A first-passage-time problem for a discrete-time Markov process

MARIO LEFEBVRE
Department of Mathematics and Industrial Engineering
Polytechnique Montréal
2500, chemin de Polytechnique, Montréal (Québec) H3T 1J4
CANADA

Abstract: We consider the discrete-time stochastic process $\{X_n, n = 0, 1, \ldots\}$ defined by $X_{n+1} = X_n - \epsilon_{n+1}$, where ϵ_{n+1} is a non-negative random variable. The aim is to compute the mean first-passage time to zero for this process, which can be used as a model for the remaining lifetime of a machine. Particular cases are solved exactly and explicitly.

Key-Words: Mean remaining lifetime, reliability, wear process, integral equation, Laplace transform.

Received: June 23, 2023. Revised: March 12, 2024. Accepted: May 14, 2024. Published: June 6, 2024.

1 Introduction

Let $\{\epsilon_n, n = 1, 2, \ldots\}$ be an infinite set of independent and identically distributed (i.i.d.) non-negative random variables. We consider the discrete-time Markov process $\{X_n, n = 0, 1, \ldots\}$ defined by

$$X_{n+1} = X_n - \epsilon_{n+1}. \tag{1}$$

This stochastic process could be used to model the *remaining lifetime* of a machine.

Remark. If we define

$$X_{n+1} = X_n + \epsilon_{n+1} \tag{2}$$

instead, then $\{X_n, n = 0, 1, \ldots\}$ could be a *wear process*.

Our aim is to determine the expected value of the *first-passage time*

$$T(x) = \min\{n \geq 0 : X_n \leq 0 \mid X_0 = x \geq 0\}. \tag{3}$$

In theory, the random variables ϵ_n could be of discrete, continuous or even mixed type. In this paper, we consider the case when they are continuous. Then, to obtain the mean

$$m(x) := E[T(x)], \tag{4}$$

one possibility is to solve an integral equation. This integral equation can sometimes be transformed into a differential equation. Another possibility is to first compute the Laplace transform of the function $m(x)$ and then try to invert this transform. We assume that $m(x)$ exists (and is finite), which should be the case because $\epsilon_n \geq 0$.

Remark. A related model that could be used for the remaining lifetime is

$$X_{n+1} = \mu + X_n + Z_{n+1}, \tag{5}$$

where $\mu < 0$ and $\{Z_n, n = 1, 2, \ldots\}$ is a *white noise* process, such that $E[Z_n] = 0$, $E[Z_n^2] = \sigma^2$ and $E[Z_n Z_m] = 0$ for any $m \neq n$. Then, $\{X_n, n = 0, 1, \ldots\}$ would be a random walk process with drift μ or an *autoregressive process of order* 1, which is denoted by AR(1). First-passage-time problems for this type of stochastic processes have been considered, in particular, by, Novikov [1] and, [2], Larralde [3], Basak and Ho [4], Novikov and Kordzakhia [5], Di Nardo [6] and, Baumgarten [7]. In these papers, the authors often assume that the random variable Z_{n+1} has a Gaussian distribution, so that $\{Z_n, n = 1, 2, \ldots\}$ is a (discrete) *Gaussian white noise*. Moreover, they generally try to compute numerical solutions to the appropriate integral equations and/or obtain bounds on $E[T(x)]$ or other quantities. Recent papers on related problems are, Rahimov *et al.* [8] and, Aliyev *et al.* [9].

2 Mean first-passage time to zero

Assume that the set S_{ϵ_1} of possible values of the random variable ϵ_1 is the interval $[0, \infty)$. Then, for any

$x > 0$, we can write that

$$\begin{aligned} m(x) &= 1 + E[m(x - \epsilon_1)] \\ &= 1 + \int_0^\infty m(x-z)\, f_{\epsilon_1}(z)\, dz \\ &= 1 + \int_0^x m(x-z)\, f_{\epsilon_1}(z)\, dz \\ &\overset{y=x-z}{=} 1 + \int_0^x m(y)\, f_{\epsilon_1}(x-y)\, dy. \quad (6) \end{aligned}$$

Therefore, to obtain the function $m(x)$, one can try to solve the above integral equation, which is an inhomogeneous Fredholm integral equation of the second kind (see, for instance, Arfken [10], p. 865 or, Arfken *et al.* [11], p. 1047).

