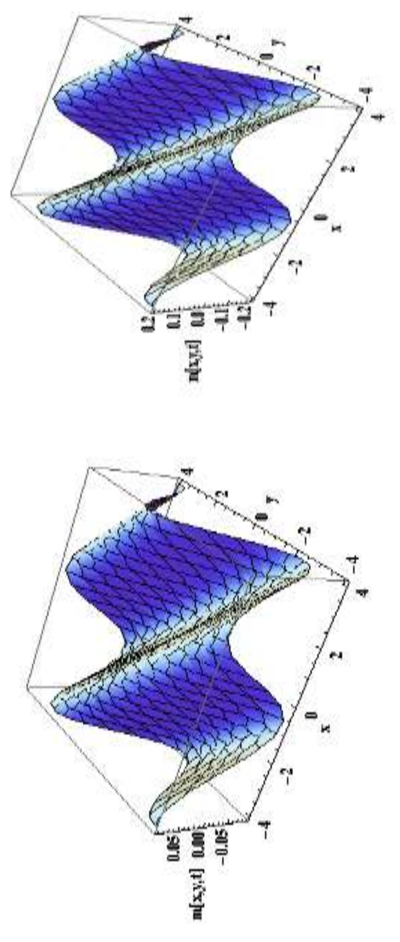

Fig.3. The 3D graph of exact solution of Eq.(41) for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.4$.

The above figures, Fig.3 illustrates the 3D graph of exact solution of Example 5.2, for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.4$.

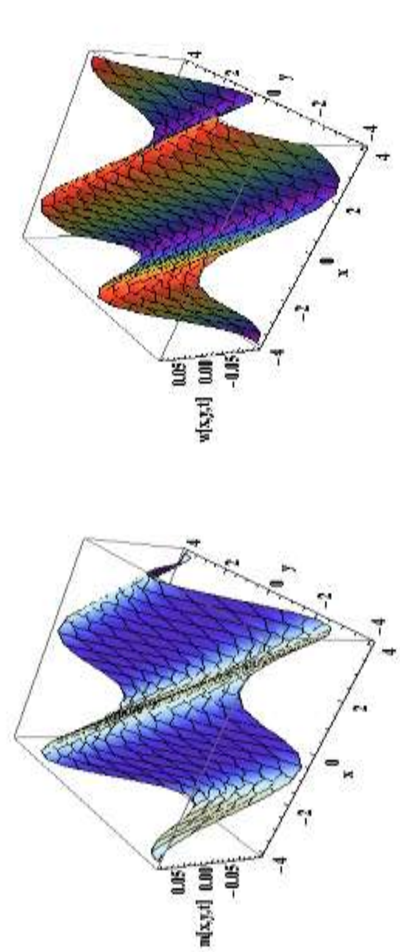

Fig.4. The 3D graph of exact solution of Eq.(41) for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.6$

Thee above figures, Fig.4 illustrates the 3D graph of exact solution of Example 5.2, for $\mu = 0$, $\alpha = 1$, $t = 2$ and $\rho_0 = 0.4$.

5. Concluding Remarks

General triple transform Adomian decomposition method is proposed in this paper as a solution to multi-dimensional fractional Navier-Stokes Eqution. Adopting this powerful method, fulfills the dual goal of managing fractional order partial differential equations, while maintaining high levels of mathematical accuracy. We merely need to change the number of iterations. Hence, it can plausibly obtaining argued that GTTADM is a powerful method in exact and numerical solutions to multi-dimensional Navier-Stokes Equation.

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