Next, we can also write that

$$\begin{aligned} m(x) &= 1 + \int_0^x m(x-z)\, f_{\epsilon_1}(z)\, dz \\ &= 1 + \int_0^\infty u_z(x) m(x-z)\, f_{\epsilon_1}(z)\, dz, \quad (7) \end{aligned}$$

where $u_z(x)$ is the Heaviside step function defined by

$$u_z(x) = \begin{cases} 0 & \text{if } x - z < 0, \\ 1 & \text{if } x - z \geq 0. \end{cases} \quad (8)$$

Let $\mathcal{L}(s)$ denote the Laplace transform of the function $m(x)$:

$$\mathcal{L}(s) := \int_0^\infty e^{-sx} m(x)\, dx, \quad (9)$$

where we assume that $s \in (0, \infty)$. Then, taking the Laplace transform of both sides of Eq. (7), we obtain that

$$\begin{aligned} \mathcal{L}(s) &= \frac{1}{s} + \int_0^\infty e^{-sx} \left[\int_0^\infty u_z(x) m(x-z) \right. \\ &\qquad \left. f_{\epsilon_1}(z)\, dz \right] dx \\ &= \frac{1}{s} + \int_0^\infty \left[\int_0^\infty e^{-sx} u_z(x) m(x-z)\, dx \right] \\ &\qquad f_{\epsilon_1}(z)\, dz \\ &= \frac{1}{s} + \int_0^\infty e^{-sz} \mathcal{L}(s)\, f_{\epsilon_1}(z)\, dz. \quad (10) \end{aligned}$$

Hence, we can state the following proposition.

Proposition 2.1. *If the moment-generating function*

$$M_{\epsilon_1}(s) := \int_0^\infty e^{-sz} f_{\epsilon_1}(z)\, dz \quad (11)$$

of the random variable ϵ_1 exists, then we can write that

$$\mathcal{L}(s) = \frac{1/s}{1 - M_{\epsilon_1}(s)}. \quad (12)$$

Remark. Proposition 2.1 gives us another possibility to determine the mean first-passage time $m(x)$, if we are able to invert the Laplace transform $\mathcal{L}(s)$. It might actually be easier to proceed this way than trying to solve the integral equation (6). Moreover, in the case when we are not able to solve Eq. (6) or to invert the Laplace transform, the function $\mathcal{L}(s)$ at least gives us some information about $m(x)$.

A third possibility is to try to transform the integral equation (6) into a differential equation. Differentiating both sides of Eq. (6) with respect to x, we deduce from Leibniz integral rule that

$$m'(x) = m(x)\, f_{\epsilon_1}(0) + \int_0^x m(y) \frac{d f_{\epsilon_1}(x-y)}{dx}\, dy. \quad (13)$$

As will be seen in the next section, it is sometimes possible to express the above integral in terms of the function $m(x)$.

3 Particular cases

In this section, we will consider various particular cases for which we are able to compute $m(x)$ explicitly and exactly.

Case I. Assume first that ϵ_1 has an exponential distribution with parameter λ, such that

$$f_{\epsilon_1}(z) = u_0(z)\, \lambda e^{-\lambda z}. \quad (14)$$

The integral equation (6) becomes

$$m(x) = 1 + \int_0^x m(y)\, \lambda e^{-\lambda(x-y)}\, dy. \quad (15)$$

With the help of the mathematical software program *Maple*, we find that the solution to the above equation is

$$m(x) = 1 + \lambda x. \quad (16)$$

Next, the moment-generating function of ϵ_1 is given by

$$M_{\epsilon_1}(s) = \frac{\lambda}{\lambda + s}, \quad (17)$$

provided that $s > -\lambda$, which holds true because we assumed that $s > 0$. Substituting this expression into Eq. (12), we get

$$\mathcal{L}(s) = \frac{\lambda + s}{s^2}. \quad (18)$$

We find that the inverse Laplace transform of $\mathcal{L}(s)$ is indeed $m(x) = 1 + \lambda x$.

Finally, we have, for $x - y \geq 0$,

$$\frac{d f_{\epsilon_1}(x-y)}{dx} = -\lambda f_{\epsilon_1}(x-y). \quad (19)$$

Therefore, making use of Eq. (6), we can rewrite Eq. (13) as follows:

$$m'(x) = m(x)\, f_{\epsilon_1}(0) - \lambda\, [m(x) - 1]\,. \tag{20}$$

That is,

$$m'(x) \equiv \lambda. \tag{21}$$

Hence, since

$$\lim_{x \downarrow 0} m(x) = 1, \tag{22}$$

we conclude that $m(x) = 1 + \lambda x$.

Remark. Notice that there is a discontinuity at the boundary, because $m(0) = 0$, but if $x > 0$ then $m(x) \geq 1$.

Thus, when $\epsilon_1 \sim \text{Exp}(\lambda)$, the three techniques considered to calculate the mean first-passage time $m(x)$ enable us to obtain this function.

Case II. Suppose now that

$$f_{\epsilon_1}(z) = u_0(z)\, \frac{\lambda\, (\lambda\, z)^{\alpha-1}\, e^{-\lambda z}}{\Gamma(\alpha)}, \tag{23}$$

where $\alpha, \lambda > 0$ and $\Gamma(\cdot)$ is the gamma function. That is, ϵ_1 has a gamma distribution with parameters α and λ. This time, *Maple* is not able to provide a solution to the corresponding integral equation for any α and λ. If we let $\alpha = 2$, *Maple* gives us the following solution:

$$m(x) = \frac{N(x)}{\lambda^4 x^4 + 12}, \tag{24}$$

where

$$\begin{aligned} N(x) \;\; := \;\; & \lambda^4 x^4 - 4\lambda^3 x^3 + 6\lambda^2 x^2 + 24 - 2\, e^{-\lambda x} \\ & (\lambda^3 x^3 + 3\lambda^2 x^2 + 6\, \lambda x + 6). \end{aligned} \tag{25}$$

Since the moment-generating function of $\epsilon_1 \sim \text{G}(2, \lambda)$ is

$$M_{\epsilon_1}(s) = \left(\frac{\lambda}{\lambda + s} \right)^2, \tag{26}$$

the Laplace transform of $m(x)$ is given by

$$\mathcal{L}(s) = \frac{1/s}{1 - \left(\frac{\lambda}{\lambda+s} \right)^2}. \tag{27}$$

The inverse Laplace transform of $\mathcal{L}(s)$ is found to be

$$m(x) = \frac{1}{4}\, e^{-2\lambda x} + \frac{1}{2}\, \lambda x + \frac{3}{4}, \tag{28}$$

which is different from the expression in Eq. (24). Figure 1 presents the functions $m(x)$ defined in Eq. (24) and Eq. (28) in the interval $[0, 1]$, when $\lambda = 1$.

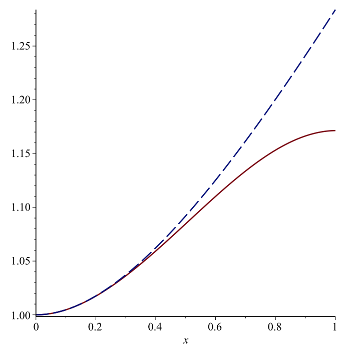

Figure 1: Functions $m(x)$ defined in Eq. (24) (solid line) and in Eq. (28) (dashed line) when $\lambda = 1$, for $0 \leq x \leq 1$.

The integral equation that we must solve when $\epsilon_1 \sim \text{G}(2, \lambda)$ is

$$m(x) = 1 + \int_0^x m(y)\, \lambda^2\, (x - y)\, e^{-\lambda(x-y)}\, dy. \tag{29}$$

We can transform it into an ordinary differential equation (ODE). We have

$$\begin{aligned} m'(x) \;\; = \;\; & \int_0^x m(y)\, \lambda^2\, e^{-\lambda(x-y)}\, dy \\ & - \lambda \int_0^x m(y)\, \lambda^2\, (x - y)\, e^{-\lambda(x-y)}\, dy, \end{aligned} \tag{30}$$

which can be rewritten as follows:

$$m'(x) = \int_0^x m(y)\, \lambda^2\, e^{-\lambda(x-y)}\, dy - \lambda\, [m(x) - 1]. \tag{31}$$

Notice that, since $m(0^+) := \lim_{x \downarrow 0} m(x) = 1$, this equation implies that $m'(0^+) = 0$.

Differentiating a second time, we get

$$\begin{aligned} m''(x) \;\; = \;\; & m(x)\, \lambda^2 - \lambda \int_0^x m(y)\, \lambda^2\, e^{-\lambda(x-y)}\, dy \\ & - \lambda m'(x) \\ = \;\; & m(x)\, \lambda^2 - \lambda\, \{m'(x) + \lambda\, [m(x) - 1]\} \\ & - \lambda m'(x). \end{aligned} \tag{32}$$

That is,

$$m''(x) + 2\, \lambda m'(x) - \lambda^2 = 0. \tag{33}$$

We can easily obtain the general solution of this second-order ODE with constant coefficients. The solution that satisfies the boundary conditions $m(0^+) = 1$ and $m'(0^+) = 0$ is the function defined in Eq. (28). If we substitute this function into the integral equation (29), we find that it does indeed satisfy this equation. However, *Maple* does not seem able to compute the right-hand side of Eq. (29) with the function $m(x)$ given in Eq. (24). Therefore, we must conclude that the solution of the integral equation provided by *Maple* is not correct.

Remark. When *Maple* is unable to give us the exact solution to a certain integral equation, we can use the option *Neumann*. This option yields an approximate solution (which is actually sometimes the exact one) known as a Neumann series solution (see, Arfken [10], p. 879). In the case of the integral equation (29), the Neumann series solution $m_N(x)$ is of the form

$$m_N(x) = 7 - e^{-x} P(x), \tag{34}$$

where $P(x)$ is a polynomial of degree 11. When we plot the function $m(x)$ defined in Eq. (28) and $m_N(x)$, we find that they practically coincide in the interval $[0, 1]$. The largest difference between the two functions is near $x = 1$, and it is smaller than 5×10^{-12}; see Figure 2.

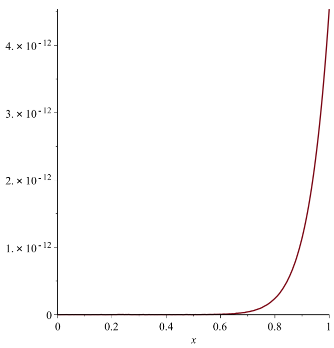

Figure 2: Difference between the function $m(x)$ defined in Eq. (28) and the Neumann series solution $m_N(x)$ given in Eq. (34) when $\lambda = 1$, for $0 \leq x \leq 1$.

Case III. Assume next that $\epsilon_1 = |Z_0|$, where $Z_0 \sim \mathrm{N}(0, 1)$, so that

$$f_{\epsilon_1}(z) = u_0(z)\sqrt{\frac{\pi}{2}}\, e^{-z^2/2}. \tag{35}$$

Maple is unable to solve the integral equation exactly with this density function. Using the option *Neumann*, it gives us a very complicated expression. We are however able to plot the Neumann series solution. It is shown in Figure 3 for $x \in [0, 1]$.

The moment-generating function of ϵ_1 is given by

$$M_{\epsilon_1}(s) = e^{s^2/2}\,\mathrm{erfc}(s/\sqrt{2}), \tag{36}$$

where "erfc" is the *complementary error function*. Unfortunately, it seems very difficult to invert the Laplace transform of $m(x)$ obtained by substituting the function $M_{\epsilon_1}(s)$ into Eq. (12).

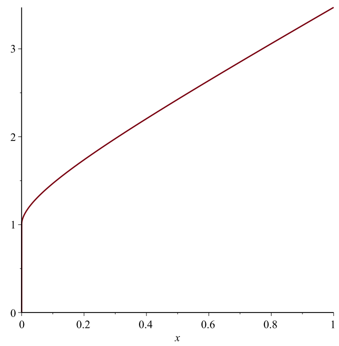

Figure 3: Neumann series solution when $f_{\epsilon_1}(z)$ is the function defined in Eq. (35), for $0 \leq x \leq 1$.

Similarly, it does not seem possible to transform the integral equation into an ODE, like we did in the two previous cases. Therefore, in this case we must content ourselves with the approximate solution provided by *Maple*, which can at least be evaluated numerically.

Case IV. In the last example that we present, we relax the assumption that $S_{\epsilon_1} = [0, \infty)$. What is important, is that $S_{\epsilon_1} = [0, d]$, with $d > X(0) = x$. Suppose that $\epsilon_1 \sim \mathrm{U}[0, x+\delta]$, where $\delta > 0$. That is, ϵ_1 is uniformly distributed over the interval $[0, x + \delta]$, so that

$$f_{\epsilon_1}(z) = \frac{1}{x+\delta} \quad \text{for } 0 \leq z \leq x + \delta. \tag{37}$$

We must then solve the integral equation

$$\begin{aligned} m(x) &= 1 + \int_0^x m(x-z)\,\frac{1}{x+\delta}\,dz \\ &= 1 + \frac{1}{x+\delta}\int_0^x m(y)\,dy. \end{aligned} \tag{38}$$

We find that

$$m(x) = 1 + \ln\left(\frac{x+\delta}{\delta}\right). \tag{39}$$

This solution can also be obtained by differentiating Eq. (38):

$$\begin{aligned} m'(x) &= -\frac{1}{(x+\delta)^2}\int_0^x m(y)\,dy + \frac{1}{x+\delta}\,m(x) \\ &= \frac{1}{x+\delta}. \end{aligned} \tag{40}$$

The solution of this simple first-order ODE that satisfies the boundary condition $m(0^+) = 1$ is indeed the function given in Eq. (39). Notice that if δ decreases to zero, then $E[T(x)]$ tends to infinity.

Remarks. (i) This example can be generalized by defining

$$T_\gamma(x) = \min\{n \geq 0 : X_n \leq \gamma \mid X_0 = x \geq \gamma\}, \tag{41}$$

where $\gamma \geq 0$. We then find that

$$m(x) = 1 + \ln\left(\frac{x+\delta-\gamma}{\delta}\right). \tag{42}$$

(ii) If we assume instead that $\epsilon_1 \sim \mathrm{U}[0,d]$, where $d > X(0) = x$, the integral equation becomes

$$m(x) = 1 + \int_0^x m(x-z)\,\frac{1}{d}\,dz = 1 + \frac{1}{d}\int_0^x m(y)\,dy. \tag{43}$$

It is easy to check that

$$m(x) = e^{x/d}. \tag{44}$$

4 Conclusion

In this note, a first-passage-time problem for stochastic processes related to autoregressive processes of order one, denoted by AR(1), has been considered. While in the case of AR(1) processes the noise term ϵ_n is assumed to have zero mean, here we assumed instead that ϵ_n is a non-negative random variable, so that $E[\epsilon_n] > 0$. Moreover, by definition, the stochastic process $\{X_n, n = 0, 1, \ldots\}$ is non-increasing and could therefore be used as a model for the remaining lifetime of a device or an engineering structure, such as a dam.

In Section 2, we gave three ways to calculate the average value of the first-passage time $T(x)$ to zero, from $X_0 = x$. Then, in Section 3, we presented explicit and exact solutions to important particular problems.

Because we assumed that ϵ_n is a continuous random variable, one way to obtain $E[T(x)]$ is by solving an integral equation. When ϵ_n is a random variable of discrete type, Eq. (6) will become a difference equation instead.

We also assumed that either $S_{\epsilon_n} = [0, \infty)$ or $S_{\epsilon_n} = [0, d]$, with $d > X_0 = x$. If $d < x$, the possibility that $E[T(x)]$ does not exist (that is, is infinite) becomes much more likely.

Finally, it would be interesting to compute the moment-generating function and/or the distribution of $T(x)$.

References:

[1] A. A. Novikov, On the first passage time of an autoregressive process over a level and an application to a "disorder" problem, *Theory of Probability and its Applications*, Vol. 25, No. 2, 1990, pp. 269–279. https://doi.org/10.1137/1135035

[2] A. A. Novikov, On distributions of first passage times and optimal stopping of AR(1) sequences, *Theory of Probability and its Applications*, Vol. 53, No. 3, 2009, pp. 419–429. https://doi.org/10.1137/S0040585X97983730

[3] H. Larralde, A first passage time distribution for a discrete version of the Ornstein-Uhlenbeck process, *Journal of Physics A: Mathematical and General*, Vol. 37, No. 12, 2004, pp. 3759–3767. https://doi.org/10.1088/0305-4470/37/12/003

[4] G. K. Basak and K.-W. R. Ho, Level-crossing probabilities and first-passage times for linear processes, *Advances in Applied Probability*, Vol. 36, No. 2, 2004, pp. 643–666. https://doi.org/10.1239/aap/1086957589

[5] A. Novikov and N. Kordzakhia, Martingales and first passage times of AR(1) sequences, *Stochastics*, Vol. 80, Nos. 2-3, 2008, pp. 197–210. https://doi.org/10.1080/17442500701840885

[6] E. Di Nardo, On the first passage time for autoregressive processes, *Scientiae Mathematicae Japonicae*, Vol. 67, No. 2, 2008, pp. 137–152. https://doi.org/10.32219/isms.67.2_137

[7] C. Baumgarten, Survival probabilities of autoregressive processes, *European Series in Applied and Industrial Mathematics (ESAIM): Probability and Statistics*, Vol. 18, 2014, pp. 145–170. https://doi.org/10.1051/ps/2013031

[8] F. H. Rahimov, I. A. Ibadova and A. D. Farhadova, Limit theorem for first passage times in the random walk described by the generalization of the autoregressive process, *Uzbek Mathematical Journal*, Vol. 2020, No. 4, 2020, pp. 102–110. https://doi.org/10.29229/uzmj.2020-4-11

[9] R. T. Aliyev, F. Rahimov and A. Farhadova, On the first passage time of the parabolic boundary by the Markov random walk, *Communications in Statistics - Theory and Methods*, Vol. 52, No. 17, 2023, pp. 6078–6087. https://doi.org/10.1080/03610926.2021.2024852

[10] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed., Academic Press, Orlando, FL, 1985.

[11] G. B. Arfken, H. J. Weber and F. E. Harris, *Mathematical Methods for Physicists. A Comprehensive Guide*, 7th ed., Elsevier/Academic Press, Amsterdam, 2013.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
The author did all the research work of this study.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
This research was supported by the Natural Sciences and Engineering Research Council of Canada.

Conflicts of Interest
The author has no conflict of interest to declare that is relevant to the content of this article.